C H A P T E R

6

Circular functions

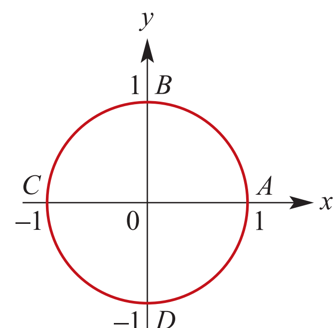
Objectives

- To measure angles in **degrees and radians**.
- To define the circular functions **sine, cosine and tangent**.
- To explore the **symmetry properties** of circular functions.
- To find **exact values** of circular functions.
- To sketch the **graphs of circular functions**.
- To **solve problems** with circular functions.

6.1 Review of circular (trigonometric) functions

Measuring angles in degrees and radians

The diagram shows a unit circle, i.e. a circle of radius 1 unit.



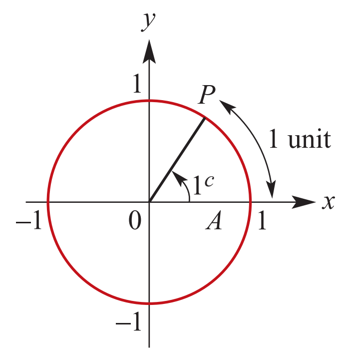
$$\begin{aligned}\text{The circumference of the unit circle} &= 2\pi \times 1\\ &= 2\pi \text{ units}\end{aligned}$$

$\therefore$ the distance in an anti-clockwise direction around the circle from:

$$A \text{ to } B = \frac{\pi}{2} \text{ units}$$
$$A \text{ to } C = \pi \text{ units}$$
$$A \text{ to } D = \frac{3\pi}{2} \text{ units}$$

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One radian (written 1^c) is the angle **subtended** at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anti-clockwise** around the circumference of the unit circle are defined as **positive**. Those formed by moving in a **clockwise** direction are said to be **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$\therefore\ 2\pi^c = 360^\circ$$

$$\therefore\ \pi^c = 180^\circ$$

$$\therefore\ 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Example 1

Convert 30° to radians.

Solution

$$\text{Since } 1^\circ = \frac{\pi^c}{180}$$

$$\therefore\ 30^\circ = \frac{30 \times \pi^c}{180}$$

$$= \frac{\pi^c}{6}$$

Example 2

Convert $\frac{\pi^c}{4}$ to degrees.

Solution

$$\text{Since } 1^c = \frac{180^\circ}{\pi}$$

$$\therefore\ \frac{\pi^c}{4} = \frac{\pi \times 180^\circ}{4 \times \pi}$$

$$= 45^\circ$$

Note: Often the symbol for radian, ^{c}, is omitted. For example, angle 45° is written as $\frac{\pi}{4}$, rather than $\frac{\pi^c}{4}$.

Exercise 6A

1 Express the following angles in radian measure in terms of π:

a 50° **b** 136° **c** 250° **d** 340° **e** 420° **f** 490°

2 Express, in degrees, the angles with the following radian measures:

a $\frac{\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{4\pi}{3}$ **d** $\frac{7\pi}{9}$ **e** 3.5π **f** $\frac{7\pi}{5}$

3 Use a calculator to convert each of the following angles from radians to degrees:

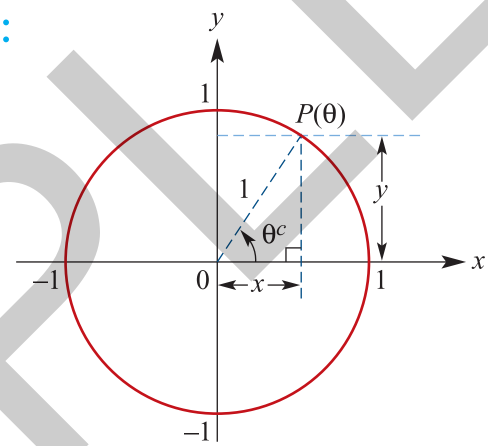
a 0.8 **b** 1.64 **c** 2.5 **d** 3.96 **e** 4.18 **f** 5.95

4 Use a calculator to express each of the following in radian measure. (Give your answer correct to two decimal places.)

a 37° **b** 74° **c** 115° **d** 122.25° **e** 340° **f** 132.5°

Defining circular functions: sine, cosine and tangent

Considering the unit circle.



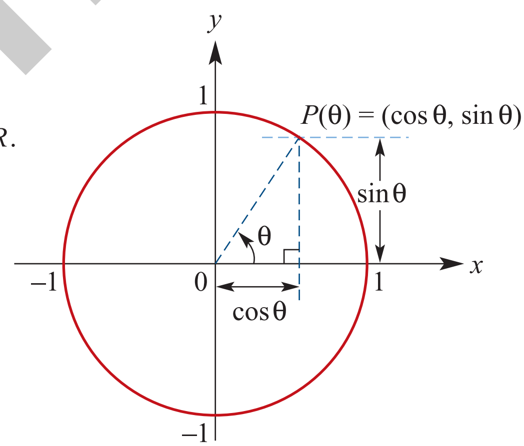
The position of point P on the circle can be described by relating the cartesian coordinates x and y and the angle, θ. The point P on the circumference corresponding to an angle θ is written $P(\theta)$.

Many different angles will give the same point, P, on the circle so the relation linking an angle to the coordinates is a many-to-one function. There are, in fact, two functions involved and they are called **sine** and **cosine** and are defined as follows:

- The x-coordinate of P, $x = \text{cosine } \theta, \theta \in R$.
- The y-coordinate of P, $y = \text{sine } \theta, \theta \in R$.

Note: These functions are usually written in an abbreviated form as follows:

- $x = \cos\theta$
- $y = \sin\theta$



Example 3

Evaluate $\sin\pi$ and $\cos\pi$.

Solution

In moving through an angle of π, the position is $P(\pi)$, which is $(-1, 0)$.

$\therefore\ \cos\pi = -1$

$\sin\pi = 0$

Example 4

Evaluate $\sin -\frac{3\pi}{2}$ and $\cos -\frac{\pi}{2}$.

Solution

$$\sin -\frac{3\pi}{2} = 1$$

$$\cos -\frac{\pi}{2} = 0$$

Example 5

With a calculator evaluate, correct to two decimal places:

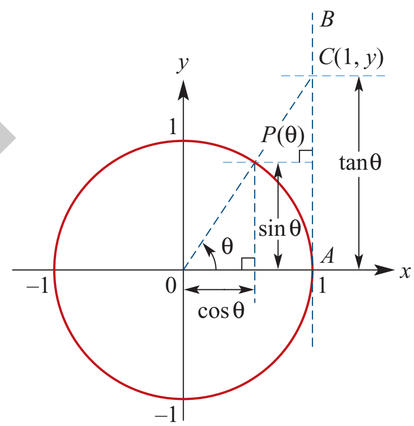
a $\sin 1.8$ **b** $\cos -2.6$ **c** $\sin 3.8$

Solution

a 0.97 **b** −0.86 **c** −0.61

Again consider the unit circle.

If a tangent to the unit circle, at A, is drawn, then the y-coordinate of C, the point of intersection of the extension of OP and the tangent, is called **tangent θ** (abbreviated to $\tan\theta$).



By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Now when $\cos\theta = 0$, $\tan\theta$ is undefined hence $\tan\theta$ is undefined when:

$$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \quad \therefore \text{domain of } \tan = R \setminus \{\theta : \cos\theta = 0\}$$

Example 6

Evaluate, using a calculator:

a $\tan 1.3$ **b** $\tan 1.9$ **c** $\tan -2.8$ **d** $\tan 59^\circ$ **e** $\tan 138^\circ$

Solution

a $\tan 1.3 = 3.6$ **b** $\tan 1.9 = -2.93$

c $\tan -2.8 = 0.36$ **d** $\tan 59^\circ = 1.66$

e $\tan 138^\circ = -0.9$

(Your calculator is in **Radian** mode for **a**, **b** and **c** and **Degree** mode for **d** and **e**.)

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Exercise 6B

1 For each of the following angles, t, determine the values of sin t and cos t:

a $t = 0$ **b** $t = \frac{3\pi}{2}$ **c** $t = -\frac{5\pi}{2}$ **d** $t = -\frac{7\pi}{2}$ **e** $t = 3\pi$ **f** $t = -4\pi$

2 Evaluate each of the following:

a $\tan(-\pi)$ **b** $\tan\frac{5\pi}{2}$ **c** $\tan\left(-\frac{7\pi}{2}\right)$ **d** $\tan 2\pi$ **e** $\tan\left(-\frac{\pi}{2}\right)$

3 Evaluate each of the following:

a $\cos(23\pi)$ **b** $\cos\left(\frac{49\pi}{2}\right)$ **c** $\cos(-35\pi)$ **d** $\cos\left(\frac{-45\pi}{2}\right)$

e $\cos(-24\pi)$ **f** $\cos(20\pi)$

4 Evaluate each of the following using a calculator. (Give answers correct to two decimal places.)

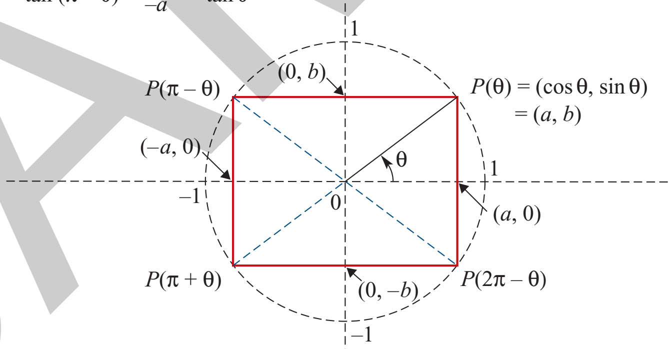
a $\sin 1.7$ **b** $\sin 2.6$ **c** $\sin 4.2$ **d** $\cos 0.4$ **e** $\cos 2.3$
f $\cos(-1.8)$ **g** $\sin(-1.7)$ **h** $\sin(-3.6)$ **i** $\tan 1.6$ **j** $\tan(-1.2)$
k $\tan 3.9$ **l** $\tan(-2.5)$

Symmetry properties of circular functions

The following relationships can be observed:

Quadrant 2
By symmetry
$\sin(\pi - \theta) = b = \sin\theta$
$\cos(\pi - \theta) = -a = -\cos\theta$
$\tan(\pi - \theta) = \frac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3
$\sin(\pi + \theta) = -b = -\sin\theta$
$\cos(\pi + \theta) = -a = -\cos\theta$
$\tan(\pi + \theta) = \frac{-b}{-a} = \tan\theta$

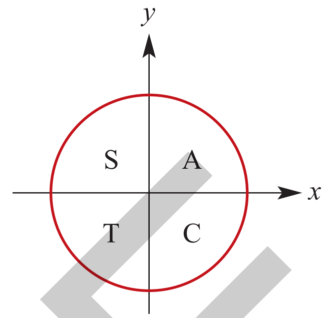
Quadrant 4
$\sin(2\pi - \theta) = -b = -\sin\theta$
$\cos(2\pi - \theta) = a = \cos\theta$
$\tan(2\pi - \theta) = -\frac{b}{a} = -\tan\theta$

Note: These relationships are true for all values of θ.

Signs of circular functions

These symmetry properties can be summarised for the signs of sin, cos and tan for the four quadrants as follows:

1st quadrant: all are positive (A).
2nd quadrant: sin is positive (S).
3rd quadrant: tan is positive (T).
4th quadrant: cos is positive (C).



Example 7

If $\sin\theta = 0.5$ and $\cos\alpha = 0.6$, find the value of:

a $\sin(\pi - \theta)$ **b** $\cos(2\pi - \alpha)$

Solution

By symmetry:

a $\sin(\pi - \theta) = \sin\theta$
$= 0.5$

b $\cos(2\pi - \alpha) = \cos\alpha$
$= 0.6$

Example 8

For the following find two values of x in the range $0 \le x \le 360$:

a $\sin x^\circ = -0.3$ **b** $\cos x^\circ = -0.7$

Solution

a First solve the equation $\sin x^\circ = 0.3$

Make sure that your calculator is in degree mode.

If $\sin x^\circ = 0.3$
$x = 17.46$

Now the value of sin is negative for $P(x)$ in the 3rd and 4th quadrants.
From the symmetry relationships:

3rd quadrant $x = 180 + 17.46$
$= 197.46$

4th quadrant $x = 360 - 17.46$
$= 342.54$

$\therefore$ if $\sin x^\circ = -0.3$, $x = 197.46, 342.54$

b First solve the equation $\cos x^\circ = 0.7$, i.e. $x = 45.57$

Now the value of cos is negative for $P(x)$ in the 2nd and 3rd quadrants.

2nd quadrant $x = 180 - 45.57$
$= 134.43$

3rd quadrant $x = 180 + 45.57$
$= 225.57$

$\therefore$ if $\cos x^\circ = -0.7$, $x = 134.43, 225.57$

Exercise 6C

1 If $\sin\theta = 0.52$, $\cos x = 0.68$ and $\tan\alpha = 0.4$ find the value of:

a $\sin(\pi - \theta)$ **b** $\cos(\pi + x)$ **c** $\sin(2\pi + \theta)$ **d** $\tan(\pi + \alpha)$
e $\sin(\pi + \theta)$ **f** $\cos(2\pi - x)$ **g** $\tan(2\pi - \alpha)$ **h** $\cos(\pi - x)$

2 Find all values of x between 0 and 2π for which:

a $\sin x = 0.6$ **b** $\cos x = 0.8$ **c** $\sin x = -0.45$ **d** $\cos x = -0.2$

3 Find all values θ between 0 and 360 for which:

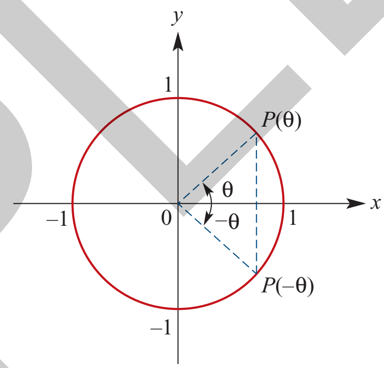
a $\sin\theta^\circ = 0.3$ **b** $\cos\theta^\circ = 0.4$ **c** $\sin\theta^\circ = -0.8$ **d** $\cos\theta^\circ = -0.5$

Further symmetry properties

Negative of angles

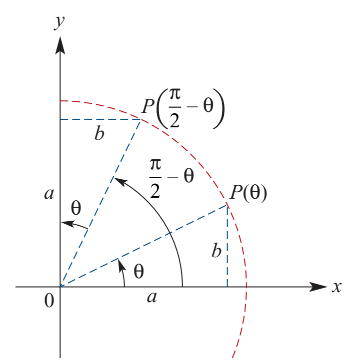
By symmetry:

$$\begin{aligned}
\cos(-\theta) &= \cos\theta \\
\sin(-\theta) &= -\sin\theta \\
\tan(-\theta) &= \frac{-\sin\theta}{\cos\theta} \\
&= -\tan\theta
\end{aligned}$$



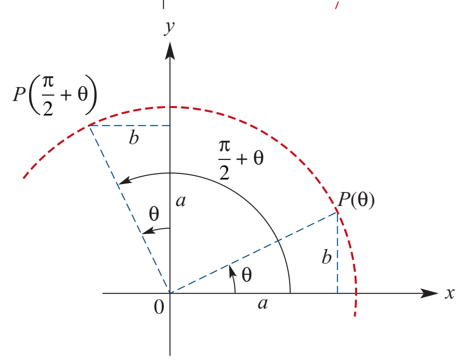
Complementary relationships

$$\begin{aligned}
\sin\left(\frac{\pi}{2} - \theta\right) &= a \\
\text{and since } a &= \cos\theta \\
\therefore\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta
\end{aligned}$$



Similarly:

$$\begin{aligned}
\cos\left(\frac{\pi}{2} - \theta\right) &= b \\
\text{and since } b &= \sin\theta \\
\therefore\ \cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta \\
\sin\left(\frac{\pi}{2} + \theta\right) &= a \\
&= \cos\theta \\
\cos\left(\frac{\pi}{2} + \theta\right) &= -b \\
&= -\sin\theta
\end{aligned}$$



Example 9

If $\sin\theta = 0.3$ and $\cos\psi = 0.8$, find the values of:

a $\sin\left(\frac{\pi}{2} - \psi\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

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Solution

a $\sin\left(\frac{\pi}{2} - \psi\right) = \cos\psi$
$= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.3$

c $\sin(-\theta) = -\sin\theta$
$= -0.3$

Exercise 6D

If $\sin x = 0.3$, $\cos\alpha = 0.6$ and $\tan\theta = 0.7$, find the values of:

1 $\cos(-\alpha)$ **2** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **3** $\tan(-\theta)$ **4** $\cos\left(\frac{\pi}{2} - x\right)$

5 $\sin(-x)$ **6** $\tan\left(\frac{\pi}{2} - \theta\right)$ **7** $\cos\left(\frac{\pi}{2} + x\right)$ **8** $\sin\left(\frac{\pi}{2} - \alpha\right)$

9 $\sin\left(\frac{3\pi}{2} + \alpha\right)$ **10** $\cos\left(\frac{3\pi}{2} - x\right)$

Exact values of circular functions

A calculator can be used to find the values of the circular functions for different values of θ. For many values of θ the calculator gives an approximation. We consider some values of θ such that sin, cos and tan can be calculated exactly.

Exact values for 0 (0°) and $\frac{\pi}{2}$ (90°)

From the unit circle:

when $\theta = 0$, $\sin\theta = 0$

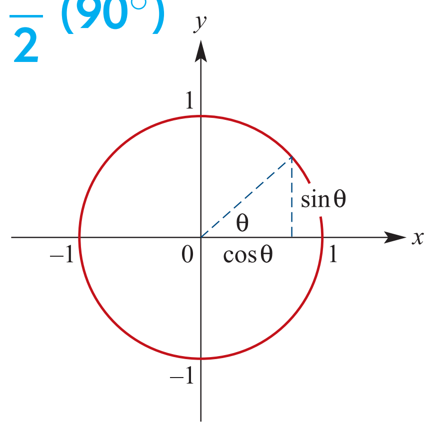
$\cos\theta = 1$

$\tan\theta = 0$

when $\theta = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

$\tan\frac{\pi}{2}$ is undefined.



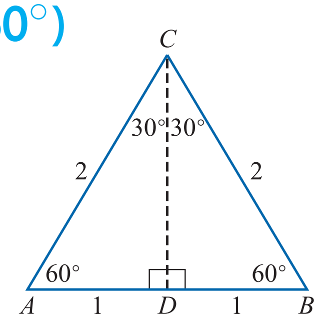
Exact values for $\frac{\pi}{6}$ (30°) and $\frac{\pi}{3}$ (60°)

Consider an equilateral triangle ABC of side length 2 units.

In ΔACD:

by the theorem of Pythagoras:

$$DC = \sqrt{AC^2 - AD^2}$$
$$= \sqrt{3}$$



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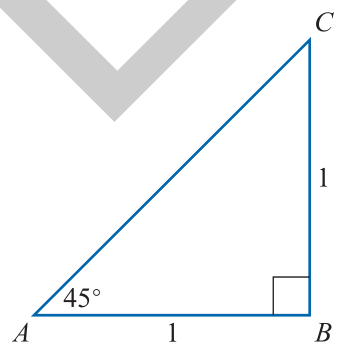
$$\begin{aligned} \sin 30^\circ &= \frac{AD}{AC} \\ &= \frac{1}{2} \\ \cos 30^\circ &= \frac{CD}{AC} \\ &= \frac{\sqrt{3}}{2} \\ \tan 30^\circ &= \frac{AD}{CD} \\ &= \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

$$\begin{aligned} \sin 60^\circ &= \frac{CD}{AC} \\ &= \frac{\sqrt{3}}{2} \\ \cos 60^\circ &= \frac{AD}{AC} \\ &= \frac{1}{2} \\ \tan 60^\circ &= \frac{CD}{AD} \\ &= \frac{\sqrt{3}}{1} \\ &= \sqrt{3} \end{aligned}$$

Exact values for $\frac{\pi}{4}$ (45°)



$$\begin{aligned} AC &= \sqrt{1^2 + 1^2} \\ &= \sqrt{2} \\ \sin 45^\circ &= \frac{BC}{AC} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned} \cos 45^\circ &= \frac{AB}{AC} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \\ \tan 45^\circ &= \frac{BC}{AB} \\ &= 1 \end{aligned}$$

As an aid to memory the exact values for circular functions can be tabulated.

Summary

θ (θ°)	sin θ	cos θ	tan θ
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined

Example 10

Evaluate:

a $\cos \frac{5\pi}{4}$ **b** $\sin \frac{11\pi}{6}$

Solution

a
$$\begin{aligned}\cos \frac{5\pi}{4} &= \cos\left(\pi + \frac{\pi}{4}\right)\\ &= -\cos \frac{\pi}{4} \text{ (by symmetry)}\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

b
$$\begin{aligned}\sin \frac{11\pi}{6} &= \sin\left(2\pi - \frac{\pi}{6}\right)\\ &= -\sin \frac{\pi}{6} \text{ (by symmetry)}\\ &= -\frac{1}{2}\end{aligned}$$

Example 11

Find all values of θ between 0 and 360 for which:

a $\cos \theta^\circ = \frac{\sqrt{3}}{2}$ **b** $\sin \theta^\circ = -\frac{1}{2}$ **c** $\cos \theta^\circ - \frac{1}{\sqrt{2}} = 0$

Solution

a $\cos \theta^\circ$ is positive

$\therefore P(\theta^\circ)$ lies in the 1st or 4th quadrant.

$$\begin{aligned}\cos \theta^\circ &= \frac{\sqrt{3}}{2}\\ \theta &= 30 \text{ or } 360 - 30\\ &= 30 \text{ or } 330\end{aligned}$$

b $\sin \theta^\circ$ is negative

$\therefore P(\theta^\circ)$ is in the 3rd or 4th quadrant.

$$\begin{aligned}\sin \theta^\circ &= -\frac{1}{2}\\ \theta &= 180 + 30 \text{ or } 360 - 30\\ &= 210 \text{ or } 330\end{aligned}$$

c $\cos \theta^\circ - \frac{1}{\sqrt{2}} = 0$

$$\therefore \cos \theta^\circ = \frac{1}{\sqrt{2}}$$

and since $\cos \theta^\circ$ is positive, $P(\theta^\circ)$ lies in the 1st or 4th quadrant.

$$\begin{aligned}\therefore \cos \theta^\circ &= \frac{1}{\sqrt{2}}\\ \theta &= 45 \text{ or } 360 - 45\\ &= 45 \text{ or } 315\end{aligned}$$

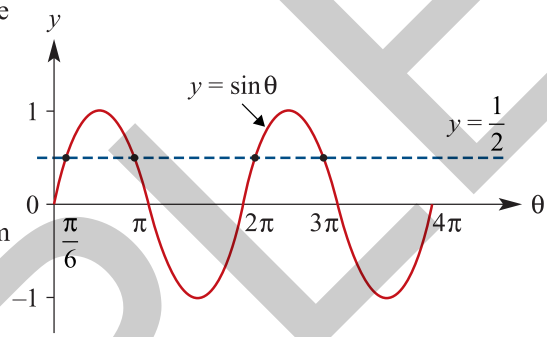
Example 12

Find all solutions to the equation $\sin\theta = \frac{1}{2}$ for $\theta \in [0, 4\pi]$.

Solution

We refer to the graph of $y = \sin\theta$ that was studied in *Essential Mathematical Methods 1 & 2 CAS*. The graph of $y = \sin\theta$ is reintroduced in the following sections.

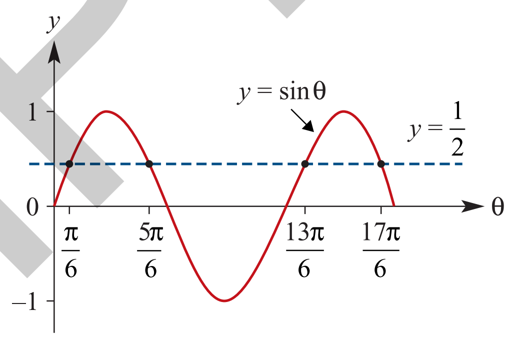
It is clear from the graph that there are four solutions in the interval $[0, 4\pi]$.



The solution for $x \in \left[0, \frac{\pi}{2}\right]$ is $x = \frac{\pi}{6}$.

This solution can be obtained from a knowledge of exact values or using $\sin^{-1}$ on your calculator.

The second solution is obtained by symmetry. The function is positive in the second quadrant and $\sin(\pi - \theta) = \sin\theta$. Therefore $x = \frac{5\pi}{6}$ is the second solution.



It can be seen that further solutions can be achieved by adding 2π, as $\sin\theta = \sin(\theta + 2\pi)$.

Thus $\theta = \frac{13\pi}{6}$ and $\frac{17\pi}{6}$ are also solutions.

Example 13

For $\pi < x < \frac{3\pi}{2}$ with $\cos x = -\sin\frac{\pi}{6}$, find the value of x $\left(\text{do not evaluate } \sin\frac{\pi}{6}\right)$.

Solution

$\cos x = \sin\left(\frac{\pi}{2} - x\right) = -\sin\frac{\pi}{6}$

Hence $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{7\pi}{6}\right)$ as $\sin(\pi + \theta) = \sin(-\theta)$

Solving the equation $\frac{\pi}{2} - x = \frac{7\pi}{6}$ gives $x = -\frac{2\pi}{3}$

This is one solution of the equation $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{7\pi}{6}\right)$. There are infinitely many.

Thus a solution in the interval $\pi < x < \frac{3\pi}{2}$ is $x = \frac{4\pi}{3}$.

Exercise 6E

Skillsheet TEACHER

1 Without using a calculator evaluate the sin, cos and tan of each of the following:

a $150°$ **b** $225°$ **c** $405°$ **d** $-120°$ **e** $-315°$ **f** $-30°$

2 Write down the exact values of:

a $\sin\frac{3\pi}{4}$ **b** $\cos\frac{2\pi}{3}$ **c** $\cos\frac{7\pi}{6}$ **d** $\sin\frac{5\pi}{6}$

e $\cos\frac{4\pi}{3}$ **f** $\sin\frac{5\pi}{4}$ **g** $\sin\frac{7\pi}{4}$ **h** $\cos\frac{5\pi}{3}$

i $\cos\frac{11\pi}{3}$ **j** $\sin\frac{200\pi}{3}$ **k** $\cos -\frac{11\pi}{3}$ **l** $\sin\frac{25\pi}{3}$

m $\sin -\frac{13\pi}{4}$ **n** $\cos -\frac{20\pi}{3}$ **o** $\sin\frac{67\pi}{4}$ **p** $\cos\frac{68\pi}{3}$

3 Find, without using a calculator, all the values of θ between 0 and 360 for each of the following:

a $\cos\theta° = \frac{1}{2}$ **b** $\sin\theta° = \frac{\sqrt{3}}{2}$ **c** $\sin\theta° = -\frac{1}{2}$

d $2\cos\theta° + 1 = 0$ **e** $2\sin\theta° = \sqrt{3}$ **f** $2\cos\theta° = -\sqrt{3}$

4 Solve each of the following for $x \in [-\pi, \pi]$:

a $\sin x = -\frac{1}{2}$ **b** $\cos x = \frac{\sqrt{3}}{2}$ **c** $\cos x = \frac{-\sqrt{3}}{2}$

5 **a** For $\frac{\pi}{2} < x < \pi$ with $\sin x = \sin\frac{\pi}{6}$, find the value of x $\left(\text{do not evaluate } \sin\frac{\pi}{6}\right)$.

b For $\frac{\pi}{2} < x < \pi$ with $\cos x = -\cos\frac{\pi}{6}$, find the value of x $\left(\text{do not evaluate } \cos\frac{\pi}{6}\right)$.

c For $\frac{\pi}{2} < x < \pi$ with $\cos x = -\sin\frac{\pi}{6}$, find the value of x $\left(\text{do not evaluate } \sin\frac{\pi}{6}\right)$.

d For $\frac{\pi}{2} < a < \pi$ with $\cos a = -\sin b$ where $0 < b < \frac{\pi}{2}$, find a in terms of b.

e For $\frac{\pi}{2} < a < \pi$ with $\sin a = \cos b$ where $0 < b < \frac{\pi}{2}$, find a in terms of b.

6 **a** For $\pi < x < \frac{3\pi}{2}$ with $\sin x = -\sin\frac{\pi}{6}$, find the value of x $\left(\text{do not evaluate } \sin\frac{\pi}{6}\right)$.

b For $\pi < x < \frac{3\pi}{2}$ with $\cos x = -\sin\frac{\pi}{6}$, find the value of x $\left(\text{do not evaluate } \sin\frac{\pi}{6}\right)$.

c For $\pi < x < \frac{3\pi}{2}$ with $\cos x = -\cos\frac{\pi}{6}$, find the value of x $\left(\text{do not evaluate } \sin\frac{\pi}{6}\right)$.

d For $\pi < a < \frac{3\pi}{2}$ with $\cos a = -\sin b$ where $0 < b < \frac{\pi}{2}$, find a in terms of b.

e For $\pi < a < \frac{3\pi}{2}$ with $\sin a = -\sin b$ where $0 < b < \frac{\pi}{2}$, find a in terms of b.

f For $\pi < a < \frac{3\pi}{2}$ with $\tan a = \tan b$ where $0 < b < \frac{\pi}{2}$, find a in terms of b.

7 Find, without using a calculator, all the values of x between 0 and 2π for each of the following:

a $\sqrt{2}\sin x - 1 = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $2\cos x + \sqrt{3} = 0$

d $2\sin x + 1 = 0$ **e** $1 - \sqrt{2}\cos x = 0$ **f** $4\cos x + 2 = 0$

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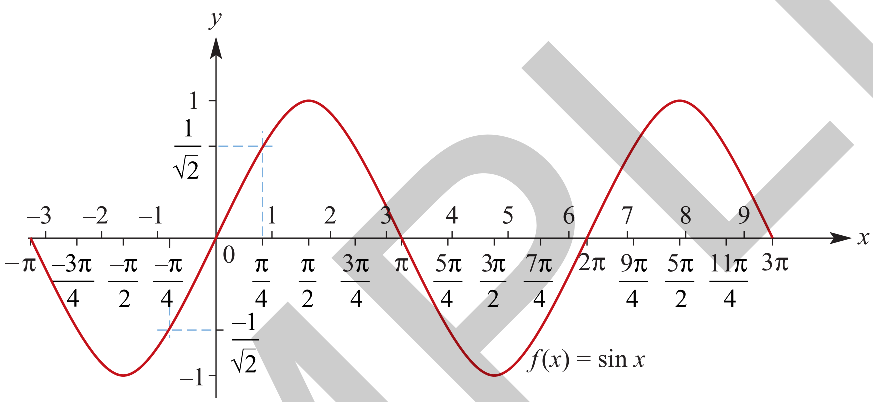
8 Find, without using a calculator, all the values of x between 0 and 2π for each of the following:

a $\sqrt{2}\sin 2x + 1 = 0$ **b** $2\cos 3x + \sqrt{3} = 0$ **c** $4\sin 2x - 2 = 0$
d $\sqrt{2}\cos 3x - 1 = 0$ **e** $10\sin 3x - 5 = 0$ **f** $2\sin 2x = \sqrt{2}$
g $4\cos 3x = -2\sqrt{3}$ **h** $-2\sin 3x = \sqrt{2}$ **i** $4\cos 2x = -2$

6.2 Graphs of sine and cosine

Graphs of sine functions

The graph of $f(x) = \sin x$ is given below. It has been plotted for $-\pi \le x \le 3\pi$.

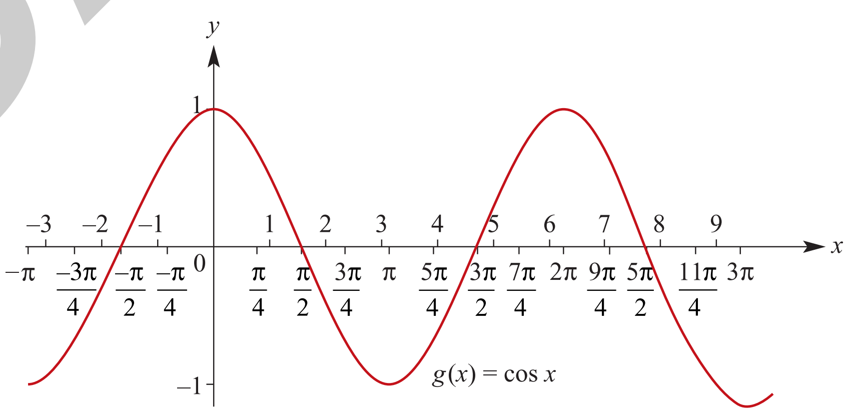


Observations from the graph

- The graph repeats itself after an interval of 2π units, i.e. $f(x + 2k\pi) = f(x)$ for all $x \in R$, $k \in Z$. A function which repeats itself regularly is called a **periodic** function and the interval between the repetitions is called the **period** of the function. Thus $\sin x$ has period of 2π units.
- The maximum and minimum values of $\sin x$ are 1 and -1 respectively. The distance between the mean position and the maximum position is called the **amplitude**. The graph of $f(x) = \sin x$ has an amplitude of 1.

Graphs of cosine functions

The graph of $g(x) = \cos x$ for $-\pi \le x \le 3\pi$ is as shown.



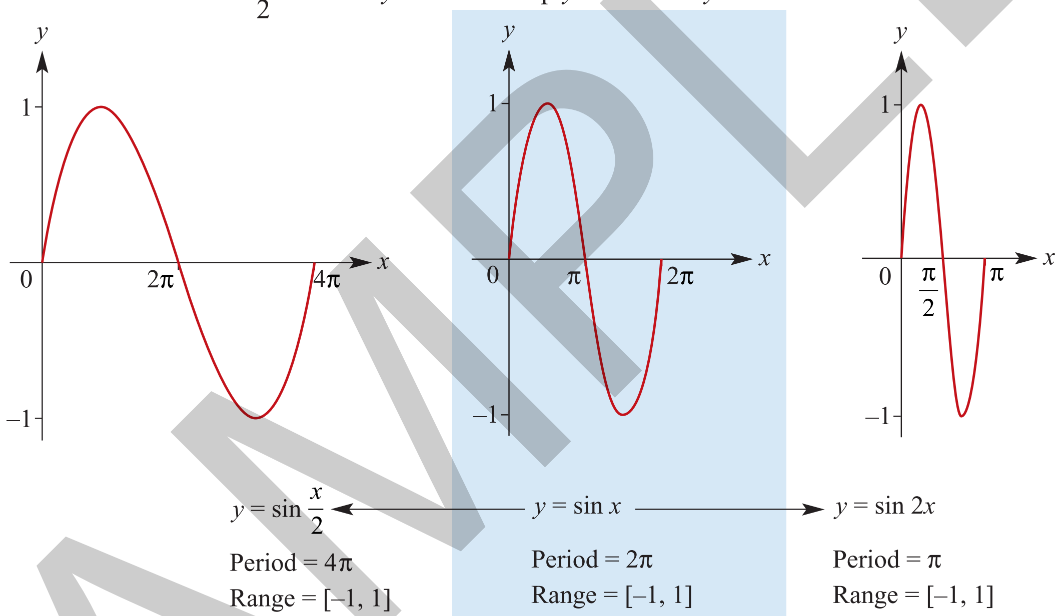
6.3 Transformations applied to graphs of $y = \sin x$ and $y = \cos x$

Dilations

A dilation of factor 2 from the y-axis has the rule $(x, y) \to (2x, y)$.

Hence $(0, 0) \to (0, 0)$, $\left(\frac{\pi}{2}, 1\right) \to (\pi, 1)$ and $(\pi, 0) \to (2\pi, 0)$. When this transformation is applied to $y = \sin x$ it will be 'stretched out' parallel to the x-axis. Let $(x, y) \to (x', y')$ under this transformation. Then $x' = 2x$ and $y' = y$, and thus $x = \frac{x'}{2}$ and $y = y'$. Hence $y = \sin x$ is mapped to $y = \sin \frac{x}{2}$.

A dilation of factor $\frac{1}{2}$ from the y-axis will map $y = \sin x$ to $y = \sin 2x$.



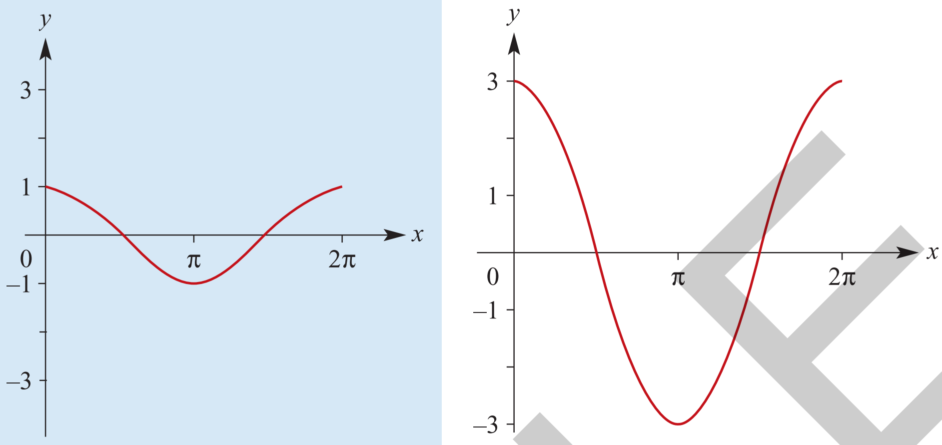
In general:

- $f: R \to R,\ f(x) = \sin(nx)$
- Period $= \dfrac{2\pi}{|n|}$
- Amplitude $= 1$
- Range $= [-1, 1]$

- $f: R \to R,\ f(x) = \cos(nx)$
- Period $= \dfrac{2\pi}{|n|}$
- Amplitude $= 1$
- Range $= [-1, 1]$

A dilation of factor 3 from the x-axis has rule $(x, y) \to (x, 3y)$. Hence $(0, 1) \to (0, 3)$, $\left(\frac{\pi}{2}, 0\right) \to \left(\frac{\pi}{2}, 0\right)$ and $(\pi, -1) \to (\pi, -3)$. When this transformation is applied to $y = \cos x$, it will be 'stretched out' parallel to the y-axis.

$y = \cos x$ is mapped to $y = 3\cos x$.



$y = \cos x$ → $y = 3\cos x$

Period $= 2\pi$ | Period $= 2\pi$

Range $= [-1, 1]$ | Range $= [-3, 3]$

In general:

- $f: R \to R,\ f(x) = a\sin(nx)$
- Period $= \dfrac{2\pi}{|n|}$
- Amplitude $= |a|$
- Range $= [-|a|, |a|]$

- $f: R \to R,\ f(x) = a\cos(nx)$
- Period $= \dfrac{2\pi}{|n|}$
- Amplitude $= |a|$
- Range $= [-|a|, |a|]$

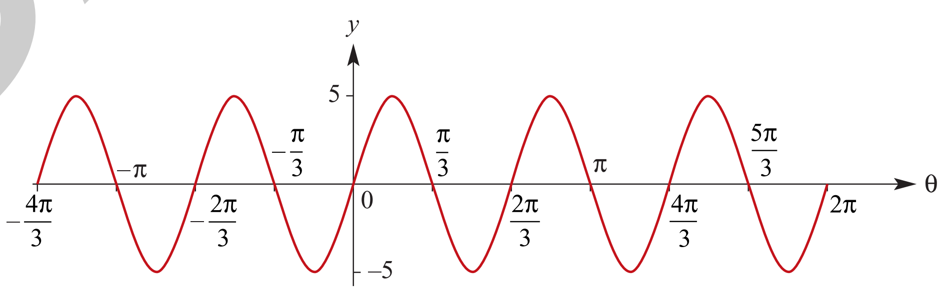
Example 14

Sketch the graph of $y = 5\sin 3\theta$ for $-\dfrac{4\pi}{3} \le \theta \le 2\pi$.

Solution

The amplitude $= 5$

The period $= \dfrac{2\pi}{3}$



The θ-axis intercepts can also be found by solving the equation.

$$5\sin 3\theta = 0$$
$$\therefore \sin 3\theta = 0$$
$$\therefore 3\theta = -4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$$
$$\theta = -\frac{4\pi}{3}, -\pi, -\frac{2\pi}{3}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$$

Exercise 6F

1 Write down **i** the period and **ii** the amplitude of each of the following:

a $3\sin\theta$ **b** $2\sin 3\theta$ **c** $\frac{1}{2}\cos 2\theta$ **d** $2\sin\frac{1}{3}\theta$

e $3\cos 4\theta$ **f** $\frac{1}{2}\sin\theta$ **g** $3\cos\frac{1}{2}\theta$

2 Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 2\sin 3\theta$ **b** $y = 2\cos 2\theta$ **c** $y = 3\sin\frac{1}{3}\theta$

d $y = \frac{1}{3}\cos 2\theta$ **e** $y = 3\sin 4\theta$

3 Sketch the graph of $f: R \to R,\ f(x) = 5\cos 3x$ for $-\pi \le x \le \pi$.

4 Sketch the graph of $f: R \to R,\ f(x) = \frac{1}{2}\sin 2x$ for $-\pi \le x \le 2\pi$.

5 Sketch the graph of $f: R \to R,\ f(x) = 2\cos\frac{3x}{2}$ for $0 \le x \le 2\pi$.

6 Find the equation of the image of the graph of $y = \sin x$ under a dilation of factor 2 from the x-axis followed by a dilation of factor 3 from the y-axis.

7 Find the equation of the image of the graph of $y = \cos x$ under a dilation of factor $\frac{1}{2}$ from the x-axis followed by a dilation of factor 3 from the y-axis.

8 Find the equation of the image of the graph of $y = \sin x$ under a dilation of factor $\frac{1}{2}$ from the x-axis followed by a dilation of factor 2 from the y-axis.

Reflections in the axes

The function with rule $f(x) = \sin x$ is an odd function, i.e. $f(-x) = -f(x)$. A reflection in the y-axis gives the 'same' result as a reflection in the x-axis when applied to the graph of $y = \sin x$.

The function with rule $f(x) = \cos x$ is an even function, i.e. $f(-x) = f(x)$. The graph of $f(x) = \cos x$ is mapped onto itself when reflected in the y-axis.

Example 15

Sketch the graph of:

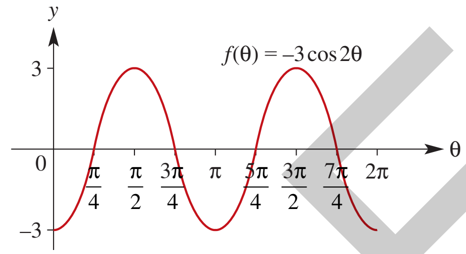
a $f(\theta) = -3\cos 2\theta \quad$ for $0 \leq \theta \leq 2\pi$

b $g(\theta) = 5\sin(-3\theta) \quad$ for $0 \leq \theta \leq 2\pi$

Solution

a Period $= \pi$

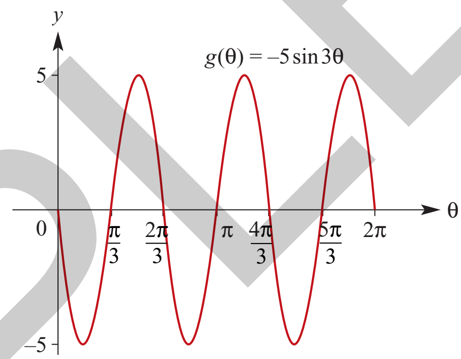
Amplitude $= 3$



b Period $= \dfrac{2\pi}{3}$

Amplitude $= |-5|$

$= 5$



Translations

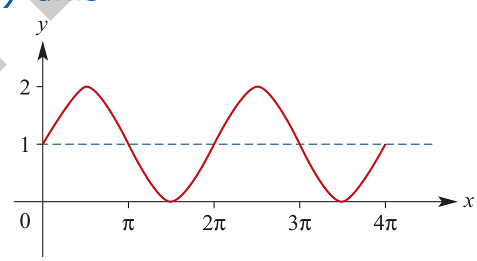
Translations in the direction of the *y*-axis

The graph of $y = \sin x + 1$ is obtained from the graph of $y = \sin x$ by a translation of 1 unit in the positive direction of the y-axis.

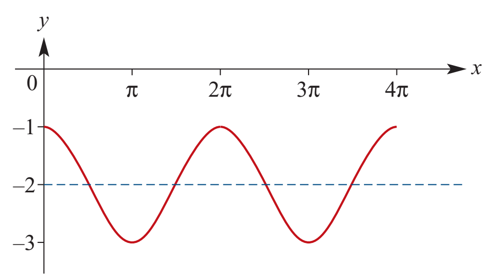
Period $= 2\pi$

Range $= [0, 2]$

Amplitude $= 1$

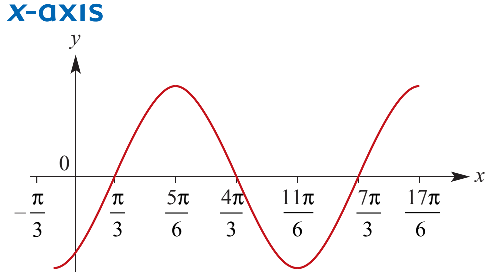


The graph of $y = \cos 2x - 2$ is obtained from the graph of $y = \cos 2x$ by a translation of 2 units in the negative direction of the y-axis.



Translations in the direction of the *x*-axis

The graph of $y = \sin\left(x - \dfrac{\pi}{3}\right)$ is obtained from the graph of $y = \sin x$ by a translation of $\dfrac{\pi}{3}$ units in the positive direction of the x-axis.



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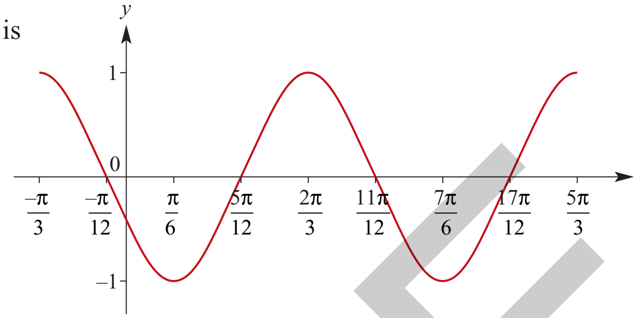
The graph of $y = \cos 2\left(x + \frac{\pi}{3}\right)$ is obtained from the graph of $y = \cos 2x$ by a translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis.



Example 16

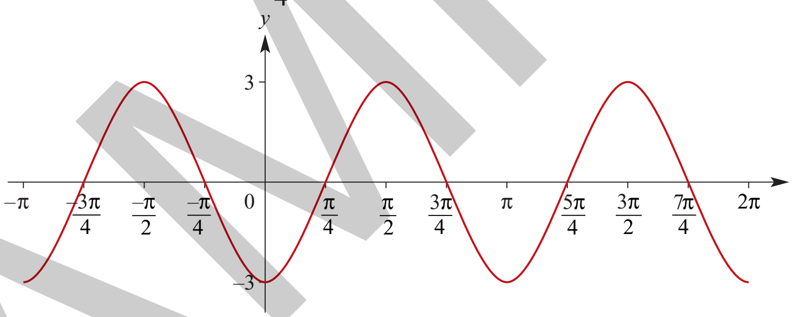
On separate axes sketch the graphs of:

a $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$ for $-\pi \le t \le 2\pi$ **b** $y = 3\cos 3\left(t + \frac{\pi}{3}\right)$ for $-\pi \le t \le \pi$

Solution

a **Note:** the transformations applied to $y = \sin t$ are:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of $\frac{\pi}{4}$ in the positive direction of the x-axis.



Period $= \pi$

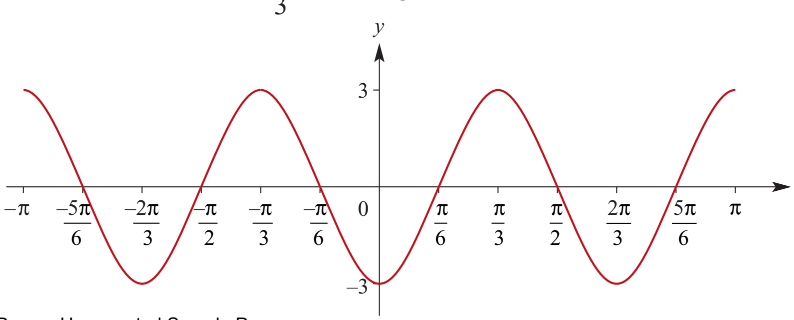
Amplitude $= 3$

Range $= [-3, 3]$

Note: This is the graph of $y = -3\cos 2t$.

b **Note:** the transformations applied to $y = \cos t$ are:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{3}$ from the y-axis
- a translation of $\frac{\pi}{3}$ in the negative direction of the x-axis.



Sketching graphs of $y = a\sin n(t \pm \varepsilon) \pm b$ and $y = a\cos n(t \pm \varepsilon) \pm b$

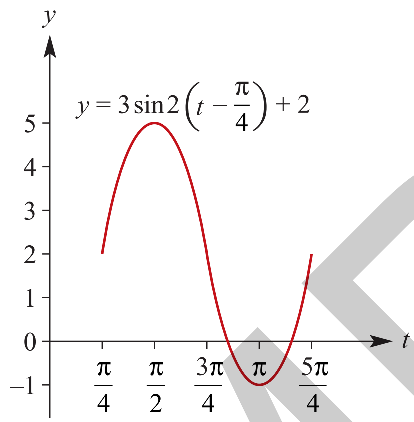
Example 17

On separate axes sketch the graphs of:

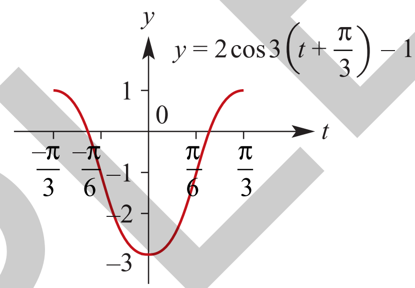
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2,\ \frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1,\ -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a



b



Observations

- The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2$ is the same shape as the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, but it is translated 2 units in the positive direction of the y-axis.
- Similarly, the graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$ is the same shape as the graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, but it is translated 1 unit in the negative direction of the y-axis.

Exercise 6G

1 Sketch the graph of each of the following, showing one complete cycle. State the period, amplitude and range in each case.

a $y = 2\sin\left(\theta - \frac{\pi}{3}\right)$
b $y = \sin 2(\theta - \pi)$
c $y = 3\sin 2\left(\theta + \frac{\pi}{4}\right)$
d $y = \sqrt{3}\sin 3\left(\theta - \frac{\pi}{2}\right)$
e $y = 2\sin 3x + 1$
f $y = 3\cos 2\left(x + \frac{\pi}{2}\right) - 1$
g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{6}\right) + 2$
h $y = 3 - 4\sin 2x$
i $y = 2 - 3\cos 2\left(\theta - \frac{\pi}{2}\right)$

2 Find the equation of the image of the graph of $y = \cos x$ under:

a a dilation of factor $\frac{1}{2}$ from the x-axis, followed by a dilation of factor 3 from the y-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis.

b a dilation of factor 2 from the x-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis.

c a dilation of factor $\frac{1}{3}$ from the x-axis, followed by a reflection in the x-axis, followed by a translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis.

3 For each of the following equations, give a sequence of transformations that takes the graph of $y = \sin x$ to the graph of the equation:

a $y = -3\sin 2x$ **b** $y = -3\sin 2\left(x - \frac{\pi}{3}\right)$

c $y = 3\sin 2\left(x - \frac{\pi}{3}\right) + 2$ **d** $y = 5 - 2\sin 2\left(x - \frac{\pi}{3}\right)$

Finding axis intercepts

Example 18

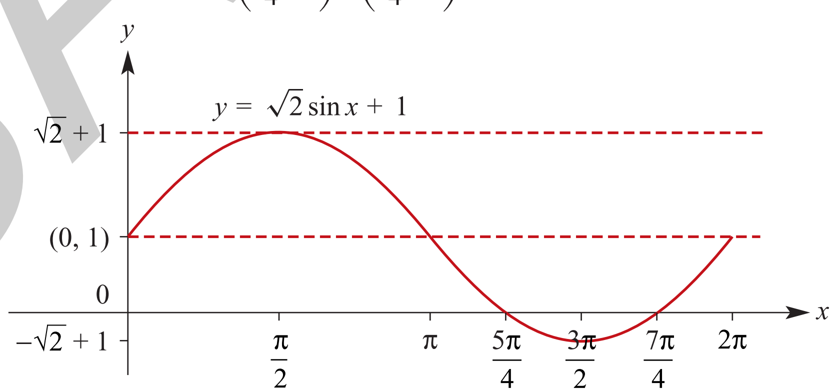
Sketch the graphs of each of the following for $x \in [0, 2\pi]$. Clearly indicate axis intercepts.

a $y = \sqrt{2}\sin x + 1$ **b** $y = 2\cos 2x - 1$ **c** $y = 2\sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3}$

Solution

a To determine the axis intercepts the equation $\sqrt{2}\sin x + 1 = 0$ must be solved.

$$\sqrt{2}\sin x + 1 = 0$$
$$\therefore \sin x = -\frac{1}{\sqrt{2}}$$
$$\therefore x = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$
$$\therefore x = \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$\therefore \text{ intercepts are } \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$$



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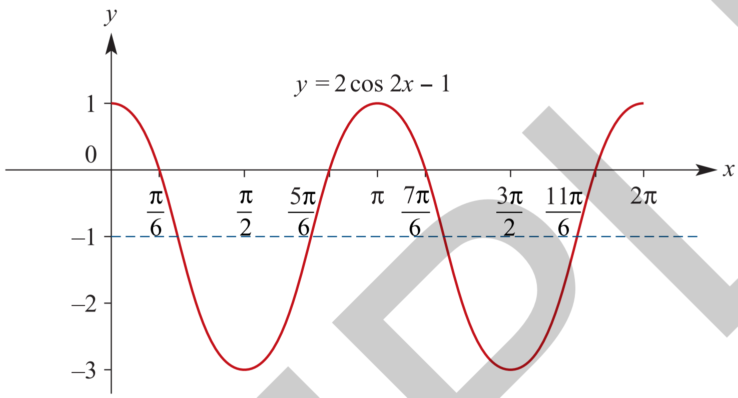
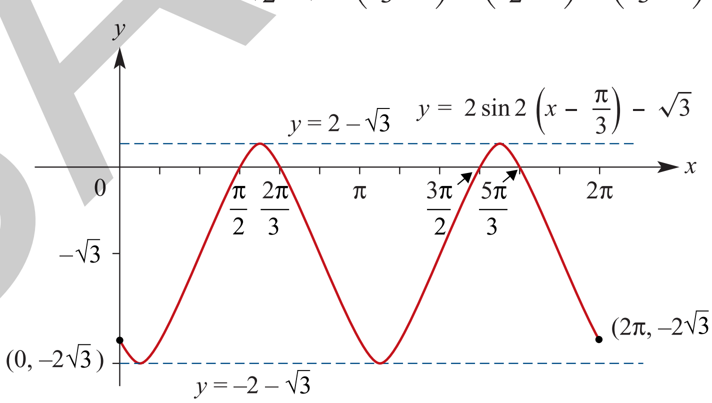
b
$$2\cos 2x - 1 = 0$$
$$\therefore \cos 2x = \frac{1}{2}$$
$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$
$$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$\therefore$ intercepts are $\left(\frac{\pi}{6}, 0\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$



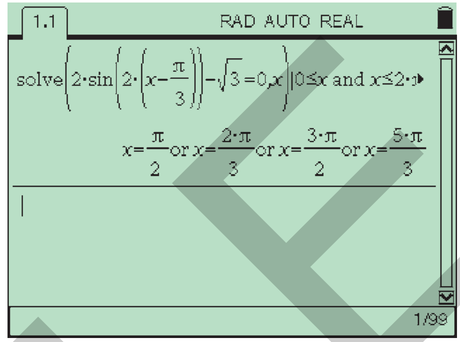
c
$$\sin 2\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$\therefore 2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$
$$\therefore x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$
$$= \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

$\therefore$ intercepts are $\left(\frac{\pi}{2}, 0\right), \left(\frac{2\pi}{3}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{5\pi}{3}, 0\right)$



Using the TI-Nspire

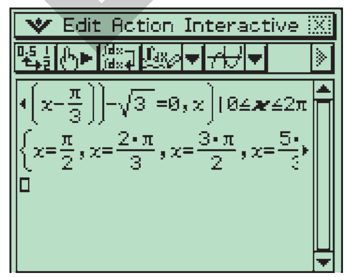
Use **solve( )** from the **Algebra** menu (menu 4 1) as shown.



Using the Casio ClassPad

To find the x-axis intercepts, enter $2\sin\left(2\left(x-\frac{\pi}{3}\right)\right)-\sqrt{3}=0|0\le x\le\pi$

Highlight *the equation part only*, then tap **Interactive Equation/inequality–solve** to find the solutions.



Exercise 6H

Skillsheet TEACHER

1 Sketch the graphs of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\cos x + 1$ **b** $y = 2\cos 2x - \sqrt{3}$ **c** $y = \sqrt{2}\cos x - 1$

d $y = 2\cos x - 2$ **e** $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 1$

2 Sketch the graphs of each of the following for $x \in [-2\pi, 2\pi]$.

a $y = 2\sin 3x - 2$ **b** $y = 2\sin\left(x - \frac{\pi}{4}\right) + 1$ **c** $y = 2\sin 2x - 3$

d $y = 1 - 2\sin x$ **e** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$ **f** $y = 2\cos\left(3x - \frac{\pi}{4}\right)$

g $y = 1 - \cos 2x$ **h** $y = -1 - \sin x$

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6.4 Addition of ordinates

Example 19

Using the same scale and axes sketch the graphs of $y_1 = 2\sin x$ and $y_2 = 3\cos 2x$ for $0 \le x \le 2\pi$. Use addition of ordinates to sketch the graph of $y = 2\sin x + 3\cos 2x$

Solution

The graphs of $y_1 = 2\sin x$ and $y_2 = 3\cos 2x$ are shown. To obtain points on the graph of $y = 2\sin x + 3\cos 2x$, the process of addition of ordinates is used.

Let $y = y_1 + y_2$ when $y_1 = 2\sin x$ and $y_2 = 3\cos 2x$

For example, at $x = 0$, $\quad y = 0 + 3 = 3$

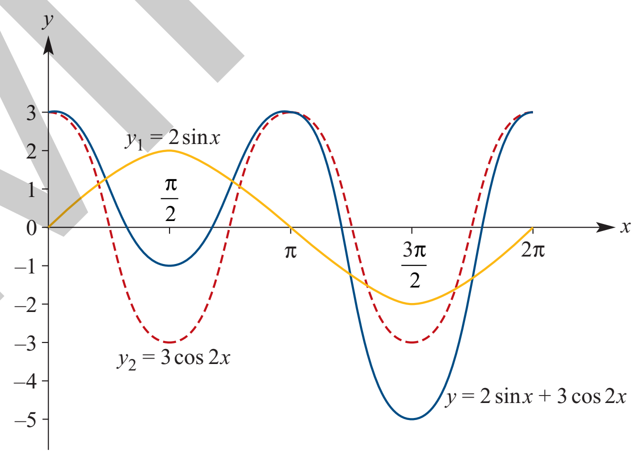
$x = \frac{\pi}{4}$, $\quad y = \frac{2}{\sqrt{2}} + 0 = \frac{2}{\sqrt{2}}$

$x = \frac{\pi}{2}$, $\quad y = 2 - 3 = -1$

$x = \pi$, $\quad y = 0 + 3 = 3$

$x = \frac{3\pi}{2}$, $\quad y = -2 - 3 = -5$

and so on.



Exercise 6I

Use addition of ordinates to sketch the graphs of each of the following for $x \in [-\pi, \pi]$:

1 $y = \sin\theta + 2\cos\theta$ **2** $y = 2\cos 2\theta + 3\sin 2\theta$ **3** $y = \frac{1}{2}\cos 2\theta - \sin\theta$

4 $y = 3\cos\theta + \sin 2\theta$ **5** $y = 2\sin\theta - 4\cos\theta$

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6.5 Determining the rule for graphs of circular functions

In previous chapters, the procedures for finding the rule for a graph known to be a polynomial, exponential or logarithmic function were introduced. In this section, we consider the procedures for finding the rules for graphs of functions known to be of the form $f(t) = A\sin(nt + b)$.

Example 20

A function has rule $f(t) = A\sin(nt)$. The amplitude is 6; the period is 10. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 10$.

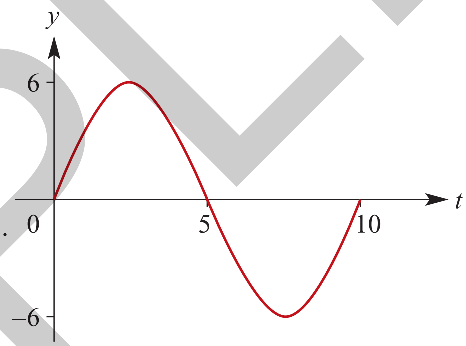
Solution

$$\text{The period} = \frac{2\pi}{n} = 10$$

$$\therefore n = \frac{\pi}{5}$$

The amplitude is 6 and therefore $A = 6$.

The function has rule $f(t) = 6\sin\dfrac{\pi t}{5}$.

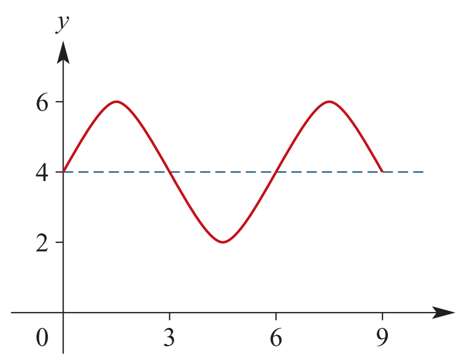


Example 21

The graph shown is that of a function with rule:

$$y = A\sin(nt) + b$$

Find A and n and b.



Solution

The amplitude is 2, and therefore $A = 2$.

The period $= 6$

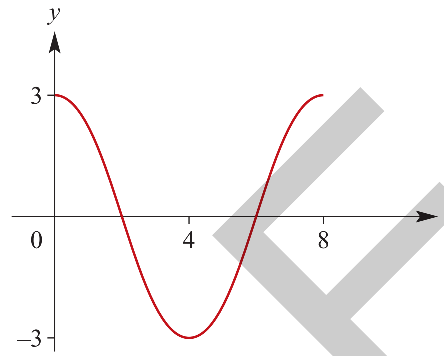
Therefore $\dfrac{2\pi}{n} = 6$ and $n = \dfrac{\pi}{3}$.

The 'centreline' is $y = 4$, and therefore $b = 4$.

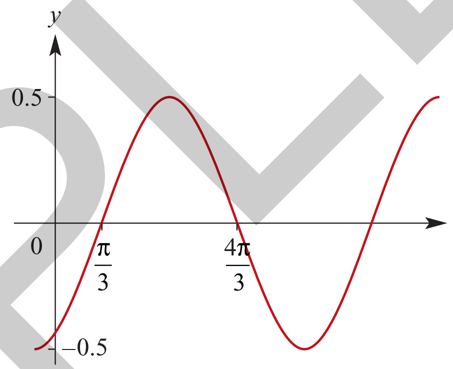
The function is $y = 2\sin\left(\dfrac{\pi t}{3}\right) + 4$

Exercise 6J

1 The graph shown has rule of the form $y = A\cos(nt)$.
Find the values of A and n.

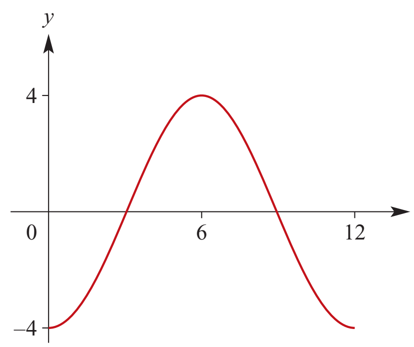


2 The graph shown has rule of the form $y = A\sin(t + \varepsilon)$.
Find the values of A and ε.



3 A function with rule $y = A\sin(nt) + b$ has range $[2, 8]$ and period $\frac{2\pi}{3}$.
Find the values of A, n and b.

4 The graph shown has rule of the form $y = A\cos(nt)$.
Find the values of A and n.



5 A function with rule $y = A\sin(nt + \varepsilon)$ has the following properties:

- range $= [-4, 4]$
- period $= 8$
- when $t = 2$, $y = 0$.

Find values for A, n and ε.

6 A function with rule $y = A\sin(nt + \varepsilon)$ has the following properties:

- range $= [-2, 2]$
- period $= 6$

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- when $t = 1$, $y = 1$.

Find values for A, n and ε.

7 A function with rule $y = A\sin(nt + \varepsilon) + d$ has the following properties:

- range $= [-2, 6]$
- period $= 8$
- when $t = 2$, $y = 2$.

Find values for A, n and ε.

8 A function with rule $y = A\sin(nt + \varepsilon) + d$ has the following properties:

- range $= [0, 4]$
- period $= 6$
- when $t = 1$, $y = 3$.

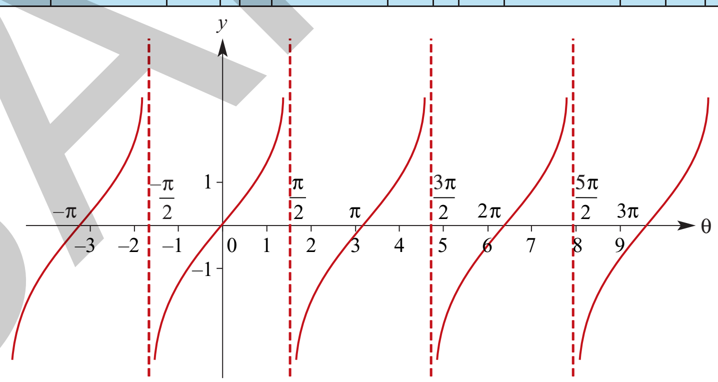
Find values for A, n, d and ε.

6.6 The function tan θ

As previously discussed, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ where $\cos\theta \neq 0$. It can be seen from this that the vertical asymptotes of the graph of $y = \tan\theta$ occur where $\cos\theta = 0$, i.e. for values of $\theta = \cdots -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$ These values can be described by $\theta = (2k+1)\dfrac{\pi}{2}$ where $k \in Z$.

A table of values for $y = \tan\theta$ is given below:

θ	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	undefined	−1	0	1	undefined	−1	0	1	undefined	−1	0	1	undefined	−1	0



Note: $\theta = -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and $\dfrac{5\pi}{2}$ are asymptotes.

Observations from the graph

- The graph repeats itself every π units, i.e. the period of tan is π.
- Range of tan is R.
- The vertical asymptotes have the equation $\theta = (2k+1)\dfrac{\pi}{2}$ where $k \in Z$.

Example 22

Sketch the graph of each of the following for $x \in [-\pi, \pi]$.

a $y = 3\tan 2x$ **b** $y = -2\tan 3x$

Solution

a $y = 3\tan 2x$

Period $= \dfrac{\pi}{2}$

Asymptotes $x = \dfrac{(2k+1)\pi}{4}, \quad k \in Z$

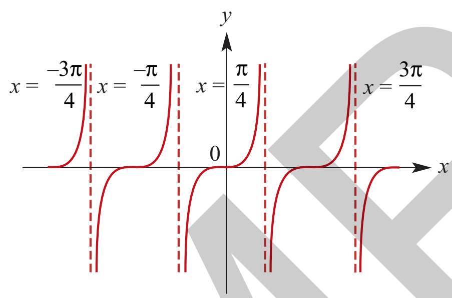
Axes intercepts $x = \dfrac{k\pi}{2}, \quad k \in Z$

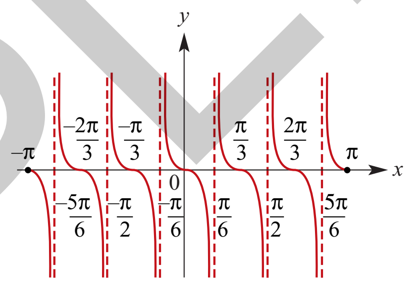
b $y = -2\tan 3x$

Period $= \dfrac{\pi}{3}$

Asymptotes $x = \dfrac{(2k+1)\pi}{6}, \quad k \in Z$

Axes intercepts $x = \dfrac{k\pi}{3}, \quad k \in Z$





Example 23

Sketch the graph of $y = 3\tan\left(2x - \dfrac{\pi}{3}\right)$ for $\dfrac{\pi}{6} \le x \le \dfrac{13\pi}{6}$.

Solution

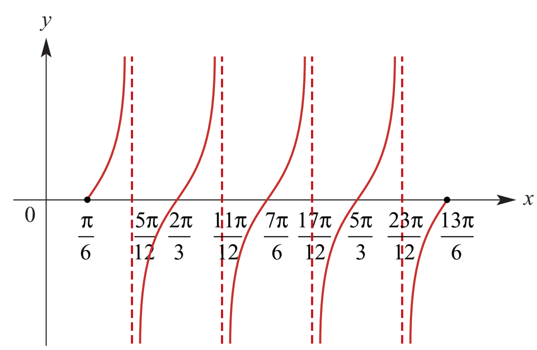
Consider $y = 3\tan\left[2\left(x - \dfrac{\pi}{6}\right)\right]$

The transformations are:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\dfrac{1}{2}$ from the y-axis
- a translation of $\dfrac{\pi}{6}$ units in the positive direction of the x-axis.

The period of the function is $\dfrac{\pi}{2}$.

The range of the function is R.



In general:

- $f\colon R\backslash\left\{x\colon x = \dfrac{(2k+1)}{n}\dfrac{\pi}{2}, k \in Z\right\} \to R,\ f(x) = a\tan(nx)$
- Period $= \dfrac{\pi}{|n|}$
- Range $= R$

For a translation of b units in the positive direction of the x-axis of $f(x) = a\tan(nx)$ the asymptotes will have equations:

$$x = \frac{(2k+1)}{n}\frac{\pi}{2} + b$$

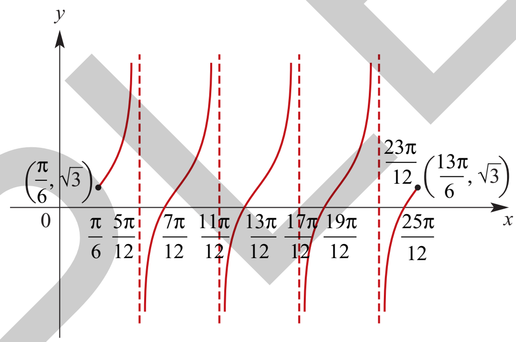
Example 24

Sketch the graph of $y = 3\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3}$ for $\frac{\pi}{6} \le x \le \frac{13\pi}{6}$.

Solution

The transformations are:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of $\frac{\pi}{6}$ units in the positive direction of the x-axis
- a translation of $\sqrt{3}$ units in the positive direction of the y-axis.



The graph is the graph obtained in Example 23 translated $\sqrt{3}$ units in the positive direction of the y-axis.

The axes intercepts are determined by solving the equation:

$$3\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3} = 0 \text{ for } \frac{\pi}{6} \le x \le \frac{13\pi}{6}$$

$$\therefore \tan\left(2x - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}$$

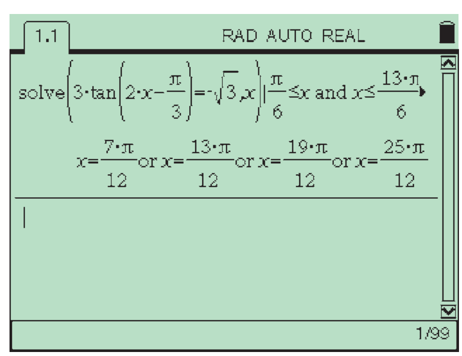
$$\therefore 2x - \frac{\pi}{3} = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$2x = \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$x = \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12} \text{ or } \frac{25\pi}{12}$$

Using the TI-Nspire

To find the x-axis intercepts, enter **solve**$\left(3\tan\left(2x - \frac{\pi}{3}\right) = -\sqrt{3}, x\right) \mid \frac{\pi}{6} \le x$ **and** $x \le \frac{13\pi}{6}$.



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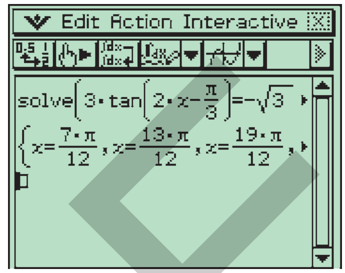
Using the Casio ClassPad

To find the x-axis intercepts, enter

$$3\tan\left(2x - \frac{\pi}{3}\right) = -\sqrt{3} \mid \frac{\pi}{6} \le x \le \frac{13\pi}{6}$$

then highlight *the equation part only*, then tap **Interactive Equation/inequality–solve** to find the solutions.



Example 25

Solve the equation $\tan\left[\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right] = -1$ for $x \in [-2\pi, 2\pi]$.

Solution

$$\tan\left[\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right] = -1$$

implies $\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{3\pi}{4}$ or $\frac{-\pi}{4}$

$$x - \frac{\pi}{4} = \frac{3\pi}{2} \text{ or } \frac{-\pi}{2}$$

Therefore $x = \frac{7\pi}{4}$ or $\frac{-\pi}{4}$

Solution of equations of the form $\sin nx = k\cos nx$

If $\sin nx = k\cos nx$, then $\tan nx = k$.

Example 26

Solve the equation $\sin 2x = \cos 2x$ for $x \in [0, 2\pi]$.

Solution

$\sin 2x = \cos 2x$

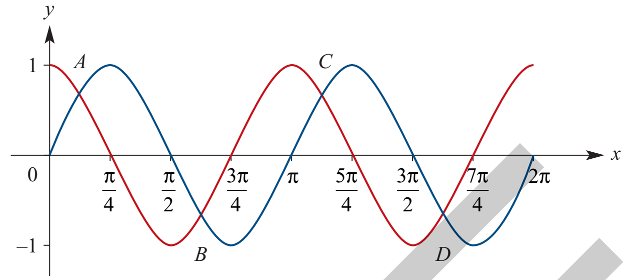
implies

$\tan 2x = 1$

$\therefore 2x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ or $\frac{9\pi}{4}$ or $\frac{13\pi}{4}$ (remember tan is positive for the first and third quadrants)

$\therefore x = \frac{\pi}{8}$ or $\frac{5\pi}{8}$ or $\frac{9\pi}{8}$ or $\frac{13\pi}{8}$

This can be shown graphically.



The points of intersection, A, B, C and D, occur when:

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8} \text{ and } \frac{13\pi}{8} \text{ respectively}$$

Example 27

On the same set of axes, sketch the graphs of $y = \sin x$ and $y = \cos x$ for $x \in [0, 2\pi]$ and find the coordinates of the points of intersection.

Solution

$\sin x = \cos x$

implies

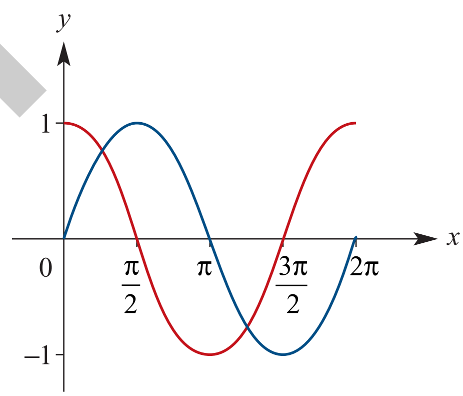
$\tan x = 1$

$\therefore\ x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

The function tan is positive in the first and third quadrant.

The coordinates of the points of intersection are:

$$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$$



We recall the exact values of tan:

$$\tan 0 = 0,\ \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}},\ \tan\frac{\pi}{4} = 1,\ \tan\frac{\pi}{3} = \sqrt{3}$$

and the symmetry properties:

- $\tan(\pi - \theta) = -\tan\theta$
- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$
- $\tan(-\theta) = -\tan\theta$

Example 28

Solve the equation $\sin 3x = \frac{1}{\sqrt{3}} \cos 3x$ for $x \in [-\pi, \pi]$.

Solution

$$\frac{\sin 3x}{\cos 3x} = \frac{1}{\sqrt{3}}$$

implies $\tan 3x = \frac{1}{\sqrt{3}}$

$$\therefore\ 3x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } -\frac{11\pi}{6} \text{ or } -\frac{5\pi}{6} \text{ or } -\frac{17\pi}{6}$$

$$\therefore\ x = \frac{\pi}{18} \text{ or } \frac{7\pi}{18} \text{ or } \frac{13\pi}{18} \text{ or } -\frac{11\pi}{18} \text{ or } -\frac{5\pi}{18} \text{ or } -\frac{17\pi}{18}$$

Exercise 6K

1 State the period for each of the following:

a $\tan 3\theta$ b $\tan \frac{\theta}{2}$ c $\tan \frac{3\theta}{2}$ d $\tan(\pi\theta)$ e $\tan\left(\frac{\pi\theta}{2}\right)$

2 Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = \tan 2x$ b $y = 2\tan 3x$ c $y = 2\tan\left(x + \frac{\pi}{4}\right)$

d $y = 3\tan x + 1$ e $y = 2\tan\left(x + \frac{\pi}{2}\right) + 1$ f $y = 3\tan 2\left(x - \frac{\pi}{4}\right) - 2$

3 Sketch the graph of $y = -2\tan(\pi\theta)$ for $-2 \le \theta \le 2$.

4 Sketch the graph of $y = \tan(-2\theta)$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

5 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sqrt{3}\sin x = \cos x$ b $\sin 4x = \cos 4x$ c $\sqrt{3}\sin 2x = \cos 2x$

d $-\sqrt{3}\sin 2x = \cos 2x$ e $\sin 3x = -\cos 3x$ f $\sin x = 0.5\cos x$

g $\sin x = 2\cos x$ h $\sin 2x = -\cos 2x$ i $\cos 3x = \sqrt{3}\sin 3x$

j $\sin 3x = \sqrt{3}\cos 3x$

6 a On the same set of axes sketch the graphs of $y = \cos 2x$ and $y = -\sin 2x$ for $x \in [-\pi, \pi]$.

b Find the coordinates of the points of intersection.

c On the same set of axes sketch the graph of $y = \cos 2x - \sin 2x$.

7 a On the same set of axes sketch the graphs of $y = \cos x$ and $y = \sqrt{3}\sin x$ for $x \in [0, 2\pi]$.

b Find the coordinates of the points of intersection.

c On the same set of axes sketch the graph of $y = \cos x + \sqrt{3}\sin x$.

8 Solve each of the following equations for $0 \le x \le 2\pi$:

a $\tan\left(2x - \frac{\pi}{4}\right) = \sqrt{3}$ b $3\tan 2x = -\sqrt{3}$ c $\tan\left(3x - \frac{\pi}{6}\right) = -1$

9 A function with rule $y = A\tan nt$ has the following properties:

- asymptotes have equations $t = (2k+1)\frac{\pi}{6}$ where $k \in Z$
- when $t = \frac{\pi}{12}$, $y = 5$.

Find values for A and n.

10 A function with rule $y = A\tan(nt)$ has the following properties:

- period is 2
- when $t = \frac{1}{2}$, $y = 6$.

Find values for A and n.

6.7 General solution of circular function equations

Solution of circular function equations has been discussed for functions over a restricted domain. In this section, we consider the general solutions of such equations over the maximal domain for each function.

In the following the convention is that:

- $\cos^{-1}$ has range $[0, \pi]$
- $\sin^{-1}$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\tan^{-1}$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For example $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Also

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

If a circular function equation has one or more solutions in one 'cycle', then it will have corresponding solutions in each 'cycle' of its domain; i.e., there will be an infinite number of solutions.

For example, if $\cos x = a$, then the solution in the interval $[0, \pi]$ is given by:

$$x = \cos^{-1}(a)$$

By the symmetry properties of the cosine function, other solutions are given by:

$-\cos^{-1}(a)$, $\pm 2\pi + \cos^{-1}(a)$, $\pm 2\pi - \cos^{-1}(a)$, $\pm 4\pi + \cos^{-1}(a)$, $\pm 4\pi - \cos^{-1}(a)$, and so on.

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In general, if $\cos(x) = a$,

$$x = 2n\pi \pm \cos^{-1}(a), \text{ where } n \in Z \text{ and } a \in [-1, 1]$$

Similarly, if $\tan(x) = a$,

$$x = n\pi + \tan^{-1}(a), \text{ where } n \in Z \text{ and } a \in R$$

If $\sin(x) = a$,

$$x = 2n\pi + \sin^{-1}(a) \text{ or } x = (2n+1)\pi - \sin^{-1}(a), \text{ where } n \in Z \text{ and } a \in [-1, 1]$$

Note: An alternative and more concise way to express the general solution of $\sin(x) = a$ is:

$$x = n\pi + (-1)^n \sin^{-1}(a), \text{ where } n \in Z \text{ and } a \in [-1, 1]$$

Example 29

Find the general solution to each of the following equations:

a $\cos(x) = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin(x) = \sqrt{2}$

Solution

a
$$\begin{aligned} x &= 2n\pi \pm \cos^{-1}(0.5) \\ &= 2n\pi \pm \frac{\pi}{3} \\ &= \frac{(6n \pm 1)\pi}{3}, \quad n \in Z \end{aligned}$$

b
$$\begin{aligned} \tan(3x) &= \frac{1}{\sqrt{3}} \\ 3x &= n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= n\pi + \frac{\pi}{6} \\ &= \frac{(6n+1)\pi}{6} \\ x &= \frac{(6n+1)\pi}{18}, \quad n \in Z \end{aligned}$$

c
$$\sin(x) = \frac{1}{\sqrt{2}}$$

$$\begin{aligned} x &= 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= 2n\pi + \frac{\pi}{4} \\ &= \frac{(8n+1)\pi}{4}, \quad n \in Z \end{aligned}$$

or

$$\begin{aligned} x &= (2n+1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= (2n+1)\pi - \frac{\pi}{4} \\ &= \frac{(8n+3)\pi}{4}, \quad n \in Z \end{aligned}$$

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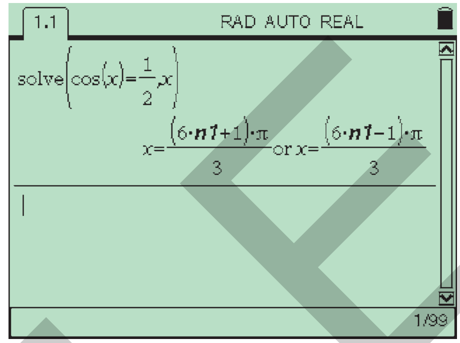
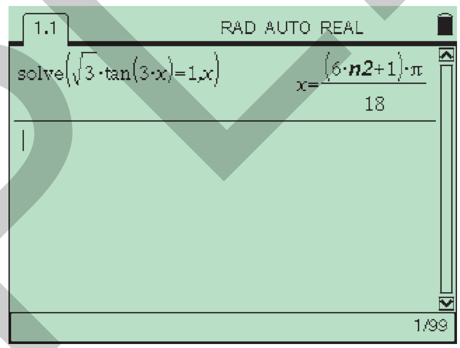
Using the TI-Nspire

Make sure the calculator is in Radian mode.

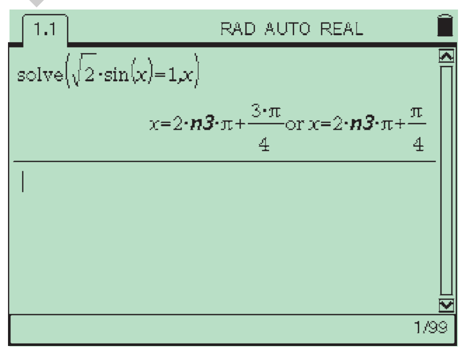
a Use **solve( )** from the **Algebra** menu (menu 3 1) and complete as shown.
Note the use of $\frac{1}{2}$ rather than 0.5 to ensure that the answer is exact.



b Complete as shown.

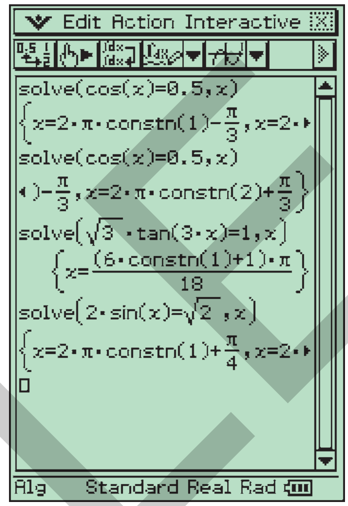


c Complete as shown.



Using the Casio ClassPad

Make sure the calculator is in Radian mode.



a Enter and highlight $\cos(x) = 0.5$ than tap **Interactive–Equation/inequality** $>$ **solve** and ensure the variable is set to x.
Note that in the screen shown, this equation has been solved twice and the solution scrolled to the right to show the other solution on the second occasion.

b Enter and highlight $\sqrt{3}\tan(3x) = 1$ and follow the instructions in part **a**.

c Enter and highlight $2\sin(x) = \sqrt{2}$ and follow the instructions in part **a**.

The calculator uses the notation constn(1) and constn(2) to represent constants when general solutions are found. The answers should be read $x = 2n\pi - \pi/3$ and $x = 2n\pi + \pi/3$.

Example 30

Find the first three positive solutions to each of the following equations:

a $\cos(x) = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin(x) = \sqrt{2}$

Solution

a The general solution (from Example 29) is given by $x = \dfrac{(6n \pm 1)\pi}{3}, n \in Z$

When $n = 0$, $x = \pm\dfrac{\pi}{3}$, and when $n = 1$, $x = \dfrac{5\pi}{3}$ or $x = \dfrac{7\pi}{3}$

The first three positive solutions of $\cos(x) = 0.5$ are $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}$

b The general solution (from Example 29) is given by $x = \dfrac{(6n + 1)\pi}{18}, n \in Z$

When $n = 0$, $x = \dfrac{\pi}{18}$, and when $n = 1$, $x = \dfrac{7\pi}{18}$, and when $n = 2$, $x = \dfrac{13\pi}{18}$

The first three positive solutions of $\sqrt{3}\tan(3x) = 1$ are $x = \dfrac{\pi}{18}, \dfrac{7\pi}{18}, \dfrac{13\pi}{18}$

c The general solution (from Example 29) is given by $x = \dfrac{(8n + 1)\pi}{4}$ or $x = \dfrac{(8n + 3)\pi}{4}, n \in Z$

When $n = 0$, $x = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$, and when $n = 1$, $x = \dfrac{9\pi}{4}$ or $\dfrac{11\pi}{4}$

The first three positive solutions of $2\sin(x) = \sqrt{2}$ are $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}$

Example 31

Find the general solution for each of the following

a $\sin(x - \frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

b $\tan(2x - \frac{\pi}{3}) = 1$

Solution

a $x - \frac{\pi}{3} = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, where $n \in Z$

$x = n\pi + (-1)^n \frac{\pi}{3} + \frac{\pi}{3}$

$x = n\pi + \frac{2\pi}{3}$ if n is even and $x = n\pi$ if n is odd.

b $\tan\left(2x - \frac{\pi}{3}\right) = 1$

$$2x - \frac{\pi}{3} = n\pi + \frac{\pi}{4}$$

$$2x = n\pi + \frac{\pi}{4} + \frac{\pi}{3}$$

$$x = \frac{1}{2}\left(n\pi + \frac{7\pi}{12}\right)$$

$$= \left(\frac{12n + 7}{24}\right)\pi, \text{ where } n \in Z$$

Exercise 6L

1 Evaluate each of the following for:

i $n = 1$ **ii** $n = 2$ **iii** $n = -2$

a $2n\pi \pm \cos^{-1}(1)$ **b** $2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

2 Find the general solution to each of the following equations:

a $\cos(x) = 0.5$ **b** $2\sin(3x) = \sqrt{3}$ **c** $\sqrt{3}\tan(x) = 3$

3 Find the first two positive solutions to each of the following equations:

a $\sin(x) = 0.5$ **b** $2\cos(2x) = \sqrt{3}$ **c** $\sqrt{3}\tan(2x) = -3$

4 It was found that a trigonometric equation had general solution $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$ where $n \in Z$. Find the solutions for the equation in the interval $[-2\pi, 2\pi]$.

5 It was found that a trigonometric equation had general solution $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$ where $n \in Z$. Find the solutions for the equation in the interval $[-\pi, 2\pi]$.

6 Find the general solution of each of the following:

a $\cos 2\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ **b** $2\tan 2\left(x + \frac{\pi}{4}\right) = 2\sqrt{3}$ **c** $2\sin\left(x + \frac{\pi}{3}\right) = -1$

7 Find the general solution to $2\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$, and hence find all the solutions for x in the interval $(-2\pi, 2\pi)$.

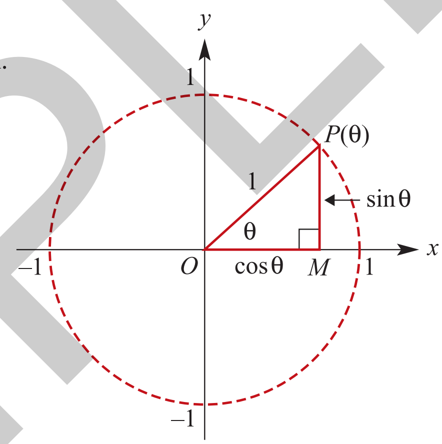
8 Find the general solution to $\sqrt{3}\tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$, and hence find all the solutions for x in the interval $[-\pi, 0]$.

9 Find the general solution to $2\sin(4\pi x) + \sqrt{3} = 0$, and hence find all the solutions for x in the interval $[-1, 1]$.

6.8 Identities

We develop several useful identities in this section.

The Pythagorean identity



Consider a point, $P(\theta)$, on the unit circle.

By the theorem of Pythagoras:

$$OP^2 = OM^2 + MP^2$$
$$\therefore\ 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Now $(\cos\theta)^2$ and $(\sin\theta)^2$ may be written as $\cos^2\theta$ and $\sin^2\theta$.

$$\therefore\ 1 = \cos^2\theta + \sin^2\theta$$

Since this is true for all values of θ, it is called an identity. In particular this is called the Pythagorean identity:

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

Addition formulas

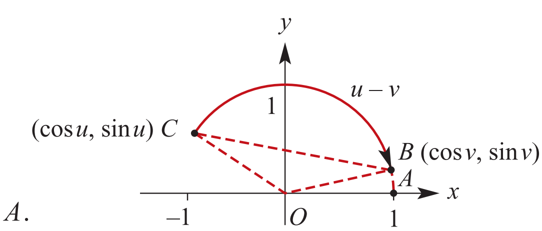
Consider a unit circle.

Let arc length $AB = v$ units

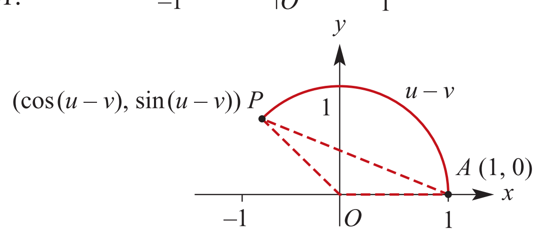
arc length $AC = u$ units

$\therefore$ arc length $CB = u - v$ units

Rotate ΔOCB so that B is coincident with A.



The point P has coordinates $(\cos(u - v), \sin(u - v))$.



Since the triangles CBO and PAO are congruent, $CB = PA$.

Apply the coordinate distance formula:

$$\begin{aligned} CB^2 &= (\cos u - \cos v)^2 + (\sin u - \sin v)^2 \\ &= 2 - 2(\cos u \cos v + \sin u \sin v) \\ PA^2 &= [\cos(u - v) - 1]^2 + (\sin(u - v) - 0]^2 \\ &= 2 - 2(\cos(u - v)) \end{aligned}$$

Equating these:

$$2 - 2(\cos u \cos v + \sin u \sin v) = 2 - 2[\cos(u - v)]$$

$$\therefore \cos(u - v) = \cos u \cos v + \sin u \sin v$$

Replacing v with $-v$, $\cos(u - (-v)) = \cos u \cos(-v) + \sin u \sin(-v)$

From symmetry properties, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$$\therefore \cos(u + v) = \cos u \cos v - \sin u \sin v$$

Example 32

Evaluate $\cos 75°$.

Solution

$$\begin{aligned} \cos 75° &= \cos(45° + 30°) \\ &= \cos 45° \cos 30° - \sin 45° \sin 30° \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3} - 1}{2\sqrt{2}} \end{aligned}$$

Replace u with $\frac{\pi}{2} - u$ in $\cos(u - v)$:

$$\therefore \cos\left[\left(\frac{\pi}{2} - u\right) - v\right] = \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v$$

Applying symmetry properties, $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

and $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$$\therefore \cos\left[\frac{\pi}{2} - (u + v)\right] = \sin u \cos v + \cos u \sin v$$

$$\therefore \sin(u + v) = \sin u \cos v + \cos u \sin v$$

Replacing v with $-v$, $\sin(u - v) = \sin u \cos(-v) + \cos u \sin(-v)$

$$\therefore \sin(u - v) = \sin u \cos v - \cos u \sin v$$

Double angle formulas

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

Replacing v with u:

$$\cos(u+u) = \cos u \cos u - \sin u \sin u$$

$$\begin{aligned}\cos 2u &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \\ &= 1 - 2\sin^2 u\end{aligned}$$

$$\begin{aligned}\text{since} \quad \cos^2 u &= 1 - \sin^2 u \\ \text{and} \quad \sin^2 u &= 1 - \cos^2 u\end{aligned}$$

Similarly replacing v with u in $\sin(u+v) = \sin u \cos v + \cos u \sin v$:

$$\therefore \sin 2u = \sin u \cos u + \cos u \sin u$$

$$\sin 2u = 2\sin u \cos u$$

Replacing v with u in $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$:

$$\therefore \tan(u+u) = \frac{\tan u + \tan u}{1 - \tan u \tan u}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u}$$

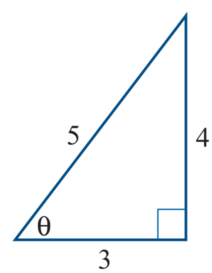
Example 33

If $\tan\theta = \frac{4}{3}$ and $0 < \theta < \frac{\pi}{2}$, evaluate $\sin 2\theta$.

Solution

$$\sin\theta = \frac{4}{5} \text{ and } \cos\theta = \frac{3}{5}$$

$$\begin{aligned}\therefore \sin 2\theta &= 2\sin\theta \cos\theta \\ &= 2 \times \frac{4}{5} \times \frac{3}{5} \\ &= \frac{24}{25}\end{aligned}$$



Exercise 6M

The following questions require the Pythagorean identity.

1 Given that $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

2 Given that $\sin x = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

3 Given that $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

6.9 Applications of circular functions

Example 34

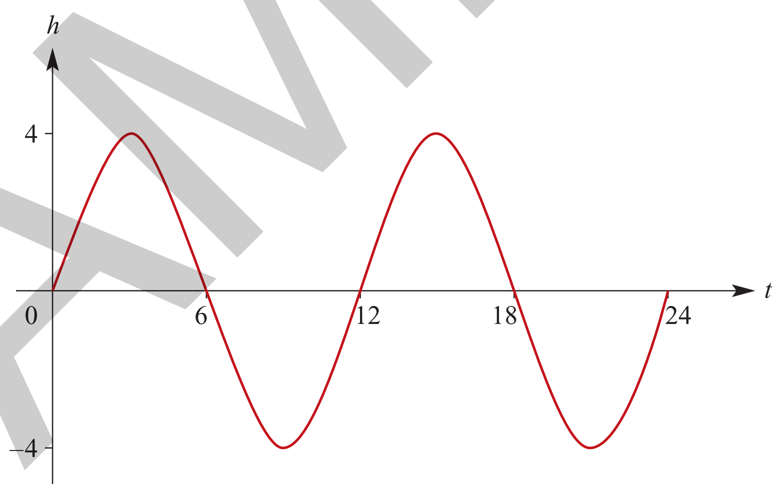
It is suggested that the height $h(t)$ metres of the tide above mean sea level on 1 January at Warnung is given approximately by the rule:

$$h(t) = 4\sin\left(\frac{\pi}{6}t\right), \text{ where } t \text{ is the number of hours after midnight}$$

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.
b When was high tide?
c What was the height of the high tide?
d What was the height of the tide at 8 a.m.?
e A boat can only cross the harbour bar when the tide is at least 1 m above mean sea level. When could the boat cross the harbour bar on 1 January?

Solution

a



Note: period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 4$.

$$4\sin\left(\frac{\pi}{6}t\right) = 4$$

$$\text{implies } \sin\left(\frac{\pi}{6}t\right) = 1$$

$$\therefore \frac{\pi}{6}t = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore t = 3, 15$$

i.e. high tide occurs at 0300 (3 a.m.) and 1500 (3 p.m.).

c The high tide has height 4 metres above the mean height.

d
$$\begin{aligned} h(8) &= 4\sin\left(\frac{8\pi}{6}\right) \\ &= 4\sin\left(\frac{4\pi}{3}\right) \\ &= 4\times\frac{-\sqrt{3}}{2} \\ &= -2\sqrt{3} \end{aligned}$$

The water is $2\sqrt{3}$ metres below the mean height at 8 a.m.

e First, consider $4\sin\frac{\pi}{6}t = 1$

Thus $\sin\frac{\pi}{6}t = \frac{1}{4}$

$$\therefore \frac{\pi}{6}t = 0.2527, 2.889, 6.5359, 9.172$$

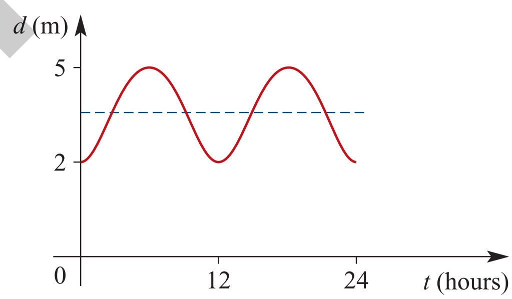
$$\therefore t = 0.4826, 5.5174, 12.4824, 17.5173$$

i.e. the water is at height 1 m at 0029, 0531, 1229 and 1731.

Thus the boat can pass across the harbour bar between 0029 and 0531 and between 1229 and 1731.

Exercise 6N

1 The graph shows the distance $d(t)$ of the tip of the hour hand of a large clock from the ceiling at time t hours.



a $d(t)$ is the rule for a sinusoidal function. Find:

i the amplitude **ii** the period

iii the rule for $d(t)$

iv the length of the hour hand

b At what times is the distance less than 3.5 metres from the ceiling?

2 The water level on a beach wall is a sinusoidal function (i.e. has a rule of the form $y = a\sin(nt + \varepsilon) + b$).

In this case, the function $d(t) = 6 + 4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$ where t is the number of hours after midnight and d is the depth of the water, in metres.

a Sketch the graph of $d(t)$ for $0 \le t \le 24$.

b What is the earliest time of day at which the water is at its highest?

c When is the water 2 m up the wall?

3 In a tidal river, the time between high tides is 12 hours. The average depth of water in the port is 5 m; at high tide the depth is 8 m. Assume that the depth of water is given by

$$h(t) = A\sin(nt + \varepsilon) + b$$

where t is the number of hours after 12:00 noon.

At 12:00 noon there is a high tide.

a Find the values of A, n, b and ε.

b At what times is the depth of the water 6 m?

c Sketch the graph of $y = h(t)$ for $0 \le t \le 24$.

4 A particle moves on a straight line, Ox, and its distance x metres from O at time t (s) is given by $x = 3 + 2\sin 3t$.

a Find its greatest distance from O.

b Find its least distance from O.

c Find the times at which it is 5 m from O for $0 \le t \le 5$.

d Find the times at which it is 3 m from O for $0 \le t \le 3$.

e Describe the motion of the particle.

5 The temperature A°C inside a house at t hours after 4 a.m. is given by $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$ for $0 \le t \le 24$, and the temperature B°C outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$ for $0 \le t \le 24$.

a Find the temperature inside the house at 8 a.m.

b Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.

c Sketch the graph of D for $0 \le t \le 24$.

d Determine when the inside temperature is less than the outside temperature.

6 Passengers on a ferris wheel ride access their seats from a platform 5 m above the ground. As each seat is filled the ferris wheel moves around so that the next seat can be filled. Once all seats are filled the ride begins and lasts for 6 minutes. The height h m of Isobel's seat above the ground t seconds after the ride has begun is given by the equation
$h = 15\sin(10t - 45)^\circ + 16.5$

a Use a calculator to sketch the graph of the equation for the first 2 minutes of the ride.

b How far above the ground is Isobel's seat at the commencement of the ride?

c After how many seconds does Isobel's seat pass the access platform?

d How many times will her seat pass the access platform in the first 2 minutes?

e How many times will her seat pass the access platform during the entire ride?

Due to a malfunction the ride stops abruptly 1 minute and 40 seconds into the ride.

f How far above the ground is Isobel stranded?

g If Isobel's brother Hamish had a seat 1.5 m above the ground at the commencement of the ride, how far above the ground was Hamish stranded?

MC TEST D&D TEST

Review

Chapter summary

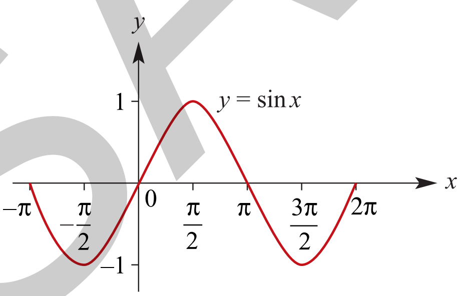
- Symmetry properties of circular functions:

$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$
$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$
$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$\sin(2\pi - \theta) = -\sin\theta$	$\sin(-\theta) = -\sin\theta$
$\cos(2\pi - \theta) = \cos\theta$	$\cos(-\theta) = \cos\theta$
$\tan(2\pi - \theta) = -\tan\theta$	$\tan(-\theta) = -\tan\theta$

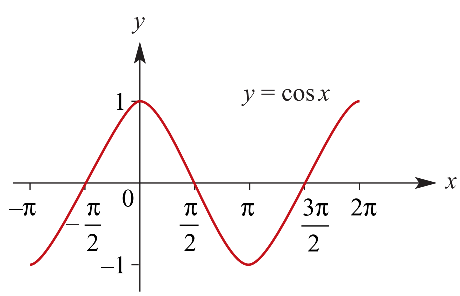
- Exact values of trigonometric functions:

$\sin 0 = 0$	$\sin 0^\circ = 0$	$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$	$\tan 30^\circ = \frac{1}{\sqrt{3}}$
$\cos 0 = 1$	$\cos 0^\circ = 1$	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$	$\sin 60^\circ = \frac{\sqrt{3}}{2}$
$\tan 0 = 0$	$\tan 0^\circ = 0$	$\cos\frac{\pi}{3} = \frac{1}{2}$	$\cos 60^\circ = \frac{1}{2}$
$\sin\frac{\pi}{2} = 1$	$\sin 90^\circ = 1$	$\tan\frac{\pi}{3} = \sqrt{3}$	$\tan 60^\circ = \sqrt{3}$
$\cos\frac{\pi}{2} = 0$	$\cos 90^\circ = 0$	$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\sin 45^\circ = \frac{1}{\sqrt{2}}$
$\sin\frac{\pi}{6} = \frac{1}{2}$	$\sin 30^\circ = \frac{1}{2}$	$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\cos 45^\circ = \frac{1}{\sqrt{2}}$
$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\tan\frac{\pi}{4} = 1$	$\tan 45^\circ = 1$

- Graphs of $y = \sin x$ and $y = \cos x$ and transformations of these graphs:



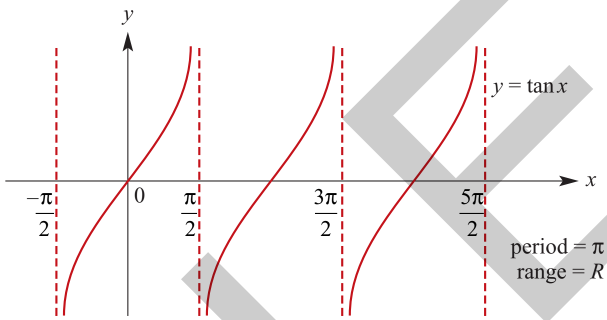
period $= 2\pi$
amplitude $= 1$
range $= [-1, 1]$



period $= 2\pi$
amplitude $= 1$
range $= [-1, 1]$

Review

- For $y = a\cos(nx)$ and $y = a\sin(nx)$:
  period $= \dfrac{2\pi}{|n|}$
  amplitude $= |a|$
  range $= [-|a|, |a|]$
- Pythagorean identity:
  $$\cos^2\theta + \sin^2\theta = 1$$
- The graph of $y = \tan x$:



- For $y = a\tan(nx)$:
  - period $= \dfrac{\pi}{|n|}$
  - range $= R$
  - asymptotes have equations $x = \dfrac{(2k+1)}{n}\dfrac{\pi}{2}, k \in Z$

Multiple-choice questions

1 The period of the graph of $y = 3\sin\left(\frac{1}{2}x - \pi\right) + 4$ is:

A π **B** 3 **C** 4π **D** $\pi + 4$ **E** 2π

2 The equation of the image of the graph of $y = \sin x$ under a transformation of a dilation of factor $\frac{1}{2}$ from the y-axis followed by a translation of $\frac{\pi}{4}$ in the positive direction of the x-axis is:

A $y = \sin\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **B** $y = \sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ **C** $y = 2\sin\left(x - \frac{\pi}{4}\right)$

D $y = \sin\left(2x - \frac{\pi}{4}\right)$ **E** $y = \sin\left[2\left(x - \frac{\pi}{4}\right)\right]$

3 The function $f: R \to R$, where $f(x) = a\sin(bx) + c$, where a, b and c are positive constants, has period:

A a **B** b **C** $\frac{2\pi}{a}$ **D** $\frac{2\pi}{b}$ **E** $\frac{b}{2\pi}$

4 The equation $3\sin x - 1 = b$, where b is a positive real number, has one solution in the interval $(0, 2\pi)$. The value of b is:

A 2 **B** 0.2 **C** 3 **D** 5 **E** 6

5 The range of the graph of $y = f(x)$, where $f(x) = 5\cos\left(2x - \frac{\pi}{3}\right) - 7$ is:

A $[-12, -2]$ **B** $[-7, 7]$ **C** $(-2, 5)$ **D** $[-2, 5]$ **E** $[-2, 12]$

6 Let $f(x) = p\cos(5x) + q$ where $p > 0$. Then $f(x) \le 0$ for all values of x if:

A $q \ge 0$ **B** $-p \le q \le p$ **C** $p \le -q$ **D** $p \ge q$ **E** $-q \le p$

7 The vertical distance of a point on a wheel from the ground as it rotates is given by $D(t) = 3 - 3\sin 6\pi t$, where t is the time in seconds. The time in seconds for a full rotation of the wheel is:

A $\frac{1}{6\pi}$ **B** $\frac{1}{3}$ **C** 6π **D** $\frac{1}{3\pi}$ **E** 3

8 A sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = -2\cos\left(\frac{x}{3}\right)$ is:

A a dilation of factor $\frac{1}{3}$ from the x-axis, followed by dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the x-axis

B a dilation of factor $\frac{1}{2}$ from the x-axis, followed by dilation of factor 3 from the y-axis, followed by a reflection in the y-axis

C a dilation of factor 2 from the x-axis, followed by dilation of factor 3 from the y-axis, followed by a reflection in the x-axis

D a dilation of factor 3 from the x-axis, followed by dilation of factor 2 from the y-axis, followed by a reflection in the x-axis

E a dilation of factor 2 from the x-axis, followed by dilation of factor $\frac{1}{3}$ from the y-axis, followed by a reflection in the x-axis

9 The equation of the image of $y = \cos x$ under a transformation of a dilation of factor 2 from the x-axis, followed by a translation of $\frac{\pi}{4}$ in the positive direction of the x-axis is:

A $y = \cos\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **B** $y = \cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ **C** $y = 2\cos\left(x + \frac{\pi}{4}\right)$

D $y = 2\sin\left(x - \frac{\pi}{4}\right)$ **E** $y = 2\cos\left(x - \frac{\pi}{4}\right)$

10 Which of the following is likely to be the rule for the graph of the circular function shown?

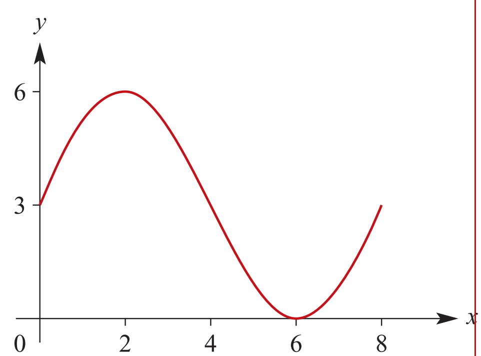
A $y = 3 + 3\cos\frac{\pi x}{4}$

B $y = 3 + 3\sin\frac{\pi x}{4}$

C $y = 3 + 3\sin 4\pi x$

D $y = 3 + 3\cos\frac{x}{4}$

E $y = 3 + 3\sin\frac{x}{4}$



Review

Short-answer questions (technology-free)

1 Solve each of the following equations for $x \in [-\pi, 2\pi]$:

a $\sin x = \frac{1}{2}$ b $2\cos x = -1$ c $2\cos x = \sqrt{3}$

d $\sqrt{2}\sin x + 1 = 0$ e $4\sin x + 2 = 0$ f $\sin 2x + 1 = 0$

g $\cos 2x = \frac{-1}{\sqrt{2}}$ h $2\sin 3x - 1 = 0$

2 Sketch the graphs of each of the following, showing one cycle. Clearly label axes intercepts.

a $f(x) = \sin 3x$ b $f(x) = 2\sin 2x - 1$ c $g(x) = 2\sin 2x + 1$

d $f(x) = 2\sin\left(x - \frac{\pi}{4}\right)$ e $f(x) = 2\sin\frac{\pi x}{3}$ f $h(x) = 2\cos\frac{\pi x}{4}$

3 Solve each of the following equations for $x \in [0, 360]$:

a $\sin x^\circ = 0.5$ b $\cos(2x)^\circ = 0$ c $2\sin x^\circ = -\sqrt{3}$

d $\sin(2x + 60)^\circ = \frac{-\sqrt{3}}{2}$ e $2\sin\left(\frac{1}{2}x\right)^\circ = \sqrt{3}$

4 Sketch the graphs of each of the following, showing one cycle. Clearly label axes intercepts.

a $y = 2\sin\left(x + \frac{\pi}{3}\right) + 2$ b $y = -2\sin\left(x + \frac{\pi}{3}\right) + 1$

c $y = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}$ d $y = -3\sin x$

e $y = \sin\left(x - \frac{\pi}{6}\right) + 3$ f $y = 2\sin\left(x - \frac{\pi}{2}\right) + 1$

5 Sketch, on the same set of axes, the curves $y = \cos x$ and $y = \sin 2x$ for the interval $0 \le x \le 2\pi$, labelling each curve carefully. State the number of solutions in this interval of the equations:

a $\sin 2x = 0.6$ b $\sin 2x = \cos x$ c $\sin 2x - \cos x = 1$

6 Sketch on separate axes for $0^\circ \le x^\circ \le 360^\circ$:

a $y = 3\cos x^\circ$ b $y = \cos 2x^\circ$ c $y = \cos(x - 30)^\circ$

7 Solve each of the following for $x \in [-\pi, \pi]$:

a $\tan x = \sqrt{3}$ b $\tan x = -1$ c $\tan 2x = -1$ d $\tan(2x) + \sqrt{3} = 0$

Extended-response questions

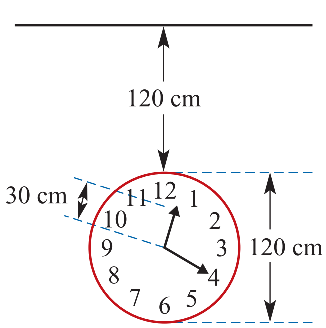
1 In a tidal river, the time between high tide and low tide is 6 hours. The average depth of water at a point in the river is 4 metres; at high tide the depth is 5 metres.

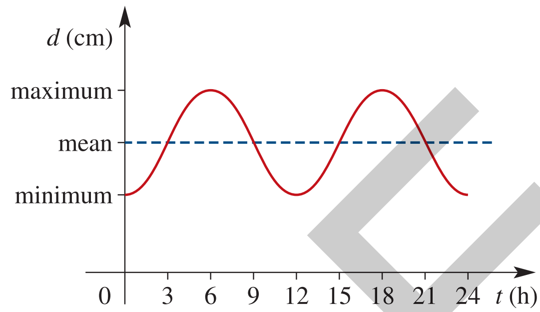
a Sketch the graph of the depth of water at the point for the time interval 0 to 24 hours if the relationship between time and depth is sinusoidal and there is high tide at noon.

b If a boat requires a depth of 4 metres of water in order to sail, how many hours before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

c If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

2





A clock hangs 120 cm below a ceiling. The clock has a diameter of 120 cm, and the hour hand is 30 cm long. The graph shows the distance from the ceiling to the tip of the hour hand over a 24-hour period.

a What are the values for the maximum, minimum and mean distance?

b An equation that determines this curve is of the form:

$$y = A \sin(nt + \varepsilon) + b$$

Find the values of A, n, ε and b.

c Find the distance from the ceiling to the tip of the hour hand at:

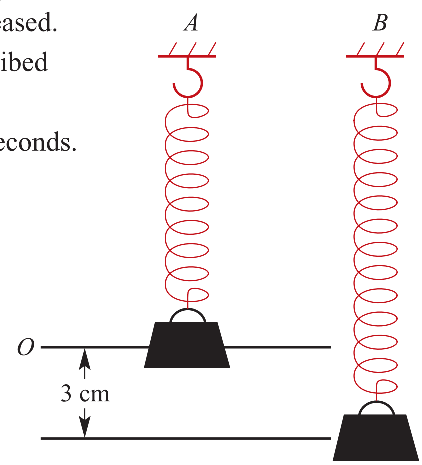
i 2:00 a.m. **ii** 11:00 p.m.

d Find the times in the morning at which the tip of the hour hand is 200 cm below the ceiling.

3 A weight is suspended from a spring as shown.

The weight is pulled down 3 cm from O and released. The vertical displacement from O at time t is described by a function of the form $y = a \cos nt$.

Let y cm be the vertical displacement at time t seconds.



t	0	0.5	1
y	-3	3	-3

It was also noted that the centre of the weight went no further than 3 cm from the centre O.

a Find the values of a and n.

b Sketch the graph of y against t.

c Find when the centre of the weight is first:

i 1.5 cm above O **ii** 1.5 cm below O

d When does the weight first reach a point 1 cm below O?

4 The manager of a reservoir and its catchment area has noted that the inflow of water into the reservoir is very predictable and in fact models the inflow with a curve with rule of the form $y = a\sin(nt + \varepsilon) + b$.

The following observations were made:

- The average inflow is $100\,000\ \text{m}^3$/day.
- The minimum daily flow is $80\,000\ \text{m}^3$/day.
- The maximum daily flow is $120\,000\ \text{m}^3$/day, and this occurs on 1 May ($t = 121$) each year.

a Find the values of a, b and n and the smallest possible positive value for ε.
b Sketch the graph of y against t.
c Find the times of year when the inflow per day is:
i $90\,000\ \text{m}^3$/day **ii** $110\,000\ \text{m}^3$/day
d Find the inflow rate on 1 June.

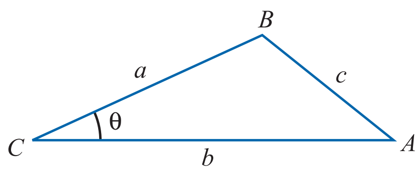
5 The number of hours of daylight at a point on the Antarctic Circle is given approximately by $d = 12 + 12\cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right)$ where t is the number of months that have elapsed since 1 January.
a Find d **i** on 21 June ($t \approx 5.7$), and **ii** on 21 March ($t \approx 2.7$)
b When will there be 5 hours of daylight?

6 The depth, $D(t)$ m, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$.
a Sketch the graph of $D(t)$ for $0 \le t \le 24$.
b Find the value of t for which $D(t) \ge 8.5$.
c Boats that need a depth of w m are permitted to enter the harbour only if the depth of the water at the entrance is at least w m for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of w that satisfies this condition.

7 The depth of water at the entrance to a harbour t hours after high tide is D m, where $D = p + q\cos(rt)^\circ$ for suitable constants p, q, r. At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.
a Show that $r = 30$ and find the values of p and q.
b Sketch the graph of D against t for $0 \le t \le 12$.
c Find how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

CAS CALCULATOR

8 The area of a triangle is given by $A = \frac{1}{2}ab\sin\theta$ and the perimeter is given by $P = a + b + \sqrt{a^2 + b^2 - 2ab\cos\theta}$



a For $a = b = 10$ and $\theta = \frac{\pi}{3}$, find:
i the area of the triangle **ii** the perimeter of the triangle

Review

b For $a = b = 10$, find the value(s) of θ for which $A = P$. (Give value(s) correct to two decimal places.)

c Show graphically that for $a = b = 6$, $P > A$ for all θ.

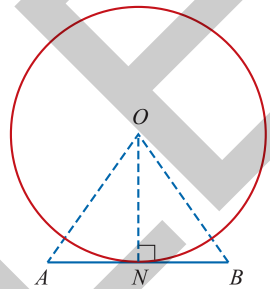
d Assume $\theta = \frac{\pi}{2}$. If $a = 6$, find the value of b such that $A = P$.

e For $a = 10$, $b = 6$, find the value(s) of θ for which $A = P$.

f If $a = b$ and $\theta = \frac{\pi}{3}$, find the value of a such that $A = P$.

CAS CALCULATOR

8 AB is one side of a regular n-sided polygon that circumscribes a circle, i.e. each edge of the polygon is tangent to the circle. The circle has radius of 1.



a Show that the area of triangle OAB is $\tan\left(\frac{\pi}{n}\right)$.

b Show that the area, A, of the polygon is given by $A = n\tan\left(\frac{\pi}{n}\right)$.

c Use a calculator to help sketch the graph of $A(x) = x\tan\left(\frac{\pi}{x}\right)$ for $x \geq 3$. Label the horizontal asymptote.

d What is the difference in area of the polygon and the circle when:

i $n = 3$? **ii** $n = 4$? **iii** $n = 12$? **iv** $n = 50$?

e State the area of an n-sided polygon that circumscribes a circle of radius r cm.

f **i** Find a formula for the n-sided regular polygon that can be inscribed in a circle of radius 1.

ii Sketch the graph of this function for $x \geq 3$.