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Chapter 4

Polynomial functions

Objectives

- To be able to use the technique of equating coefficients.
- To introduce the **functions of the form $f(x) = a(x + h)^n + k$** and to sketch graphs of this form through the use of **transformations**.
- To **divide polynomials**.
- To use the **factor theorem** to solve cubic equations and quartic equations.
- To use the **remainder theorem**.
- To draw and use **sign diagrams**.
- To **find equations** for given graphs of polynomials.
- To apply **polynomial functions** to problem solving.

4.1 Polynomials

In an earlier chapter, linear functions were discussed. This family of functions is a member of a larger family of polynomial functions.

A function with rule $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, a_2, a_3, \ldots, a_n$ are real constants and n is a positive integer, is called a **polynomial** in x over the real numbers. The numbers $a_0, a_1, a_2, a_3, \ldots, a_n$ are called the **coefficients** of the polynomial. Assuming $a_n \neq 0$, the term a_nx^n is called the **leading term**. The integer n of the leading term is the **degree** of the polynomial.

For example:

- $f(x) = a_0 + a_1x$ is a degree one polynomial (a linear function).
- $f(x) = a_0 + a_1x + a_2x^2$ is a degree two polynomial (a quadratic function).
- $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ is a degree three polynomial (a cubic function).
- $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ is a degree four polynomial (a quartic function).

The polynomials above are written with ascending powers of x. They can also be written with descending powers of x, for example:

$$f(x) = 3x + 1,\ f(x) = x^2 + 2x + 3,\ f(x) = x^3 + 4x^2 + 3x + 1$$

Polynomials are often written in factorised form, e.g. $(3x + 2)^2$, $4(x - 1)^3 + 2$.

Although it is fairly simple to expand such polynomials when the degree is small, say two or three, the binomial theorem discussed in Appendix A facilitates the expansion of polynomials of larger degree.

For example:

$$\begin{aligned}(2x+3)^4 &= \tbinom{4}{0}(2x)^4(3)^0 + \tbinom{4}{1}(2x)^3(3)^1 + \tbinom{4}{2}(2x)^2(3)^2 \\ &\quad + \tbinom{4}{3}(2x)^1(3)^3 + \tbinom{4}{4}(2x)^0(3)^4 \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81\end{aligned}$$

Equating coefficients

If $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $Q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ are equal, i.e. if $P(x) = Q(x)$ for all x, then the degree of $P(x)$ = degree of $Q(x)$ and $a_0 = b_0, a_1 = b_1, a_2 = b_2, \ldots,$ etc.

Example 1

If $x^2 + 6x + 4 = a(x+3)^2 + b$ for all $x \in R$, find the values of a and b.

Solution

Expanding the right-hand side of the equation gives:

$$\begin{aligned}a(x+3)^2 + b &= a(x^2 + 6x + 9) + b \\ &= ax^2 + 6ax + 9a + b\end{aligned}$$

If $x^2 + 6x + 4 = ax^2 + 6ax + 9a + b$ for all $x \in R$, then by equating coefficients:

$$\begin{aligned}&(\text{coefficient of } x^2) && 1 = a \\ &(\text{coefficient of } x) && 6 = 6a \\ &(\text{coefficient of } x^0\text{, the constant term of the polynomial}) && 4 = 9a + b\end{aligned}$$

Hence $a = 1$ and $b = -5$.

Example 2

a If $x^3 + 3x^2 + 3x + 8 = a(x+1)^3 + b$ for all $x \in R$, find the values of a and b.

b Show that $x^3 + 6x^2 + 6x + 8$ cannot be written in the form $a(x+c)^3 + b$ for real numbers a, b and c.

Solution

a The expansion of the right-hand side of the equation gives:

$$\begin{aligned}a(x+1)^3 + b &= a(x^3 + 3x^2 + 3x + 1) + b \\ &= ax^3 + 3ax^2 + 3ax + a + b\end{aligned}$$

If $x^3 + 3x^2 + 3x + 8 = ax^3 + 3ax^2 + 3ax + a + b$ for all $x \in R$, equating coefficients gives:

$$\begin{aligned} &(\text{coefficient of } x^3) && 1 = a \\ &(\text{coefficient of } x^2) && 3 = 3a \\ &(\text{coefficient of } x) && 3 = 3a \\ &(\text{coefficient of } x^0\text{, the constant term of the polynomial}) && 8 = a + b \end{aligned}$$

Hence $a = 1$ and $b = 7$.

b The expansion of the right-hand side of the equation gives:

$$\begin{aligned} a(x+c)^3 + b &= a(x^3 + 3cx^2 + 3c^2x + c^3) + b \\ &= ax^3 + 3cax^2 + 3c^2ax + c^3a + b \end{aligned}$$

If $x^3 + 6x^2 + 6x + 8 = ax^3 + 3cax^2 + 3c^2ax + c^3a + b$ for all $x \in R$:

$$\begin{aligned} &(\text{coefficient of } x^3) && 1 = a && (1) \\ &(\text{coefficient of } x^2) && 6 = 3ca && (2) \\ &(\text{coefficient of } x) && 6 = 3c^2a && (3) \\ &(\text{coefficient of } x^0\text{, the constant term of the polynomial}) && 8 = c^3a + b && (4) \end{aligned}$$

From (1), $a = 1$
From (2), $c = 2$
For (3), the right-hand side $3c^2a$ equals 12, which is a contradiction.

Division of polynomials

The division of polynomials was introduced in *Essential Mathematical Methods 1 & 2 CAS*.

The general result for polynomial division is:

For non-zero polynomials, $P(x)$ and $D(x)$, if $P(x)$ (the dividend) is divided by $D(x)$ (the divisor), then there are unique polynomials, $Q(x)$ (the quotient) and $R(x)$ (the remainder), such that

$$P(x) = D(x)Q(x) + R(x)$$

Either the degree of $R(x) < D(x)$, or $R(x) = 0$

When $R(x) = 0$, then $D(x)$ is called a divisor of $P(x)$ and $P(x) = D(x)Q(x)$

The following example illustrates the process of dividing.

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Example 3

Divide $3x^4 - 9x^2 + 27x - 8$ by $x - 2$.

Solution

$$
\begin{array}{r@{}l}
 & 3x^3 + 6x^2 + 3x + 33 \\
\cline{2-2}
x - 2 & \big)\, 3x^4 + 0x^3 - 9x^2 + 27x - 8 \\
 & \underline{3x^4 - 6x^3} \\
 & \phantom{)3x^4 - {}} 6x^3 - 9x^2 \\
 & \phantom{)3x^4 - {}} \underline{6x^3 - 12x^2} \\
 & \phantom{)3x^4 - 6x^3 - {}} 3x^2 + 27x \\
 & \phantom{)3x^4 - 6x^3 - {}} \underline{3x^2 - 6x} \\
 & \phantom{)3x^4 - 6x^3 - 9x^2 + {}} 33x - 8 \\
 & \phantom{)3x^4 - 6x^3 - 9x^2 + {}} \underline{33x - 66} \\
 & \phantom{)3x^4 - 6x^3 - 9x^2 + 27x - {}} 58
\end{array}
$$

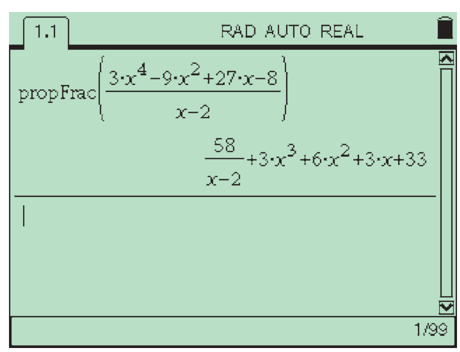
Thus $3x^4 - 9x^2 + 27x - 8 = (x - 2)(3x^3 + 6x^2 + 3x + 33) + 58$ or, equivalently,

$$\frac{3x^4 - 9x^2 + 27x - 8}{x - 2} = 3x^3 + 6x^2 + 3x + 33 + \frac{58}{x - 2}$$

In this example $3x^4 - 9x^2 + 27x - 8$ is the dividend, $x - 2$ is the divisor and the remainder is 58.

Using the TI-Nspire

Use **propFrac** (menu 3 7 1) as shown.

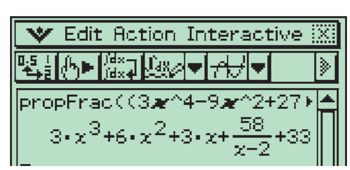


Using the Casio ClassPad

Enter and highlight

$(3x^4 - 9x^2 + 27x - 8)/(x - 2)$ and tap

Interactive > **Transformation** > **propFrac**.



The remainder theorem and factor theorem

The following two results are recalled from *Essential Mathematical Methods 1 & 2 CAS*.

The remainder theorem

If a polynomial $P(x)$ is divided by $ax + b$ the remainder is $P\left(\frac{-b}{a}\right)$.

This is easily proved by observing if

$$P(x) = D(x)(ax + b) + R(x) \text{ and } x = \frac{-b}{a}$$

$$P\left(\frac{-b}{a}\right) = D\left(\frac{-b}{a}\right)\left(a \times \frac{-b}{a} + b\right) + R\left(\frac{-b}{a}\right)$$

and thus

$$P\left(\frac{-b}{a}\right) = R\left(\frac{-b}{a}\right)$$

The factor theorem

An immediate consequence of the remainder theorem is the **factor theorem**.

$ax + b$ is a factor of the polynomial $P(x)$ if and only if $P\left(\frac{-b}{a}\right) = 0$.

Example 4

Find the remainder when $P(x) = 3x^3 + 2x^2 + x + 1$ is divided by $2x + 1$.

Solution

By the remainder theorem the remainder is:

$$\begin{aligned} P\left(-\frac{1}{2}\right) &= 3\left(-\frac{1}{2}\right)^3 + 2\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1 \\ &= -\frac{3}{8} + \frac{2}{4} - \frac{1}{2} + 1 \\ &= -\frac{3}{8} + 1 = \frac{5}{8} \end{aligned}$$

Example 5

Given that $x + 1$ and $x - 2$ are factors of $6x^4 - x^3 + ax^2 - 6x + b$, find the values of a and b.

Solution

Let $P(x) = 6x^4 - x^3 + ax^2 - 6x + b$.

By the factor theorem, $P(-1) = 0$ and $P(2) = 0$.

Hence,

$$6 + 1 + a + 6 + b = 0 \qquad (1)$$

and $$96 - 8 + 4a - 12 + b = 0 \qquad (2)$$

Rearranging gives:

$$a + b = -13 \qquad (1')$$
$$4a + b = -76 \qquad (2')$$

Subtract $(1')$ from $(2')$:

$$3a = -63$$

Therefore $a = -21$ and, from $(1')$, $b = 8$.

Solving polynomial equations

The factor theorem may be used in the solution of equations.

Example 6

Factorise $P(x) = x^3 - 4x^2 - 11x + 30$ and hence solve the equation $x^3 - 4x^2 - 11x + 30 = 0$

Solution

$P(1) = 1 - 4 - 11 + 30 \neq 0$

$P(-1) = -1 - 4 + 11 + 30 \neq 0$

$P(2) = 8 - 16 - 22 + 30 = 0$

$\therefore x - 2$ is a factor.

Dividing $x^3 - 4x^2 - 11x + 30$ by $x - 2$ reveals that:

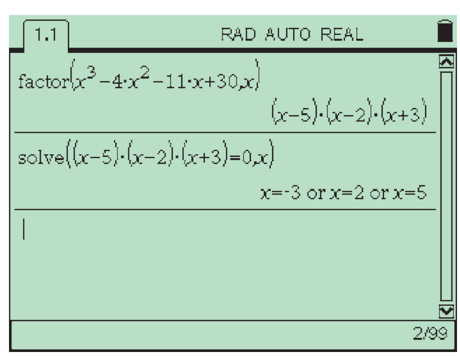
$$\begin{aligned} P(x) &= (x - 2)(x^2 - 2x - 15) \\ &= (x - 2)(x - 5)(x + 3) \end{aligned}$$

Therefore $x - 2 = 0$ or $x - 5 = 0$ or $x + 3 = 0$

$\therefore x = 2$ or $x = 5$ or or $x = -3$

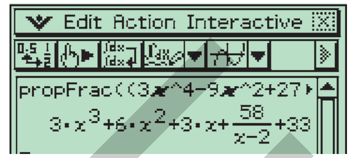
Using the TI-Nspire

Use **factor( )** (menu 3 2) and **solve( )** (menu 3 1) as shown.



Using the Casio ClassPad

Enter and highlight $x^3 - 4x^2 - 11x + 30$ then tap **Interactive** > **Transformation** > **factor**.



Copy and paste the answer to the next entry line and tap **Interactive** > **Equation/inequality** > **solve** and ensure the variable set is x.

Exercise 4A

Skillsheet TEACHER

1 Find the values of A and B such that $A(x + 3) + B(x + 2) = 4x + 9$ for all real numbers.

2 Find the values of A, B and C in each of the following if:

a $x^2 - 4x + 10 = A(x + B)^2 + C$ for all $x \in R$

b $4x^2 - 12x + 14 = A(x + B)^2 + C$ for all $x \in R$

c $x^3 - 9x^2 + 27x - 22 = A(x + B)^3 + C$ for all $x \in R$

3 For each of the following, divide the first term by the second:

a $2x^3 - 7x^2 + 15x - 3,\ x - 3$

b $5x^5 + 13x^4 - 2x^2 - 6,\ x + 1$

c $x^4 - 9x^3 + 25x^2 - 8x - 2,\ x^2 - 2$

4 a Find the remainder when $x^3 + 3x - 2$ is divided by $x + 2$.

b Find the value of a for which $(1 - 2a)x^2 + 5ax + (a - 1)(a - 8)$ is divisible by $(x - 2)$ but not by $(x - 1)$.

5 Given that $f(x) = 6x^3 + 5x^2 - 17x - 6$:

a Find the remainder when $f(x)$ is divided by $x - 2$.

b Find the remainder when $f(x)$ is divided by $x + 2$.

c Factorise $f(x)$ completely.

6 a Prove that the expression $x^3 + (k - 1)x^2 + (k - 9)x - 7$ is divisible by $x + 1$ for all values of k.

b Find the value of k for which the expression has a remainder of 12 when divided by $x - 2$.

7 The function $f(x) = 2x^3 + ax^2 - bx + 3$ has a factor $(x + 3)$. When $f(x)$ is divided by $(x - 2)$, the remainder is 15.

a Calculate the values of a and b.

b Find the other two factors of $f(x)$.

8 The expression $4x^3 + ax^2 - 5x + b$ leaves remainders of -8 and 10 when divided by $(2x - 3)$ and $(x - 3)$ respectively. Calculate the values of a and b.

9 Find the remainder when $(x + 1)^4$ is divided by $x - 2$

10 Let $P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$

a Show that neither $(x - 1)$ nor $(x + 1)$ is a factor of $P(x)$

b Given that $P(x)$ can be written in the form $(x^2 - 1)Q(x) + ax + b$ where $Q(x)$ is a polynomial and a and b are constants, hence or otherwise, find the remainder when $P(x)$ is divided by $x^2 - 1$

11 a Show that both $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are factors of $x^4 + x^3 - x^2 - 3x - 6$

b Hence write down one quadratic factor of $x^4 + x^3 - x^2 - 3x - 6$, and find a second quadratic factor.

12 Solve each of the following equations for x:

a $(2 - x)(x + 4)(x - 2)(x - 3) = 0$

b $x^3(2 - x) = 0$

c $(2x - 1)^3(2 - x) = 0$

d $(x + 2)^3(x - 2)^2 = 0$

e $x^4 - 4x^2 = 0$

f $x^4 - 9x^2 = 0$

g $12x^4 + 11x^3 - 26x^2 + x + 2 = 0$

h $x^4 + 2x^3 - 3x^2 - 4x + 4 = 0$

i $6x^4 - 5x^3 - 20x^2 + 25x - 6 = 0$

13 Find the x-axis intercepts and the y-axis intercepts of the graphs of each of the following:

a $y = x^3 - x^2 - 2x$

b $y = x^3 - 2x^2 - 5x + 6$

c $y = x^3 - 4x^2 + x + 6$

d $y = 2x^3 - 5x^2 + x + 2$

e $y = x^3 + 2x^2 - x - 2$

f $y = 3x^3 - 4x^2 - 13x - 6$

g $y = 5x^3 + 12x^2 - 36x - 16$

h $y = 6x^3 - 5x^2 - 2x + 1$

i $y = 2x^3 - 3x^2 - 29x - 30$

14 The expressions $px^4 - 5x + q$ and $x^4 - 2x^3 - px^2 - qx - 8$ have a common factor $x - 2$. Find the values of p and q.

15 Find the remainder when $f(x) = x^4 - x^3 + 5x^2 + 4x - 36$ is divided by $x + 1$.

16 Factorise each of the following polynomials using a calculator to help find at least one linear factor:

a $x^3 - 11x^2 - 125x + 1287$

b $x^3 - 9x^2 - 121x + 1089$

c $2x^3 - 9x^2 - 242x + 1089$

d $4x^3 - 367x + 1287$

17 Factorise each of the following:

a $x^4 - x^3 - 43x^2 + x + 42$

b $x^4 + 4x^3 - 27x - 108$

18 Factorise each of the following polynomials, using a calculator to help find at least one linear factor:

a $2x^4 - 25x^3 + 57x^2 + 9x + 405$ **b** $x^4 + 13x^3 + 40x^2 + 81x + 405$
c $x^4 + 3x^3 - 4x^2 + 3x - 135$ **d** $x^4 + 4x^3 - 35x^2 - 78x + 360$

4.2 Quadratic functions

The following is a summary of material assumed to have been covered in *Essential Mathematical Methods 1 & 2 CAS*.

- The general expression of a quadratic function is $y = ax^2 + bx + c,\ x \in R$.
- To sketch the graph of a quadratic function (called a parabola) use the following:
  - If $a > 0$, the function has a minimum value.
  - If $a < 0$, the function has a maximum value.
  - The value of c gives the y-axis intercept.
  - The equation of the axis of symmetry is $x = -\dfrac{b}{2a}$
  - The x-axis intercepts are determined by solving the equation $ax^2 + bx + c = 0$
- A quadratic equation may be solved by:
  - factorising

    e.g., $$\begin{aligned} 2x^2 + 5x - 12 &= 0 \\ (2x - 3)(x + 4) &= 0 \\ \therefore x &= \frac{3}{2} \text{ or } -4 \end{aligned}$$
  - completing the square

    e.g., $x^2 + 2x - 4 = 0$

    Add and subtract $\left(\dfrac{b}{2}\right)^2$ to 'complete the square'.

    $$\begin{aligned} x^2 + 2x + 1 - 1 - 4 &= 0 \\ \therefore (x + 1)^2 - 5 &= 0 \\ \therefore (x + 1)^2 &= 5 \\ \therefore x + 1 &= \pm\sqrt{5} \\ \therefore x &= -1 \pm \sqrt{5} \end{aligned}$$
  - using the general quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

    e.g., $$\begin{aligned} -3x^2 - 12x - 7 &= 0 \\ x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)} \\ &= \frac{6 \pm \sqrt{15}}{-3} \end{aligned}$$

- Using the discriminant of the quadratic function $f(x) = ax^2 + bx + c$
  - If $b^2 - 4ac > 0$, the graph of the function has two x-axis intercepts.
  - If $b^2 - 4ac = 0$, the graph of the function touches the x-axis.
  - If $b^2 - 4ac < 0$, the graph of the function does not intersect the x-axis.

Example 7

Sketch the graph of $f(x) = -3x^2 - 12x - 7$ by using the quadratic formula to calculate the x-axis intercepts.

Solution

Since $c = -7$ the y-axis intercept is $(0, -7)$.

Find the turning point coordinates.

$$\begin{aligned}\text{Axis of symmetry, } x &= -\frac{b}{2a}\\ &= -\frac{(-12)}{2 \times -3}\\ &= -2\end{aligned}$$

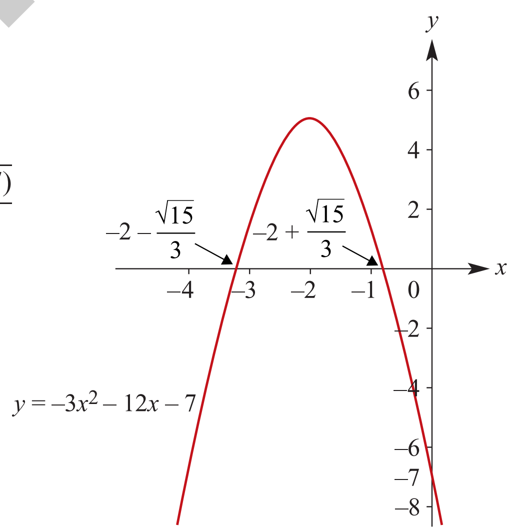
and

$$\begin{aligned} f(-2) &= -3(-2)^2 - 12(-2) - 7\\ &= 5\end{aligned}$$

$\therefore$ turning point coordinates are $(-2, 5)$.

Calculate the x-axis intercepts.

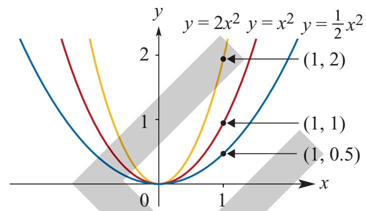
$$\begin{aligned}-3x^2 - 12x - 7 &= 0\\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)}\\ &= \frac{12 \pm \sqrt{60}}{-6}\\ &= \frac{12 \pm 2\sqrt{15}}{-6}\\ &= \frac{6 \pm \sqrt{15}}{-3}\\ &\approx \frac{6 \pm 3.87}{-3} \text{ (to 2nd decimal place)}\\ &= -3.29 \text{ or } -0.71\end{aligned}$$



Use of 'completing the square' to sketch quadratics

For $a > 0$, the graph of the function $f(x) = ax^2$ is obtained from the graph of $f(x) = x^2$ by a dilation of factor a from the x-axis.

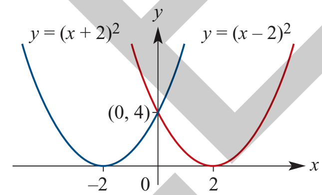
The graphs on the right are those of $y = x^2$, $y = 2x^2$ and $y = \frac{1}{2}x^2$, i.e. $a = 1, 2$ and $\frac{1}{2}$.



For $h > 0$, the graph of $f(x) = (x + h)^2$ is obtained from the graph of $f(x) = x^2$ by a translation of h units in the negative direction of the x-axis.

For $h < 0$, the graph of $f(x) = (x + h)^2$ is obtained from the graph of $f(x) = x^2$ by a translation of $-h$ units in the positive direction of the x-axis.

The graphs of $y = (x + 2)^2$ and $y = (x - 2)^2$ are shown:



For $k > 0$, the graph of

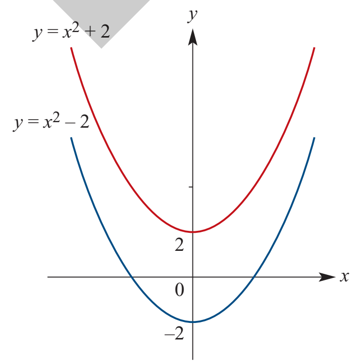
$$f(x) = x^2 + k$$

is obtained from the graph of

$$f(x) = x^2$$

by a translation of k units in the positive direction of the y-axis.

For $k < 0$, the translation is in the negative direction of the y-axis.



For example, the graph of the function

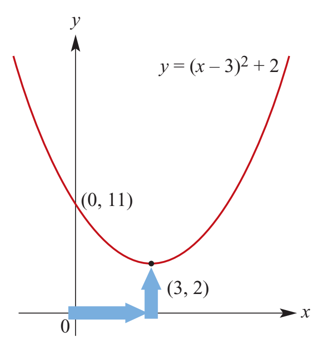
$$f(x) = (x - 3)^2 + 2$$

is obtained by translating the graph of the function

$$f(x) = x^2$$

by 3 units in the positive direction of the x-axis and 2 units in the positive direction of the y-axis.

Note that the vertex is at (3, 2).



The graph of the function

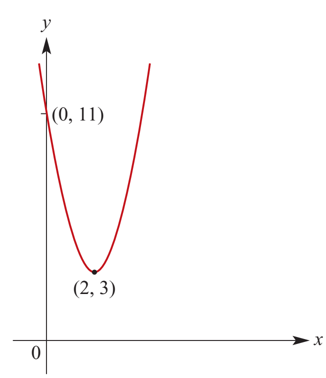
$$f(x) = 2(x - 2)^2 + 3$$

is obtained from the graph of

$$f(x) = x^2$$

by the following transformations:

- dilation of factor 2 from the x-axis
- translation of 2 units in the positive direction of the x-axis
- translation of 3 units in the positive direction of the y-axis.



By completing the square, any quadratic function can be written in the form:

$$y = a(x + h)^2 + k$$

Example 8

Solve $2x^2 - 4x - 5 = 0$ by expressing $2x^2 - 4x - 5$ in the form $y = a(x + h)^2 + k$
Use this to help you sketch the graph of $f(x) = 2x^2 - 4x - 5$

Solution

$$\begin{aligned} f(x) &= 2x^2 - 4x - 5 \\ &= 2\left(x^2 - 2x - \frac{5}{2}\right) \\ &= 2\left(x^2 - 2x + 1 - 1 - \frac{5}{2}\right) \quad \text{(adding and subtracting } \left(\frac{b}{2}\right)^2 \text{ to 'complete} \\ &= 2\left[(x^2 - 2x + 1) - \frac{7}{2}\right] \quad \text{the square')} \\ &= 2\left[(x - 1)^2 - \frac{7}{2}\right] \\ &= 2(x - 1)^2 - 7 \end{aligned}$$

Therefore, the graph of $f(x) = 2x^2 - 4x - 5$ may be obtained from the graph of $y = x^2$ by the following transformations:

- dilation of factor 2 from the x-axis
- translation of 1 unit in the positive direction of the x-axis
- translation of 7 units in the negative direction of the y-axis.

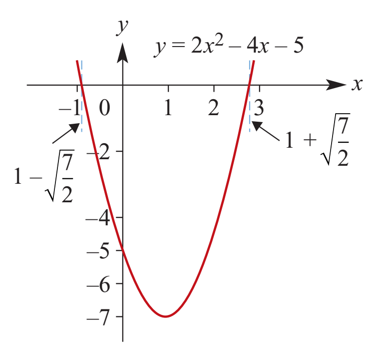
The x-axis intercepts can also be determined by this process.

To solve

$$\begin{aligned} 0 &= 2x^2 - 4x - 5 \\ \therefore 0 &= 2(x - 1)^2 - 7 \\ \therefore (x - 1)^2 &= \frac{7}{2} \\ \therefore x - 1 &= \pm\sqrt{\frac{7}{2}} \\ \therefore x &= 1 \pm \sqrt{\frac{7}{2}} \\ \therefore x &= 1 + \sqrt{\frac{7}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{7}{2}} \\ &\approx 2.87 \quad \text{or} \quad \approx -0.87 \end{aligned}$$

This information can now be used to sketch the graph.

- Since $c = -5$ the y-axis intercept is (0, –5).
- Turning point coordinates are (1, –7).
- x-axis intercepts are (2.87, 0) and (–0.87, 0) to two decimal places.



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Exercise 4B

Skillsheet TEACHER

1 Without sketching the graphs of the following functions, determine whether they cross, touch or do not intersect the x-axis:

a $f(x) = x^2 - 5x + 2$ **b** $f(x) = -4x^2 + 2x - 1$ **c** $f(x) = x^2 - 6x + 9$
d $f(x) = 8 - 3x - 2x^2$ **e** $f(x) = 3x^2 + 2x + 5$ **f** $f(x) = -x^2 - x - 1$

2 Sketch the graphs of the following functions:

a $f(x) = 2(x-1)^2$ **b** $f(x) = 2(x-1)^2 - 2$ **c** $f(x) = -2(x-1)^2$
d $f(x) = 4 - 2(x+1)^2$ **e** $f(x) = 4 + 2\left(x + \frac{1}{2}\right)^2$ **f** $f(x) = 2(x+1)^2 - 1$
g $f(x) = 3(x-2)^2 - 4$ **h** $f(x) = (x+1)^2 - 1$ **i** $f(x) = 5x^2 - 1$
j $f(x) = 2(x+1)^2 - 4$

3 Express each of the following functions in the form $y = a(x+h)^2 + k$ and hence find the maximum or minimum value and the range in each case:

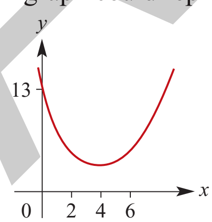
a $f(x) = x^2 + 3x - 2$ **b** $f(x) = x^2 - 6x + 8$ **c** $f(x) = 2x^2 + 8x - 6$
d $f(x) = 4x^2 + 8x - 7$ **e** $f(x) = 2x^2 - 5x$ **f** $f(x) = 7 - 2x - 3x^2$
g $f(x) = -2x^2 + 9x + 11$

4 Sketch the graphs of the following functions, clearly labelling the intercepts and turning points:

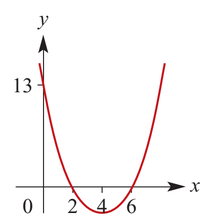
a $y = -x^2 + 2x$ **b** $y = x^2 - 6x + 8$ **c** $y = -2x^2 + 8x - 6$
d $y = -x^2 - 5x - 6$ **e** $f(x) = x^2 + 3x - 2$ **f** $f(x) = 2x^2 + 4x - 7$
g $f(x) = 5x^2 - 10x - 1$ **h** $f(x) = -2x^2 + 4x - 1$
i $y = 2.5x^2 + 3x + 0.3$ **j** $y = -0.6x^2 - 1.3x - 0.1$

5 **a** Which of the graphs shown could represent the graph of the equation $y = (x-4)^2 - 3$?
b Which graph could represent $y = 3 - (x-4)^2$?

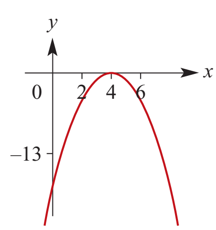
A



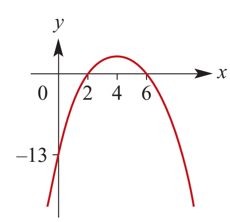
B



C



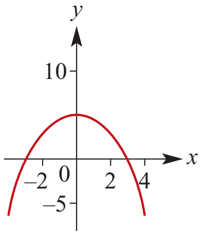
D



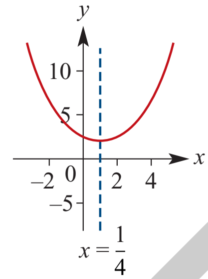
6 Which of the curves shown could be defined by each of the following?

a $y = \frac{1}{3}(x+4)(8-x)$ **b** $y = x^2 - \frac{x}{2} + 1$
c $y = -10 + 2(x-1)^2$ **d** $y = \frac{1}{2}(9 - x^2)$

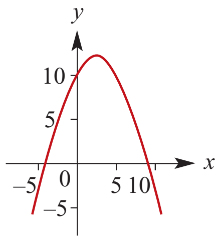
A



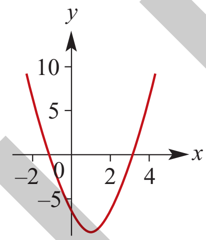
B



C



D



7 For which values of m does the equation $mx^2 - 2mx + 3 = 0$ have:

a two solutions for x? **b** one solution for x?

8 Show that the equation $(k + 1)x^2 - 2x - k = 0$ has a solution for all values of k.

9 For which values of k does the equation $kx^2 - 2kx = 5$ have:

a two solutions for x? **b** one solution for x?

10 Show that the equation $ax^2 - (a + b)x + b = 0$ has a solution for all values of a and b.

4.3 Determining the rule for a parabola

Excel

Given sufficient information about a curve, a rule for the function of the graph may be determined.

For example, if the coordinates of three points on a parabola of the form

$$y = ax^2 + bx + c$$

are known, the rule for the parabola may be found, i.e. the values of a, b and c may be found.

Sometimes a more specific rule is known. For example, the curve may be a dilation of $y = x^2$. It is then known to be of the $y = ax^2$ family, and the coordinates of one point (with the exception of the origin) will be enough to determine the value for a.

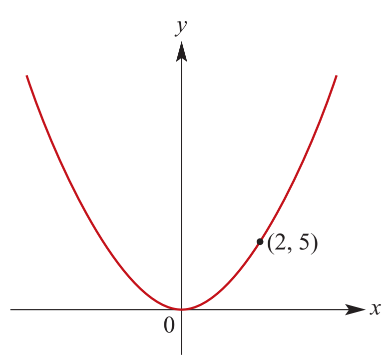
In the following it is assumed that each of the graphs is that of a parabola, and each rule is that of a quadratic function in x.

1 This is of the form $y = ax^2$ (since the graph has its vertex at the origin).

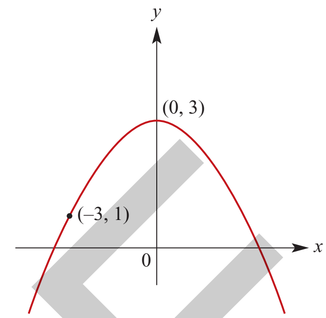
When $x = 2$, $y = 5$

$$\therefore\ 5 = a(2)^2 \Rightarrow a = \frac{5}{4}$$

The rule is $y = \dfrac{5}{4}x^2$



2 This is of the form $y = ax^2 + c$ (since the graph is symmetric about the y-axis).



For (0, 3) $\quad 3 = a(0) + c$ implies $c = 3$

For $(-3, 1)$ $\quad 1 = a(-3)^2 + 3$

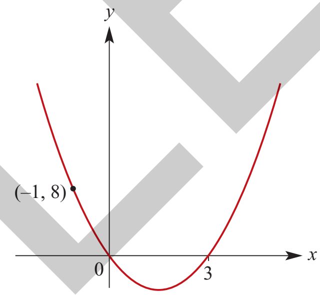
$$1 = 9a + 3$$

$$\therefore a = -\frac{2}{9}$$

$\therefore$ the rule is $y = -\frac{2}{9}x^2 + 3$

3 This is of the form $y = ax(x - 3)$

As the point $(-1, 8)$ is on the parabola:



$$8 = -a(-1 - 3)$$

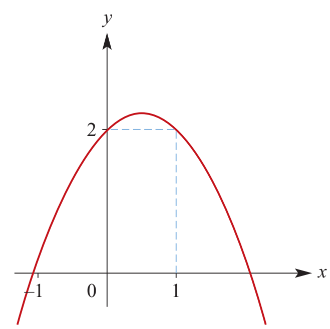
And hence $4a = 8$

Therefore $a = 2$

The rule is $y = 2x(x - 3)$

4 This is of the form $y = ax^2 + bx + c$

As the point with coordinates (–1, 0) is on the parabola:



$$0 = a - b + c \qquad (1)$$

The y-axis intercept is 2 and therefore $c = 2$ (2)

As the point with coordinates (1, 2) is on the parabola:

$$2 = a + b + c \qquad (3)$$

Substitute $c = 2$ in (1) and (3) $\quad 0 = a - b + 2$

From (1) $\quad -2 = a - b \quad$ (1a)

From (3) $\quad 0 = a + b \quad$ (3a)

Subtract (3a) from (1a) $\quad -2 = -2b$

$$\therefore b = 1$$

Substitute $b = 1$ and $c = 2$ in (1).

$$\therefore \ 0 = a - 1 + 2$$

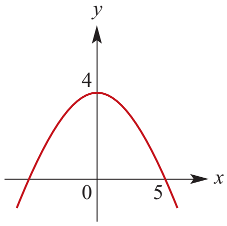
$$0 = a + 1 \Rightarrow a = -1$$

$\therefore$ the quadratic rule is $y = -x^2 + x + 2$

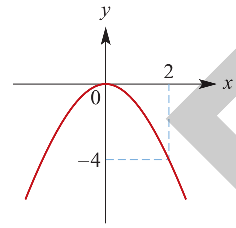
Exercise 4C

1 Determine the equation of each of the following parabolas:

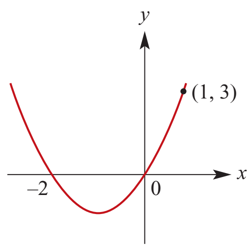
a



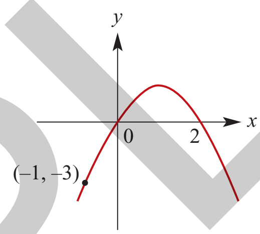
b



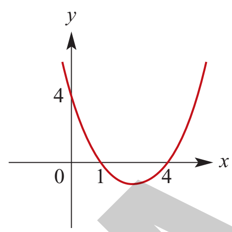
c



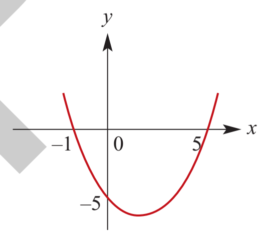
d



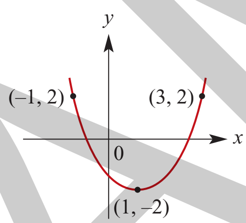
e



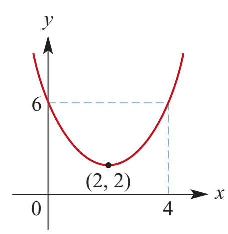
f



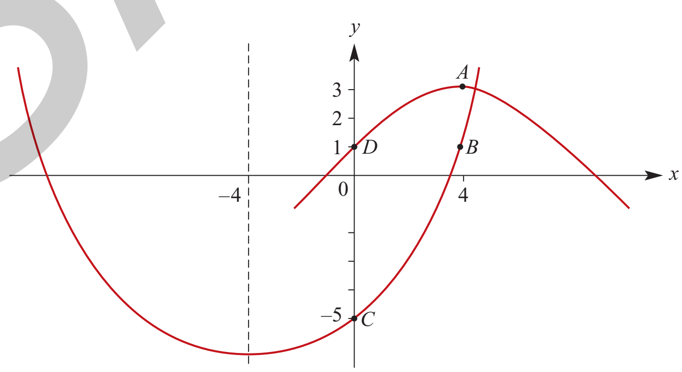
g



h



2 Find quadratic expressions representing the two curves shown in the diagram, given that the coefficient of x in each case is 1. A is (4, 3), B is (4, 1), C is (0,−5) and D is (0, 1).



3 The graph of the quadratic function $f(x) = A(x + b)^2 + B$ has a vertex at $(-2, 4)$ and passes through the point (0, 8). Find the values of A, b and B.

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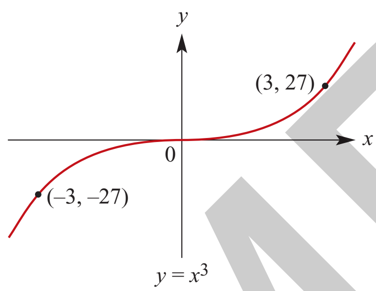
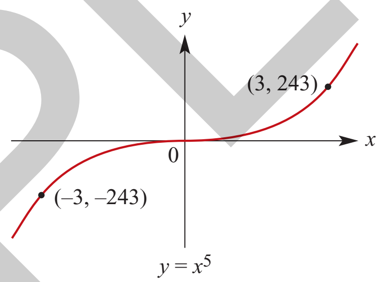
4.4 Functions of the form $f: R \to R$, $f(x) = a(x+h)^n + k$, where n is a natural number

In the previous section it was shown that every quadratic polynomial can be written in the form $a(x+h)^2 + k$. This is not true for polynomials of higher degree. This was shown in Example 2 of this chapter. However, there are many polynomials that can be written in this form. In Chapter 3 the family of power functions was introduced. In this section the sub-family of power functions with rules of the form $f(x) = x^n$ where n is a natural number are considered.

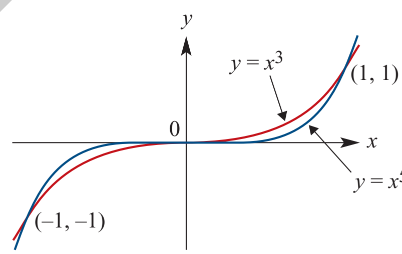
$f(x) = x^n$ where n is an odd positive integer

The diagrams below shows the graphs of $y = x^3$ and $y = x^5$





The diagram on the right shows both functions graphed on the one set of axes for a smaller domain.



The following properties can be observed for a function $f(x) = x^n$ where n is an odd integer:

- $f(0) = 0$
- $f(1) = 1$ and $f(-1) = -1$
- $-f(x) = f(-x)$, i.e. f is an odd function.
  $f(-x) = (-x)^n = (-1)^n(x)^n = -f(x)$ as n is odd.
- As $x \to \infty, f(x) \to \infty$
- As $x \to -\infty, f(x) \to -\infty$

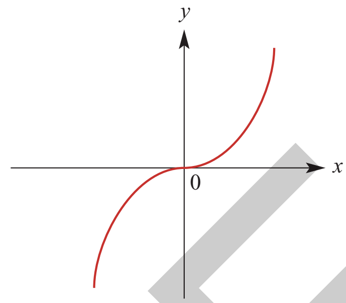
From *Essential Mathematical Methods 1 & 2 CAS* you will recall that the gradient function of $f(x) = x^n$ has rule nx^{n-1}. Hence the gradient is 0 when $x = 0$. As $n - 1$ is even for n odd, $nx^{n-1} > 0$ for all non-zero x. That is, the gradient of the graph of $y = f(x)$ is positive for all non-zero x and zero when $x = 0$. Recall that the stationary point at $x = 0$, for functions of this form, is called a stationary point of inflexion.

Comparing the graphs of $y = x^n$ and $y = x^m$ where n and m are odd and $n > m$

- $x^n < x^m$ when $0 < x < 1$ $\quad (x^n - x^m = x^m(x^{n-m} - 1) < 0)$
- $x^n = x^m$ when $x = 0$
- $x^n > x^m$ when $-1 < x < 0$ $\quad (x^n - x^m = x^m(x^{n-m} - 1) > 0$ as $n - m$ is even, m is odd and x is negative and greater than $-1)$

- $x^n = x^m$ for $x = 1$ and $x^n = x^m$ for $x = -1$
- $x^n > x^m$ for $x > 1$
- $x^n < x^m$ for $x < -1$

It should be noted that the appearance of graphs is dependent on the scales on the y- and x-axes. Odd power functions are often depicted as shown.



Transformations of graphs of functions with rule $f(x) = x^n$ where n is an odd positive integer

Transformations of these graphs result in graphs with rules of the form $y = a(x + h)^n + k$ where a, h and k are real constants.

Example 9

Sketch the graph of $f(x) = (x - 2)^3 + 1$

Solution

This can be done by noting that a translation $(x, y) \to (x + 2, y + 1)$ maps the graph of $y = x^3$ to $y = (x - 2)^3 + 1$

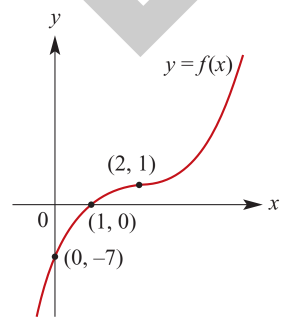
Note: The point with coordinates (2, 1) is a point of zero gradient.

For the axes intercepts, consider this:

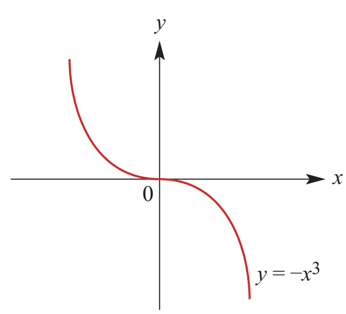
- When $x = 0$, $y = (-2)^3 + 1 = -7$
- When $y = 0$, $0 = (x - 2)^3 + 1$

$$\therefore -1 = (x - 2)^3$$
$$\therefore -1 = x - 2$$

and hence $x = 1$



The reflection in the x-axis described by the transformation rule $(x, y) \to (x, -y)$ and applied to $y = x^3$ results in the graph of $y = -x^3$



Example 10

Sketch the graph of $y = -(x - 1)^3 + 2$

Solution

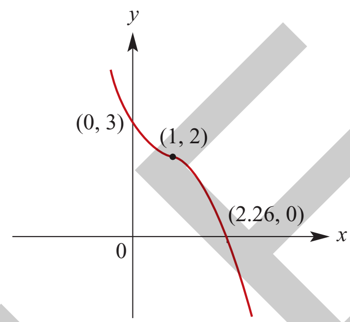
A reflection in the x-axis followed by a translation with rule $(x, y) \rightarrow (x+1, y+2)$ applied to the graph of $y = x^3$ results in the graph $y = -(x-1)^3 + 2$

Note: (1, 2) is a point of zero gradient.

For the axes intercepts consider this:

- When $x = 0$, $y = -(-1)^3 + 2 = 3$
- When $y = 0$, $0 = -(x-1)^3 + 2$

$$\therefore (x-1)^3 = 2$$
$$\therefore x - 1 = 2^{\frac{1}{3}}$$
$$\text{and hence } x = 1 + 2^{\frac{1}{3}}$$
$$\approx 2.26$$



Example 11

Sketch the graph of $y = 2(x+1)^3 + 2$

Solution

A dilation of factor 2 from the x-axis followed by the translation with rule

$$(x, y) \rightarrow (x-1, y+2)$$

maps the graph of $y = x^3$ to the graph of

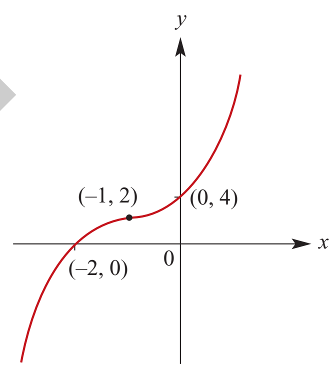
$$y = 2(x+1)^3 + 2$$

Note: $(-1, 2)$ is a point of zero gradient.

For the axes intercepts consider this:

- When $x = 0$, $y = 2 + 2 = 4$
- When $y = 0$, $0 = 2(x+1)^3 + 2$

$$\therefore -1 = (x+1)^3$$
$$\therefore -1 = x + 1$$
$$\text{and hence } x = -2$$



Example 12

The graph of $y = a(x+h)^3 + k$ has a point of zero gradient at (1, 1) and passes through the point (0, 4). Find the values of a, h and k.

Solution

$$h = -1 \text{ and } k = 1$$

Therefore as the graph passes through (0, 4):

$$4 = -a + 1$$
$$\therefore a = -3$$

Example 13

a Find the rule for the image of the graph of $y = x^5$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

b Find a sequence of transformations which takes the graph of $y = x^5$ to the graph of $y = 6 - 2(x+5)^5$

Solution

a $(x, y) \to (-x, y) \to (-2x, y) \to (-2x + 2, y + 3)$

Let (x', y') be the image of (x, y) under this transformation.

Hence $x' = -2x + 2$ and $y' = y + 3$

Thus $-2x = x' - 2$, which implies $x = \dfrac{x' - 2}{-2}$ and $y = y' - 3$

Therefore, the graph of $y = x^5$ maps to the graph of $y' - 3 = \left(\dfrac{x' - 2}{-2}\right)^5$,

i.e., to the graph of $y = -\dfrac{1}{32}(x - 2)^5 + 3$

b Rearrange $y = 6 - 2(x+5)^5$ to $\dfrac{y' - 6}{-2} = (x' + 5)^5$

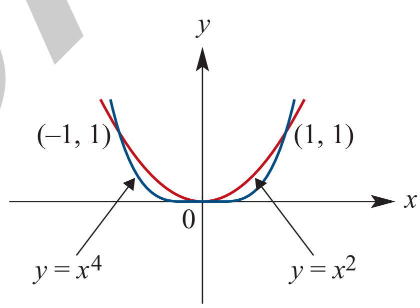
Therefore, $y = \dfrac{y' - 6}{-2}$ and $x = x' + 5$ and $y' = -2y + 6$ and $x' = x - 5$

The sequence of transformations is:

- reflection in the x-axis
- dilation of factor 2 from the x-axis
- translation of 5 units in the negative direction of the x-axis and 6 units in the positive direction of the y-axis.

$f(x) = x^n$ where n is an even positive integer

The graphs of $y = x^2$ and $y = x^4$ are shown on the one set of axes.



The following properties can be observed for a function $f(x) = x^n$ where n is an even integer:

- $f(0) = 0$
- $f(1) = 1$ and $f(-1) = 1$
- $f(x) = f(-x)$, i.e. f is an even function, $f(-x) = (-x)^n$
$$= (-1)^n(x)^n$$
$$= f(x) \text{ as } n \text{ is even}$$
- As $x \to \infty$, $f(x) \to \infty$
- As $x \to -\infty$, $f(x) \to \infty$

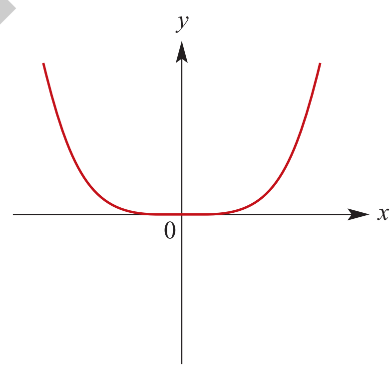
From *Essential Mathematical Methods 1 & 2 CAS* you will recall that the gradient function of $f(x) = x^n$ has rule nx^{n-1}. Hence the gradient is 0 when $x = 0$. As $n - 1$ is odd for n even, $nx^{n-1} > 0$ for $x > 0$ and $nx^{n-1} < 0$ for $x < 0$. That is, the gradient of the graph of $y = f(x)$ is positive for x positive, negative for x negative and zero when $x = 0$.

Comparing the graphs of $y = x^n$ and $y = x^m$ where n and m are even and $n > m$

- $x^n < x^m$ when $0 < x < 1$ $\quad (x^n - x^m = x^m(x^{n-m} - 1) < 0)$
- $x^n = x^m$ when $x = 0$
- $x^n < x^m$ when $-1 < x < 0$ $\quad (x^n - x^m = x^m(x^{n-m} - 1) < 0$ as $n - m$ is even, m is even and x is negative and greater than $-1)$
- $x^n = x^m$ for $x = 1$ and $x^n = x^m$ for $x = -1$
- $x^n > x^m$ for $x > 1$
- $x^n > x^m$ for $x < -1$

It should be noted that the appearance of graphs is dependent on the scales on the y- and x-axes.

Power functions of even degree are often depicted as shown.



Example 14

The graph of $y = a(x + h)^4 + k$ has a turning point at (2, 2) and passes through the point (0, 4). Find the values of a and h and k.

Solution

$$h = -2 \text{ and } k = 2$$

Therefore as the graph passes through (0, 4):

$$4 = 16a + 2$$

$$\therefore a = \frac{1}{8}$$

Exercise 4D

1 Sketch the graphs of each of the following. State the coordinates of the point of zero gradient and the axes intercepts.

a $f(x) = 2x^3$ **b** $g(x) = -2x^3$ **c** $h(x) = x^5 + 1$
d $f(x) = x^3 - 4$ **e** $f(x) = (x+1)^3 - 8$ **f** $f(x) = 2(x-1)^3 - 2$
g $g(x) = -2(x-1)^3 + 2$ **h** $h(x) = 3(x-2)^3 - 4$ **i** $f(x) = 2(x-1)^3 + 2$
j $h(x) = -2(x-1)^3 - 4$ **k** $f(x) = (x+1)^5 - 32$ **l** $f(x) = 2(x-1)^5 - 2$

2 The graph of $y = a(x+h)^3 + k$ has a point of zero gradient at (0, 4) and passes through the point (1, 1). Find the values of a, h and k.

3 The graph of $y = a(x+h)^4 + k$ has a turning point at (1, 7) and passes through the point (0, 23). Find the values of a, h and k.

4 Find the equation of the image of $y = x^3$ under each of the following transformations:

a a dilation of factor 3 from the x-axis
b a translation with rule $(x, y) \to (x-1, y+1)$
c a reflection in the x-axis followed by a translation with rule

$$(x, y) \to (x+2, y-3)$$

d a dilation of factor 2 from the x-axis followed by a translation with rule

$$(x, y) \to (x-1, y-2)$$

e a dilation of factor 3 from the y-axis.

5 By applying suitable transformations to $y = x^4$, sketch the graph of each of the following:

a $y = 3(x-1)^4 - 2$ **b** $y = -2(x+2)^4$ **c** $y = (x-2)^4 - 6$
d $y = 2(x-3)^4 - 1$ **e** $y = 1 - (x+4)^4$ **f** $y = -3(x-2)^4 - 3$

4.5 The general cubic function

Excel

Not all cubic functions are of the form $f(x) = a(x+h)^3 + k$. In this section the general cubic function is considered. The form of a general cubic function is:

$$f(x) = ax^3 + bx^2 + cx + d$$

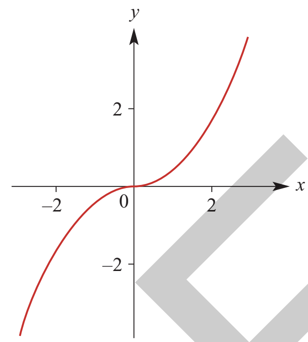
Excel

It is impossible to fully investigate cubic functions without the use of calculus. Cubic functions will be revisited in Chapter 9.

The 'shapes' of cubic graphs vary. Below is a gallery of cubic graphs demonstrating the variety of 'shapes' that are possible.

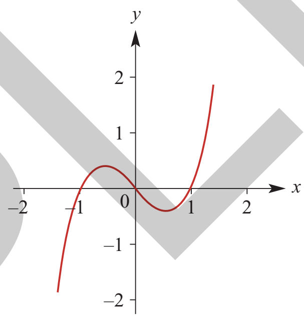
1 $f(x) = x^3 + x$

Note: (0, 0) is not a point of zero gradient. There is one root, 0.



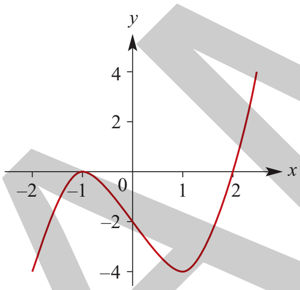
2 $f(x) = x^3 - x$

Note: The turning points do not occur symmetrically between consecutive x-axis intercepts as they do for quadratics. Differential calculus must be used to determine them. There are three roots: 1, 0 and -1.

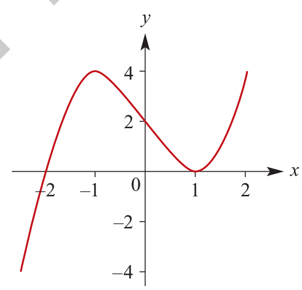


3 $f(x) = x^3 - 3x - 2$

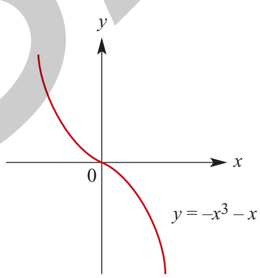
Note: There are two roots: –1 and 2.

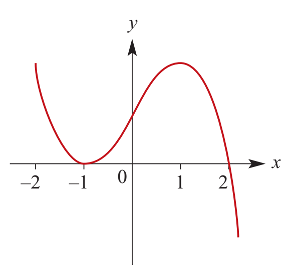


4 $f(x) = x^3 - 3x + 2$



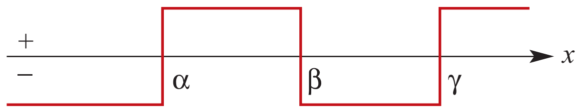
5 The graphs of $f(x) = -x^3 - x$ and $f(x) = -x^3 + 3x + 2$ are shown. They are the reflection in the y-axis of the graphs of **1** and **4** respectively.





Sign diagrams

A sign diagram is a number line diagram that shows when an expression is positive or negative. For a cubic function with rule $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$ when $\gamma > \beta > \alpha$, the sign diagram is as shown:



Example 15

Draw a sign diagram for the expression $x^3 - 4x^2 - 11x + 30$

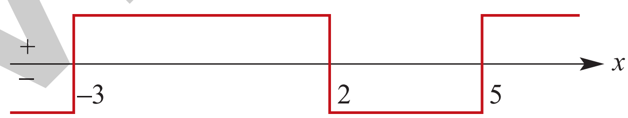
Solution

From Example 6, $f(x) = (x - 2)(x - 5)(x + 3)$

We note: $f(x) > 0$ for $x > 5$
$f(x) < 0$ for $2 < x < 5$
$f(x) > 0$ for $-3 < x < 2$
$f(x) < 0$ for $x < -3$

Also, $f(2) = f(5)$
$= f(-3)$
$= 0$

Hence the sign diagram may be drawn.



Example 16

For the cubic function with rule $f(x) = -x^3 + 19x - 30$:

a Sketch the graph of $y = f(x)$ using a calculator to find the values of the coordinates of the turning points, correct to two decimal places.

b Sketch the graph of $y = \frac{1}{2} f(x - 1)$

Solution

a $f(x) = -x^3 + 19x - 30 = (3 - x)(x - 2)(x + 5)$
$= -(x + 5)(x - 2)(x - 3)$

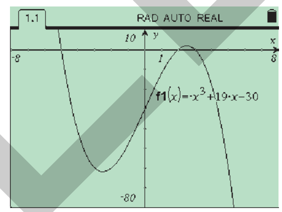
We note: $f(x) < 0$ for $x > 3$
$f(x) > 0$ for $2 < x < 3$
$f(x) < 0$ for $-5 < x < 2$
$f(x) > 0$ for $x < -5$

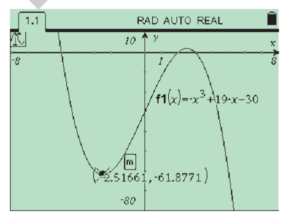
$$\begin{aligned} \text{Also, } f(-5) &= f(2) \\ &= f(3) \\ &= 0 \end{aligned}$$

Hence the sign diagram may be drawn.

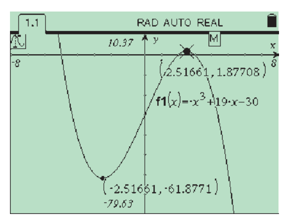
The x-axis intercepts are at $x = -5$, $x = 2$ and $x = 3$ and the y-axis intercept is at $y = -30$.

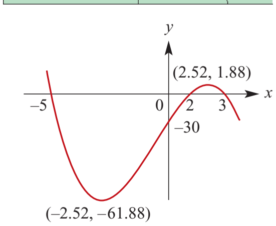
The window has to be adjusted carefully to see the x-axis intercepts and the turning points. The local minimum turning point may be found by selecting **3:Minimum** from the **F5** menu and then entering the lower bound -5 and the upper bound 0 and pressing $\boxed{\text{ENTER}}$. The local minimum is found to occur at the point with coordinates $(-2.52, -61.88)$ with values given correct to two decimal places.



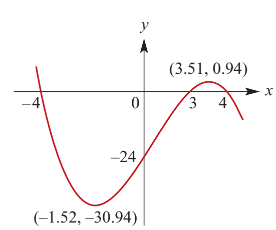


The local maximum turning point may be found by selecting **4:Maximum** from the **F5** menu and then entering the lower bound 2 and the upper bound 3 and pressing $\boxed{\text{ENTER}}$. The local maximum is found to occur at the point with coordinates $(2.52, 1.88)$ with values given correct to two decimal places.





b The rule for the transformation is $(x, y) \rightarrow \left(x + 1, \frac{1}{2}y\right)$ By the transformation $(-5, 0) \rightarrow (-4, 0)$, $(2, 0) \rightarrow (3, 0)$, $(3, 0) \rightarrow (4, 0)$, $(0, -30) \rightarrow (1, -15)$, $(2.52, 1.88) \rightarrow (3.52, 0.94)$ and $(-2.52, -61.88) \rightarrow (-1.52, -30.94)$



Exercise 4E

1 Draw sign diagrams for each of the following expressions:

a $(3 - x)(x - 1)(x - 6)$ **b** $(3 + x)(x - 1)(x + 6)$

c $(x - 5)(x + 1)(2x - 6)$ **d** $(4 - x)(5 - x)(1 - 2x)$

2 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = x^3 - 2x^2 + 1$

b On the same screen plot the graphs of:

i $y = f(x - 2)$

ii $y = f(x + 2)$

iii $y = 3f(x)$

4.6 Polynomials of higher degree

PowerPoint

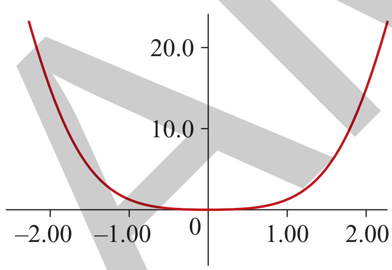
The techniques that have been developed for cubic functions may now be applied to quartic functions and in general to functions of higher degree. It is clear that a polynomial $P(x)$ of degree n has at most n solutions for the equation $P(x) = 0$. It is possible for the graphs of polynomials of even degree to have no x-axis intercepts, for example $P(x) = x^2 + 1$, but graphs of polynomials of odd degree have at least one x-axis intercept.

The general form for a quartic function is:

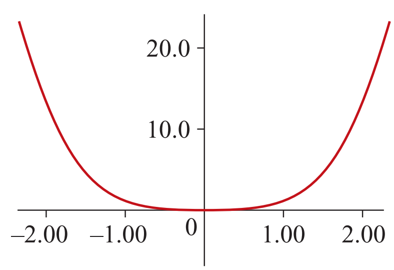
$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

A gallery of quartic functions produced with a graphing package is shown below.

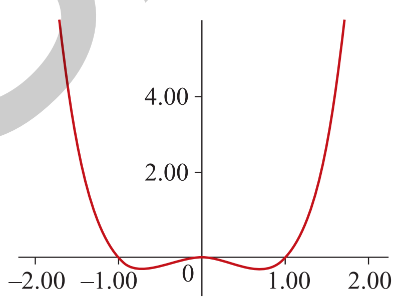
1 $f(x) = x^4$



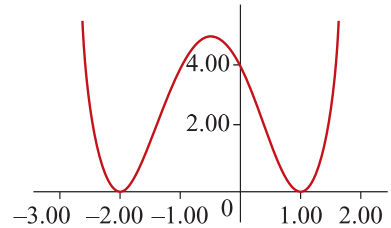
2 $f(x) = x^4 + x^2$



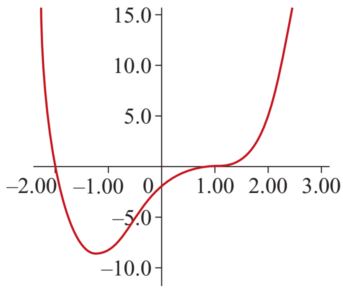
3 $f(x) = x^4 - x^2$



4 $f(x) = (x - 1)^2(x + 2)^2$



5 $f(x) = (x-1)^3(x+2)$



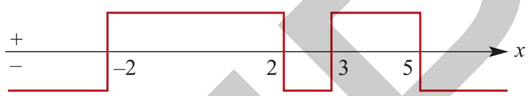
Example 17

Draw a sign diagram for the quartic expressions:

a $(2-x)(x+2)(x-3)(x-5)$ **b** $x^4 + x^2 - 2$

Solution

a



b Let $P(x) = x^4 + x^2 - 2$

$$P(1) = 1 + 1 - 2$$
$$= 0$$

$\therefore x - 1$ is a factor.

$$\begin{array}{r l}
 & x^3 + x^2 + 2x + 2 \\
x - 1 \big) & x^4 + 0x^3 + x^2 + 0x - 2 \\
 & \underline{x^4 - x^3} \\
 & \quad x^3 + x^2 \\
 & \quad \underline{x^3 - x^2} \\
 & \qquad 2x^2 + 0x \\
 & \qquad \underline{2x^2 - 2x} \\
 & \qquad\quad 2x - 2 \\
 & \qquad\quad \underline{2x - 2} \\
 & \qquad\qquad 0
\end{array}$$

$$\begin{aligned}
\therefore P(x) &= (x-1)(x^3 + x^2 + 2x + 2) \\
&= (x-1)[x^2(x+1) + 2(x+1)] \\
&= (x-1)(x+1)(x^2+2)
\end{aligned}$$

Example 18

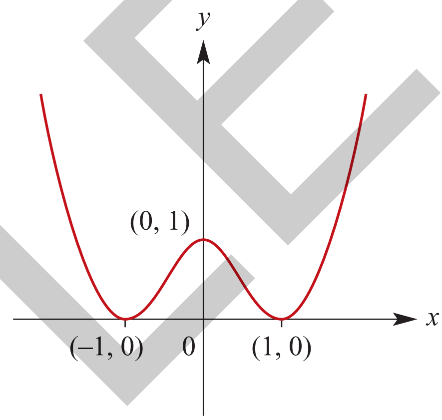
Find the coordinates of the points where the graph of $y = p(x)$, $p(x) = x^4 - 2x^2 + 1$, crosses the x- and y-axes, and hence sketch the graph.

Solution

$$p(1) = 1 - 2 + 1 = 0$$

$\therefore x - 1$ is a factor.

$$\begin{aligned} p(x) &= (x-1)(x^3 + x^2 - x - 1) \\ &= (x-1)[x^2(x+1) - (x+1)] \\ &= (x-1)(x+1)(x^2-1) \\ &= (x-1)^2(x+1)^2 \end{aligned}$$



Alternatively, note that:

$$\begin{aligned} p(x) &= (x^2-1)^2 \\ &= [(x-1)(x+1)]^2 \\ &= (x-1)^2(x+1)^2 \end{aligned}$$

Therefore, the x-axis intercepts are $(1, 0)$ and $(-1, 0)$.

When $x = 0$, $y = 1$.

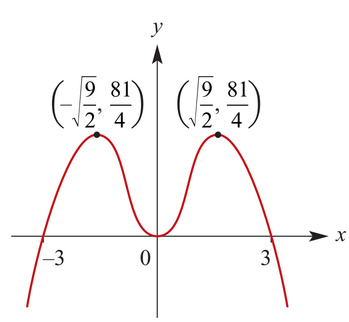
Exercise 4F

1 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = x^4 - 2x^3 + x + 1$

b On the same screen plot the graphs of:

i $y = f(x-2)$ **ii** $y = f(2x)$ **iii** $y = f\left(\frac{x}{2}\right)$

2 The graph of $y = 9x^2 - x^4$ is as shown. Sketch the graph of each of the following by applying suitable transformations.



a $y = 9(x-1)^2 - (x-1)^4$ **b** $y = 18x^2 - 2x^4$

c $y = 18(x+1)^2 - 2(x+1)^4$

d $y = 9x^2 - x^4 - \dfrac{81}{4}$

e $y = 9x^2 - x^4 + 1$

(Do not find the x-axis intercepts for **e**.)

4.7 Determining rules for the graphs of polynomials

It is first worth noting that the graph of a polynomial function of degree n is completely determined by any $n + 1$ points on the curve.

For example, for a cubic function with rule $y = f(x)$, if it is known that $f(a_1) = b_1$, $f(a_2) = b_2$, $f(a_3) = b_3$, $f(a_4) = b_4$, then the rule can be determined.

Finding the rule for a parabola has been discussed in Section 4.3 of this chapter.

Example 19

For the cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$, it is known that the points with coordinates $(-1, -18)$, $(0, -5)$, $(1, -4)$ and $(2, -9)$ lie on the graph of the cubic. Find the values of a, b, c and d.

Solution

Excel

The following equations can be formed.

$$-a + b - c + d = -18 \quad (1)$$
$$d = -5 \quad (2)$$
$$a + b + c + d = -4 \quad (3)$$
$$8a + 4b + 2c + d = -9 \quad (4)$$

Adding (1) and (3) gives $2b + 2d = -22$ and as $d = -5$, $b = -6$.

There are now only two unknowns.

(3) becomes $a + c = 7$ $\quad (3')$

and (4) becomes $8a + 2c = 20$ $\quad (4')$

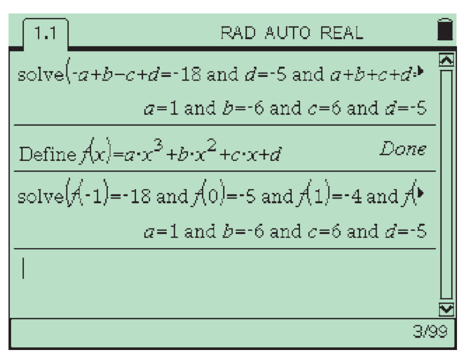
Multiply $(3')$ by 2 and subtract from $(4')$.

$6a = 6$ which gives $a = 1$ and $c = 6$

Using the TI-Nspire

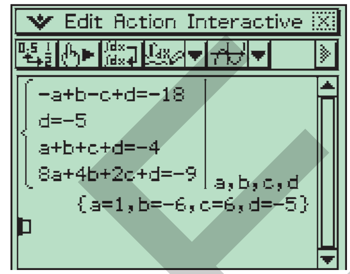
Enter **solve**$(-a + b - c + d = -18$ **and** $d = -5$ **and** $a + b + c + d = -4$ **and** $8a + 4b + 2c + d = -9, \{a, b, c, d\})$.

Alternatively, **Define** $f(x) = ax^3 + bx^2 + cx + d$ and then **solve**$(f(-1) = -18$ **and** $f(0) = -5$ **and** $f(1) = -4$ **and** $f(2) = -9, \{a, b, c, d\})$.



Both methods are shown on the screen to the right.

Using the Casio ClassPad

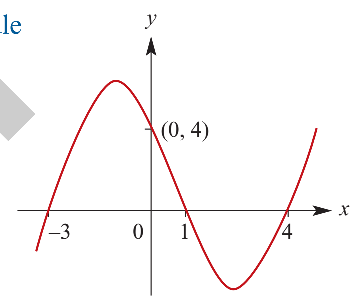


Press Keyboard and select 2D menu (if necessary tap ←). Tap {☐ three times to produce the template to enter four simultaneous equations. Enter the equations and set the variables in the variable box to a,b,c,d then tap EXE.
An alternative method is to define $f(x) = ax^3 + bx^2 + cx + d$ then enter $f(-1) = -18$, $f(0) = 5$, $f(1) = -4$ and $f(2) = -9$ as the simultaneous equations to be solved with variables a,b,c,d as above.

The following examples provide further procedures for finding the rules of cubic functions. It should be noted that a similar facility is available for quartics.

Example 20

The graph shown is that of a cubic function. Find the rule for this cubic function.



Solution

From the graph the function can be seen to be of the form $y = a(x - 4)(x - 1)(x + 3)$.

It remains to find the value of a. The point with coordinates (0, 4) is on the graph. Hence:

$$4 = a(-4)(-1)3$$

$$\therefore a = \frac{1}{3}$$

and $$y = \frac{1}{3}(x - 4)(x - 1)(x + 3)$$

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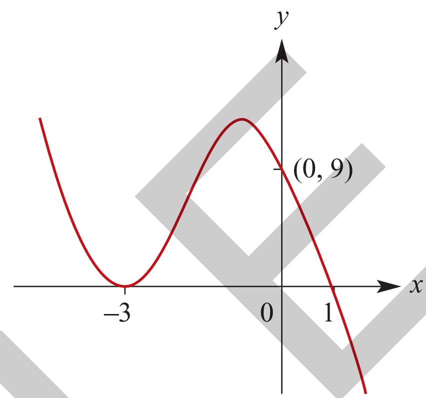
Example 21

The graph shown is that of a cubic function.
Find the rule for this cubic function.



Solution

From the graph it can be seen to be of the form $y = k(x - 1)(x + 3)^2$. It remains to find the value of k. As the value (0, 9) is on the graph:

$$9 = k(-1)(9)$$
$$\therefore k = -1$$
and $$y = -(x - 1)(x + 3)^2$$

Note: Because the graph reveals that there are two factors, $(x + 3)$ and $(x - 1)$, there must be a third linear factor $(ax + b)$, but this implies that there is an intercept $-\frac{b}{a}$.

Thus $ax + b = k_1(x + 3)$ or $k_2(x - 1)$. A consideration of the signs reveals that $(x + 3)$ is a repeated factor.

Example 22

The graph of a cubic function passes through the point with coordinates (0, 1), (1, 4), (2, 17) and (−1, 2). Find the rule for this cubic function.

Solution

The cubic function will have a rule of the form:

$$y = ax^3 + bx^2 + cx + d$$

The values of a, b, c and d have to be determined.

As the point (0, 1) is on the graph, $d = 1$.

By using the points (1, 4), (2, 17) and (−1, 2), three simultaneous equations are produced:

$$4 = a + b + c + 1$$
$$17 = 8a + 4b + 2c + 1$$
$$2 = -a + b - c + 1$$

These become:

$$3 = a + b + c \qquad (1)$$
$$16 = 8a + 4b + 2c \qquad (2)$$
$$1 = -a + b - c \qquad (3)$$

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Add (1) and (3) to find:

$$2b = 4$$

i.e., $b = 2$

Substitute in (1) and (2):

$$1 = a + c \qquad (4)$$

$$8 = 8a + 2c \qquad (5)$$

Multiply (4) by 2 and subtract from (5).

Hence $6 = 6a$

and $a = 1$

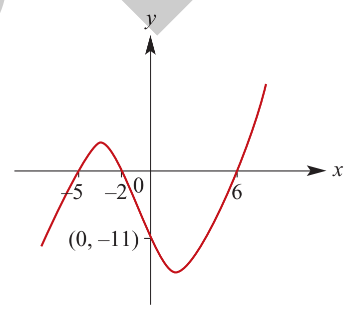
From (4) $c = 0$

Thus $y = x^3 + 2x^2 + 1$

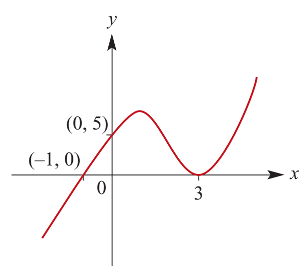
Exercise 4G

Skillsheet TEACHER

1 Determine the rule for the cubic function with graph as shown:



2 Determine the rule for the cubic function with graph as shown:



3 Find the rule for the cubic function that passes through the following points:

a (0, 1), (1, 3), (−1, −1) and (2, 11) **b** (0, 1), (1, 1), (−1, 1) and (2, 7)

c (0, −2), (1, 0), (−1, −6) and (2, 12)

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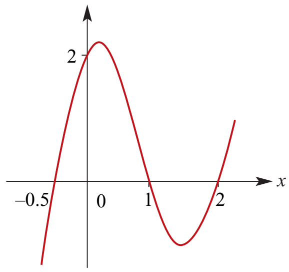
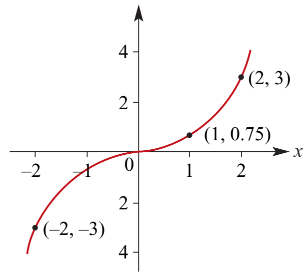
4 Find expressions which define the following cubic curves:

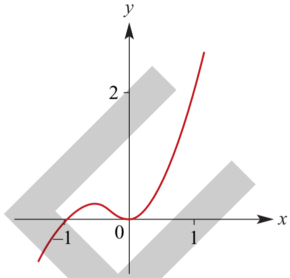
a



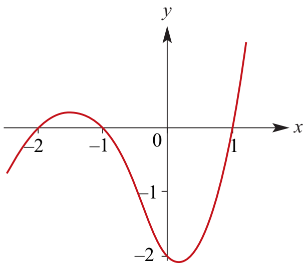
b



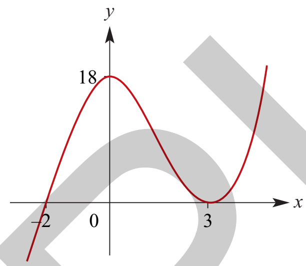
c



d



e



5 Find the equation of the cubic function for which the graph passes through the points with coordinates:

a (0, 135), (1, 156), (2, 115), (3, 0)
b (−2, −203), (0, 13), (1, 25), (2, −11)

6 Find the equation of the quartic function for which the graph passes through the points with coordinates:

a (−1, 43), (0, 40), (2, 70), (6, 1618), (10, 670)
b (−3, 119), (−2, 32), (−1, 9), (0, 8), (1, 11)
c (−3, 6), (−1, 2), (1, 2), (3, 66), (6, 1227)

4.8 Solution of literal equations and systems of equations

Literal equations

Solving literal linear equations and simultaneous equations was undertaken in Section 2.2. In this section other non-linear expressions are considered. They certainly can be solved with a CAS calculator but full setting out is shown here.

Example 23

Solve each of the following literal equations for x:

a $x^2 + kx + k = 0$ **b** $x^3 - 3ax^2 + 2a^2x = 0$ **c** $x(x^2 - a) = 0$

Solution

a Completing the square gives

$$x^2 + kx + \frac{k^2}{4} + k - \frac{k^2}{4} = 0$$

$$\left(x + \frac{k}{2}\right)^2 = \frac{k^2}{4} - k$$

$$x + \frac{k}{2} = \pm\sqrt{\frac{k^2 - 4k}{4}}$$

$$x = \frac{-k \pm \sqrt{k^2 - 4k}}{2}$$

A real solution exists only for $k^2 - 4k \geq 0$; that is for $k \geq 4$ or $k \leq 0$

b $x^3 - 3ax^2 + 2a^2x = 0$

$x(x^2 - 3ax + 2a^2) = 0$

$x(x - a)(x - 2a) = 0$

$x = 0$ or $x = a$ or $x = 2a$

c $x(x^2 - a) = 0$ implies $x = 0$ or $x = \sqrt{a}$ or $x = -\sqrt{a}$

In the following, the property that for suitable values of a, b and an odd natural number n, $b^n = a$ is equivalent to $b = a^{\frac{1}{n}}$ and also $b = \sqrt[n]{a}$. Also $b^{\frac{p}{q}} = a$ is equivalent to $b = a^{\frac{q}{p}}$ for suitable values of a and b and integers q and p.

Care must be taken with the application of these. For example $x^2 = 2$ is equivalent to $x = \pm\sqrt{2}$ and $x^{\frac{2}{3}} = 4$ is equivalent to $x = 8$ or $x = -8$.

Example 24

Solve each of the following equations for x:

a $ax^3 - b = c$ **b** $x^{\frac{3}{5}} = a$

c $x^{\frac{1}{n}} = a$ where n is a natural number and x is a positive real number

d $a(x + b)^3 = c$ **e** $ax^{\frac{1}{5}} = b$ **f** $x^5 - c = d$

Solution

a $ax^3 - b = c$

$ax^3 = b + c$

$$x^3 = \frac{b + c}{a}$$

$$x = \sqrt[3]{\frac{b + c}{a}}$$

or $x = \left(\dfrac{b + c}{a}\right)^{\frac{1}{3}}$

b $x^{\frac{3}{5}} = a$ is equivalent to $x = a^{\frac{5}{3}}$

c $x^{\frac{1}{n}} = a$ is equivalent to $x = a^n$

d $a(x + b)^3 = c$

$$(x + b)^3 = \frac{c}{a}$$

$$x + b = \left(\frac{c}{a}\right)^{\frac{1}{3}}$$

$$x = \left(\frac{c}{a}\right)^{\frac{1}{3}} - b$$

Back to Menu

e $ax^{\frac{1}{5}} = b$

$x^{\frac{1}{5}} = \dfrac{b}{a}$

$x = \left(\dfrac{b}{a}\right)^5$

f $x^5 - c = d$

$x^5 = c + d$

$x = (c + d)^{\frac{1}{5}}$

Simultaneous equations

In this section, methods for finding the coordinates of the points of intersection of different graphs are discussed.

Example 25

Find the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ with the straight line with equation $y = x + 4$

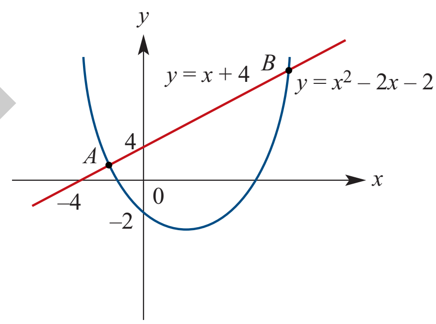
Solution

Consider $x + 4 = x^2 - 2x - 2$

Then $0 = x^2 - 3x - 6$

$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times -6 \times 1}}{2}$$

$$= \frac{3 \pm \sqrt{33}}{2}$$



The points of intersection have coordinates

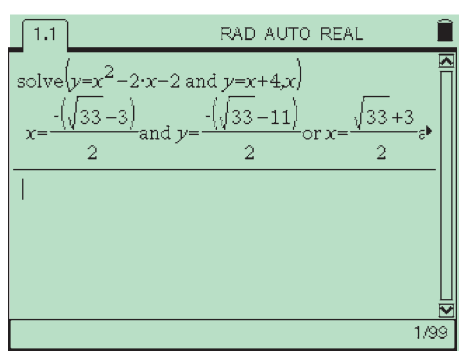
$$A\left(\frac{3 - \sqrt{33}}{2}, \frac{11 - \sqrt{33}}{2}\right) \text{ and } B\left(\frac{3 + \sqrt{33}}{2}, \frac{11 + \sqrt{33}}{2}\right)$$

Using the TI-Nspire

Use **solve( )** from the **Algebra** menu (menu 3 1) as shown.

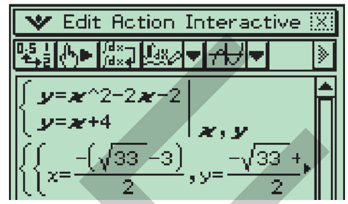
The word ‘and’ can be typed directly or found in the catalog (menu 3 1).

Use the **up arrow** (▲) to move up to the answer and use the **right arrow** (▶) to display the remaining part of the answer.



Using the Casio ClassPad

In the simultaneous equation entry screen, enter the equations $y = x^2 - 2x - 2$ and $y = x + 4$ and set the variables as x, y. Note the ▸ at the end of the answer line indicates that you must scroll to the right to see all the solutions.



Example 26

Find the coordinates of the points of intersection of the circle with equation $(x - 4)^2 + y^2 = 16$ and the line with equation $x - y = 0$

Solution

Rearrange $x - y = 0$ to make y the subject.
Substitute $y = x$ into the equation of the circle.

i.e., $(x - 4)^2 + x^2 = 16$

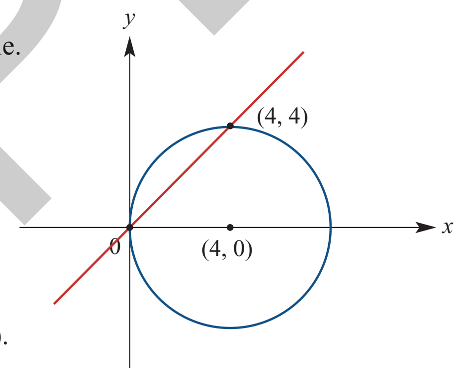
$\therefore\ x^2 - 8x + 16 + x^2 = 16$

i.e., $2x^2 - 8x = 0$

$2x(x - 4) = 0$

$x = 0$ or $x = 4$

The points of intersection are (0, 0) and (4, 4).



Example 27

Find the point of contact of the line with equation $\frac{1}{9}x + y = \frac{2}{3}$ and the curve with equation $xy = 1$

Solution

Rewrite the equations as $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = \frac{1}{x}$

Consider $-\frac{1}{9}x + \frac{2}{3} = \frac{1}{x}$

$\therefore\ -x + 6 = \frac{9}{x}$

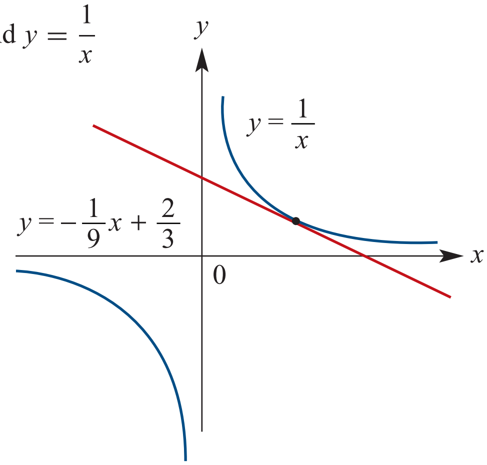
and $-x^2 + 6x = 9$

Therefore $x^2 - 6x + 9 = 0$

and $(x - 3)^2 = 0$

i.e. $x = 3$

The point of intersection is $\left(3, \frac{1}{3}\right)$.

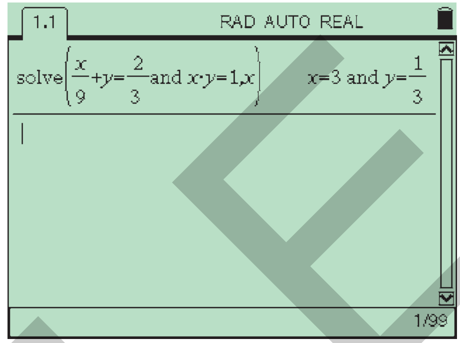


Back to Menu

Using the TI-Nspire

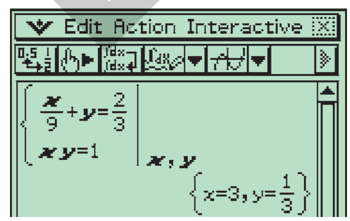
Use **solve( )** from the **Algebra** menu (menu 3 1) as shown.

Note that the multiplication sign is required between x and y.



Using the Casio ClassPad

In the simultaneous equation entry screen, enter the equations $\frac{x}{9} + y = \frac{2}{3}$ **and** $xy = 1$ and set the variables as x, y. Note that for this example, since the fraction entry is on the same screen as the simultaneous equation in the keyboard menu, the fractions can be entered using the fraction entry key.



Exercise 4H

1 Solve each of the following literal equations for x:

a $kx^2 + x + k = 0$ **b** $x^3 - 7ax^2 + 12a^2x = 0$

c $x(x^3 - a) = 0$ **d** $x^2 - kx + k = 0$

e $x^3 - ax = 0$ **f** $x^4 - a^4 = 0$

g $(x - a)^5(x - b) = 0$ **h** $(a - x)^4(a - x^3)(x^2 - a) = 0$

2 Solve each of the following equations for x:

a $ax^3 - b = c$ **b** $x^{\frac{3}{7}} = a$

c $x^{\frac{1}{n}} + c = a$ where n is a natural number and x is a positive real number

d $a\,(x + b)^3 = c$ **e** $ax^{\frac{1}{3}} = b$ **f** $x^3 - c = d$

3 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = x$

b $y - 2x^2 = 0$, $y - x = 0$

c $y = x^2 - x$, $y = 2x + 1$

4 Find the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$
$x + y = 16$

b $x^2 + y^2 = 125$
$x + y = 15$

c $x^2 + y^2 = 185$
$x - y = 3$

d $x^2 + y^2 = 97$
$x + y = 13$

e $x^2 + y^2 = 106$
$x - y = 4$

5 Find the coordinates of the points of intersection for each of the following:

a $x + y = 28$
$xy = 187$

b $x + y = 51$
$xy = 518$

c $x - y = 5$
$xy = 126$

6 Find the coordinates of the points of intersection of the straight line with equation $y = 2x$ and the circle with equation $(x - 5)^2 + y^2 = 25$

7 Find the coordinates of the points of intersection of the curves with equation $y = \dfrac{1}{x-2} + 3$ and $y = x$

8 Find the coordinates of the points of intersection of the line with equation $\dfrac{y}{4} - \dfrac{x}{5} = 1$ and the circle with equation $x^2 + 4x + y^2 = 12$

9 Find the coordinates of the points of intersection of the curve with equation $y = \dfrac{1}{x+2} - 3$ and the line with equation $y = -x$

10 Find the coordinates of the point where the line $4y = 9x + 4$ touches the parabola with equation $y^2 = 9x$

11 Find the coordinates of the point where the line with equation $y = 2x + 3\sqrt{5}$ touches the circle $x^2 + y^2 = 9$

12 Find the coordinates of the point where the straight line with equation $y = \dfrac{1}{4}x + 1$ touches the curve with equation $y = -\dfrac{1}{x}$

13 Find the coordinates of the points of intersection of the curve with equation $y = \dfrac{2}{x-2}$ and the line $y = x - 1$

14 Solve the simultaneous equations:

a $5x - 4y = 7$ and $xy = 6$

b $2x + 3y = 37$ and $xy = 45$

c $5x - 3y = 18$ and $xy = 24$

15 What is the condition for $x^2 + ax + b$ to be divisible by $x + c$?

16 Solve the simultaneous equations $y = x + 2$ and $y = \dfrac{160}{x}$

17 **a** Solve the simultaneous equations $y = mx$ and $y = \dfrac{1}{x} + 5$ for x in terms of m.

b Find the value of m for which the graphs of $y = mx$ and $y = \dfrac{1}{x} + 5$ touch and give the coordinates of this point.

c For which values of m do the graphs not meet?

MC TEST D&D TEST

Chapter summary

- A function $P(x) = a_0 + a_1x + \cdots + a_nx^n$, where $a_0, a_1, \ldots, a_n$ are constants, is called a **polynomial** in x. The numbers $a_0, a_1, \ldots a_n$ are the **coefficients**. Assuming $a_n \neq 0$, the term a_nx^n is the **leading term**. The integer n is the **degree** of the polynomial.
  - Degree one polynomials are called **linear** functions.
  - Degree two polynomials are called **quadratic** functions.
  - Degree three polynomials are called **cubic** functions.
  - Degree four polynomials are called **quartic** functions.
- **The remainder theorem**: If the polynomial $P(x)$ is divided by $ax + b$, the remainder is $P\left(-\frac{b}{a}\right)$
- **The factor theorem**: $(ax + b)$ is a factor of $P(x)$ if and only if $P\left(-\frac{b}{a}\right) = 0$
- To sketch the graph of a quadratic function $f(x) = ax^2 + bx + c$ (called a parabola), use the following:
  - If $a > 0$, the function has a minimum value.
  - If $a < 0$, the function has a maximum value.
  - The value of c gives the y-axis intercept.
  - The equation of the axis of symmetry is $x = -\frac{b}{2a}$
  - The x-axis intercepts are determined by solving the equation $ax^2 + bx + c = 0$
- A quadratic equation may be solved by:
  - factorising
  - completing the square
  - using the general quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Using the discriminant of the quadratic function $f(x) = ax^2 + bx + c$:
  - If $b^2 - 4ac > 0$, the graph of the function has two x-axis intercepts.
  - If $b^2 - 4ac = 0$, the graph of the function touches the x-axis.
  - If $b^2 - 4ac < 0$, the graph of the function does not intercept the x-axis.
- By completing the square, any quadratic function can be written in the form $y = A(x + b)^2 + B$
  From this it can be seen that the graph of any quadratic function may be obtained by a composition of transformations applied to the graph of $y = x^2$.

Multiple-choice questions

1 The equation $5x^2 - 10x - 2$ in turning point form $a(x + h)^2 + k$ by completing the square is:

A $(5x + 1)^2 + 5$ **B** $(5x - 1)^2 - 5$ **C** $5(x - 1)^2 - 5$
D $5(x + 1)^2 - 2$ **E** $5(x - 1)^2 - 7$

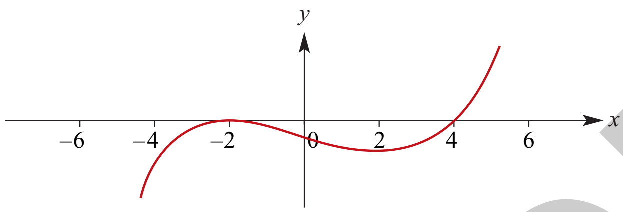
2 The value(s) of m that will give the equation $mx^2 + 6x - 3 = 0$ two real roots is/are:

A $m = -3$ **B** $m = 3$ **C** $m = 0$ **D** $m > -3$ **E** $m < -3$

3 $x^3 + 27$ factorised over R is equal to:

A $(x + 3)^3$ **B** $(x - 3)^3$ **C** $(x + 3)(x^2 - 6x + 9)$
D $(x - 3)(x^2 + 3x + 9)$ **E** $(x + 3)(x^2 - 3x + 9)$

4 The equation of the graph shown is:



A $y = x(x - 2)(x + 4)$ **B** $y = x(x + 2)(x - 4)$ **C** $y = (x + 2)^2(x - 4)$
D $y = (x + 2)(x - 4)^2$ **E** $y = (x + 2)^2(x - 4)^2$

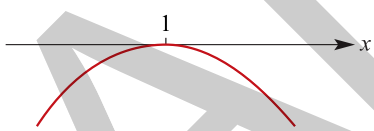
5 $x - 1$ is a factor of $x^3 + 3x^2 - 2ax + 1$. The value of a is:

A 2 **B** 5 **C** $\frac{2}{5}$ **D** $-\frac{2}{5}$ **E** $\frac{5}{2}$

6 $6x^2 - 8xy - 8y^2$ is equal to:

A $(3x + 2y)(2x - 4y)$ **B** $(3x - 2y)(6x + 4y)$ **C** $(6x - 4y)(x + 2y)$
D $(3x - 2y)(2x + 4y)$ **E** $(6x + y)(x - 8y)$

7 A part of the graph of the third-degree polynomial function $f(x)$, near the point $(1, 0)$, is shown below.



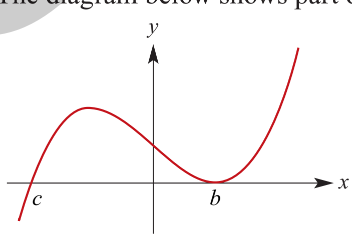
Which of the following could be the rule for $f(x)$?

A $f(x) = x^2(x - 1)$ **B** $f(x) = (x - 1)^3$ **C** $f(x) = -x(x - 1)^2$
D $f(x) = x(x - 1)^2$ **E** $f(x) = -x(x + 1)^2$

8 The coordinates of the turning point of the graph of the function $p(x) = 3((x - 2)^2 + 4)$ are:

A $(-2, 12)$ **B** $(-2, 4)$ **C** $(2, -12)$ **D** $(2, 4)$ **E** $(2, 12)$

9 The diagram below shows part of the graph of a polynomial function.



A possible equation for the rule of the function is:

A $y = (x + c)(x - b)^2$ **B** $y = (x - b)(x - c)^2$
C $y = (x - c)(b - x)^2$ **D** $y = -(x - c)(b - x)^2$
E $y = (x + b)^2(x - c)$

10 The number of roots to the equation $(x^2 + a)(x - b)(x + c) = 0$, where a, b and $c \in R^+$, is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

Short-answer questions (technology-free)

1 Sketch the graphs of each of the following quadratic functions. Clearly indicate coordinates of the vertex and axes intercepts.

a $h(x) = 3(x - 1)^2 + 2$ **b** $h(x) = (x - 1)^2 - 9$ **c** $f(x) = x^2 - x + 6$
d $f(x) = x^2 - x - 6$ **e** $f(x) = 2x^2 - x + 5$ **f** $h(x) = 2x^2 - x - 1$

2 The points with coordinates (1, 1) and (2, 5) lie on a parabola with equation of the form $y = ax^2 + b$. Find the values of a and b.

3 Solve the equation $3x^2 - 2x - 10 = 0$ by using the quadratic formula.

4 Sketch the graphs of each of the following. State the coordinates of the point of zero gradient and the axes intercepts:

a $f(x) = 2(x - 1)^3 - 16$ **b** $g(x) = -(x + 1)^3 + 8$ **c** $h(x) = -(x + 2)^3 - 1$
d $f(x) = (x + 3)^3 - 1$ **e** $f(x) = 1 - (2x - 1)^3$

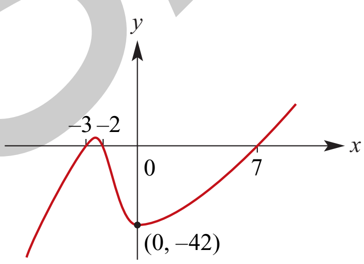
5 Draw a sign diagram for each of the following:

a $y = (x + 2)(2 - x)(x + 1)$ **b** $y = (x - 3)(x + 1)(x - 1)$
c $y = x^3 + 7x^2 + 14x + 8$ **d** $y = 3x^3 + 10x^2 + x - 6$

6 Without actually dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 + 3x^2 - 4x + 2$, $x + 1$ **b** $x^3 - 3x^2 - x + 6$, $x - 2$
c $2x^3 + 3x^2 - 3x - 2$, $x + 2$

7 Determine the rule for the cubic function shown in the graphs:



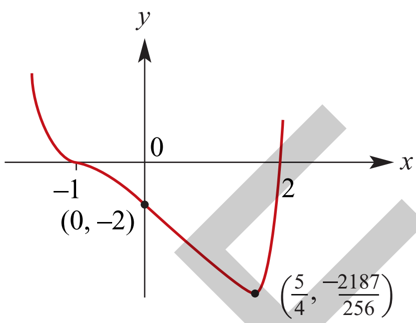
8 The graph of $f(x) = (x + 1)^3(x - 2)$ is shown. Sketch the graph of:

a $y = f(x - 1)$

b $y = f(x + 1)$

c $y = f(2x)$

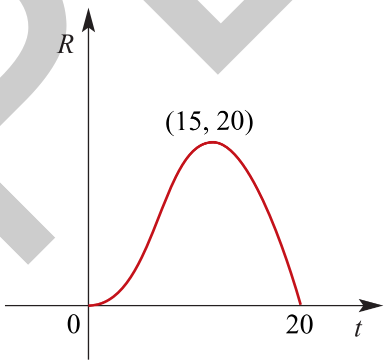
d $y = f(x) + 2$



9 Find the rule for the cubic function, the graph of which passes through the points (1, 1), (2, 4), (3, 9) and (0, 6).

Extended-response questions

1 The rate of flow of water, R mL/min, into a vessel is described by the quartic expression $R = kt^3(20 - t)$, $0 \leq t \leq 20$, where t minutes is the time elapsed from the beginning of the flow. The graph is shown.



a Find the value of k.

b Find the rate of flow when $t = 10$.

c The flow is adjusted so that a new expression for flow is:

$$R_{new} = 2kt^3(20 - t),\ 0 \leq t \leq 20$$

i Sketch the graph of R_{new} against t for $0 \leq t \leq 20$.

ii Find the rate of flow when $t = 10$.

d Water is allowed to run from the vessel and it is found that the rate of flow from the vessel is given by

$$R_{out} = -k(t - 20)^3(40 - t) \text{ for } 20 \leq t \leq 40$$

i Sketch the graph of R_{out} against t for $20 \leq t \leq 40$.

ii Find the rate of flow when $t = 30$.

Hint: Note that the graph of R_{new} against t is given by a dilation of factor 2 from the x-axis. The graph of R_{out} against t is given by a translation with rule $(t, R) \to (t + 20, R)$, followed by a reflection in the t-axis.

CAS CALCULATOR

2 A large gas container is being deflated. The volume V (in m^3) at time t hours is given by:

$$V = 4(9 - t)^3 \text{ where } 0 \leq t \leq 9$$

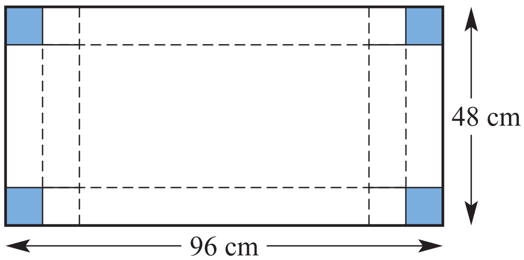
a Find the volume when:

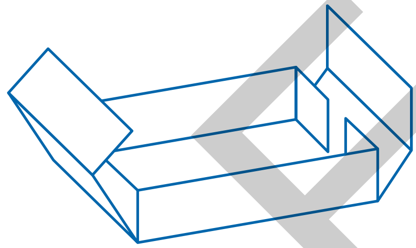
i $t = 0$ **ii** $t = 9$

b Sketch the graph of V against t for $0 \leq t \leq 9$.

c At what time is the volume 512 m^3?

3 A reinforced box is made by cutting congruent squares of side length x cm from the four corners of a rectangular piece of cardboard that measures 48 cm by 96 cm. The flaps are folded up.



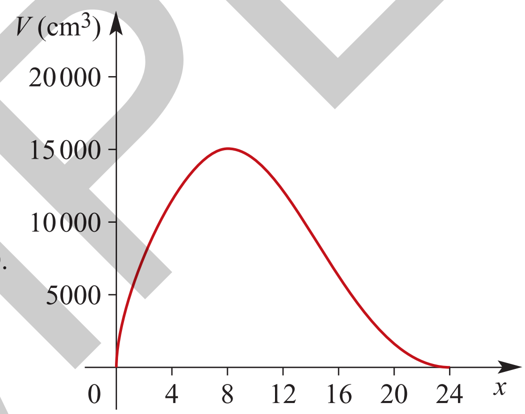


a Find an expression for V, the volume of the box formed.

b A graph of V against x is as shown.

i What is the domain of the function V?

ii From the graph, find the maximum volume of the box and the value of x for which this occurs (approximate values are required).



c Find the volume of the box when $x = 10$.

d It is decided that $0 \leq x \leq 5$.
Find the maximum volume possible.

e If $5 \leq x \leq 15$, what is the minimum volume of the box?

4 A hemispherical bowl of radius 6 cm contains water. The volume of water in the hemispherical bowl where the depth of the water is x cm and is given by:

$$V = \frac{1}{3}\pi x^2(18 - x)\ \text{cm}^3$$

a Find the volume of water when:

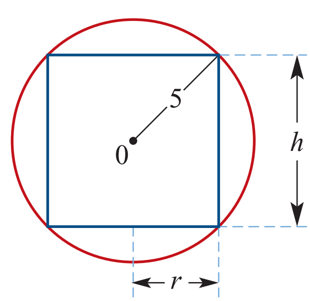
i $x = 2$ **ii** $x = 3$ **iii** $x = 4$

b Find the volume when the hemispherical bowl is full.

c Sketch the graph of V against x.

d Find the depth of water when the volume is equal to $\frac{325\pi}{3}$ cm^3.

5 A metal worker is required to cut a circular cylinder from a solid sphere of radius 5 cm. A cross-section of the sphere and the cylinder is shown in the diagram.



a Express r in terms of h, where r cm is the radius of the cylinder and h cm is the height of the cylinder.
Hence show that the volume, V cm^3, of the cylinder is given by $V = \frac{1}{4}\pi h(100 - h^2)$.

b Sketch the graph of V against h for $0 < h < 10$.
The coordinates of the maximum point are $\approx (5.77, 302.3)$.
c Find the volume of the cylinder if $h = 6$.
d Find the height and radius of the cylinder if the volume of the cylinder is 48π cm^3.

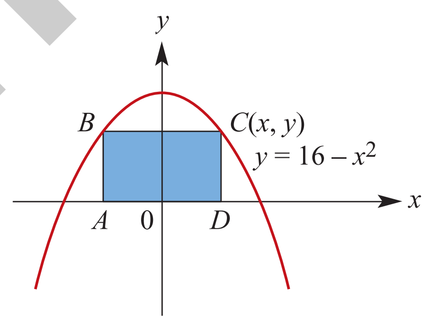
CAS CALCULATOR

6 An open tank is to be made from a sheet of metal 84 cm by 40 cm by cutting congruent squares of side length x cm from each of the corners.

a Find the volume V cm^3 of the box in terms of x.
b State the maximal domain for V when it is considered as a function of x.
c Plot the graph of V against x using a calculator.
d Find the volume of the tank when:
i $x = 2$ **ii** $x = 6$ **iii** $x = 8$ **iv** $x = 10$
e Find the value(s) of x, correct to two decimal places, for which the capacity of the tank is 10 litres.
f Find, correct to two decimal places, the maximum capacity of the tank in cubic centimetres.

CAS CALCULATOR

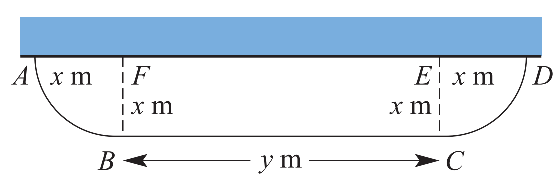
7 The rectangle is defined by vertices B and C on the curve with equation $y = 16 - x^2$ and vertices A and D on the x-axis.



a **i** Find the area, A, of the rectangle in terms of x.
ii State the implied domain for the function defined by the rule given in **i**.
b Find:
i the value of A when $x = 3$
ii the value, correct to two decimal places, of x when $A = 25$
c A cuboid has volume V given by the rule $V = xA$
i Find V in terms of x.
ii Find the value, correct to two decimal places, of x such that $V = 100$.

CAS CALCULATOR

8 The plan of a garden adjoining a wall is shown. The rectangle $BCEF$ is of length y m and width x m. The borders of the two end sections are quarter circles of radius x m and centres at E and F. A fence is erected along the curves AB and CD and the straight line CB.



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a Find the area A of the garden in terms of x and y.

b If the length of the fence is 100 m, find:

  i y in terms of x

  ii A in terms of x

  iii the maximal domain of the function for which the rule has been obtained in **ii**

c Find, correct to two decimal places, the value(s) of x if the area of the garden is to be 1000 m^2.

d It is decided to build the garden up to a height of $\frac{x}{50}$ metres. If the length of the fence is 100 m, find correct to two decimal places:

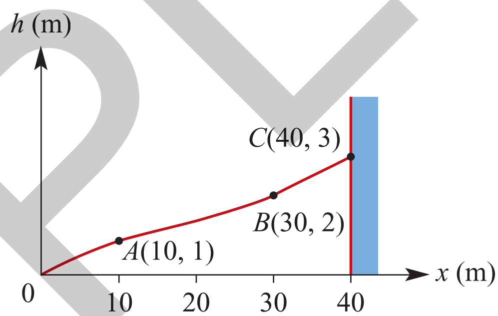
  i the volume V m^3 of soil needed in terms of x

  ii the volume V m^3 of soil needed for a garden of area 1000 m^2

  iii the value(s) of x for which 500 m^3 of soil is required

9 CAS CALCULATOR

A mound of earth is piled up against a wall. The cross-section is as shown. The coordinates of several points on the surface are given.



a Find the equation for the cubic function for which the graph passes through the points O, A, B and C.

b For what value of x is the height of the mound 1.5 metres?

c The coefficient of x^3 for the function is 'small'. Consider the quadratic formed when the x^3 term is deleted. Compare the graph of the resulting quadratic function with the graph of the cubic function.

d The mound moves and the curve describing the cross-section now passes through the points $O(0, 0)$, $A(10, 0.3)$, $B(30, 2.7)$ and $D(40, 2.8)$.

Find the equation of the cubic function for which the graph passes through these points.

e Let $y = f(x)$ be the function obtained in **a**.

  i Sketch the graph of the hybrid function:

$$g(x) = \begin{cases} f(x) & 0 \le x \le 40 \\ f(80 - x) & 40 < x \le 80 \end{cases}$$

  ii Comment on the appearance of the graph of $y = g(x)$