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CHAPTER

11

Differentiation of transcendental functions

Objectives

- To differentiate **exponential functions**.
- To differentiate **natural logarithmic functions**.
- To find the derivatives of **circular functions**.
- To apply the differentiation of transcendental functions to **solving problems**.

11.1 Differentiation of e^x

In this section we investigate the derivative of functions of the form $f(x) = a^x$. It is found that the number e, Euler's number, has the special property that $f'(x) = f(x)$ where $f(x) = e^x$.

Let $f: R \to R$, $f(x) = 2^x$

To find the derivative of $f(x)$ we recall that:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \\ &= 2^x \lim_{h \to 0} \frac{2^h - 1}{h} \\ &= 2^x f'(0) \end{aligned}$$

A calculator can be used to see $f'(0) = \lim_{h \to 0} \dfrac{2^{0+h} - 2^0}{h} \approx 0.693$. This is done by entering $Y_1 = (2\hat{\ }X - 1)/X$ and considering values of the function for $X \to 0$.

Thus $f'(x) \approx 0.693 f'(0)$

Let $g: R \to R$, $g(x) = 3^x$. Then as above for f it may be shown that $g'(x) = 3^x g'(0)$

It can be shown that $g'(0) \approx 1.0986$ and hence $g'(x) = 3^x g'(0) \approx 1.0986 \times 3^x$. The question arises of the existence of a number, b, between 2 and 3 such that if $h(x) = b^x$, $h'(x) = b^x$, i.e. $h'(0) = 1$

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By using a calculator the limit as $h \to 0$ of $\dfrac{b^h - 1}{h}$ for various values of b between 2 and 3 can be investigated.

This investigation is continued through the spreadsheet shown below. Start by taking values for b between 2.71 and 2.72 (column A) and finding $f'(0)$ for these values (column B). From these results it may be seen that the required value of b lies between 2.718 and 2.719; in columns D and E the investigation is continued in a similar way with values between 2.718 and 2.719. From this the required value of b is seen to lie between 2.7182 and 2.7183.

	A	B	C	D	E
1	0.00000001	is the value of h			
2		$f'(0)$			$f'(0)$
3	2.710	0.996948635		2.7180	0.99989632129649
4	2.711	0.997317584		2.7181	0.99993311408753
5	2.712	0.997686378		2.7182	0.99996990687856
6	2.713	0.998055039		2.7183	1.00000669966950
7	2.714	0.998423566		2.7184	1.00004347025610
8	2.715	0.998791960		2.7185	1.00008026304720
9	2.716	0.999160221		2.7186	1.00011705583820
10	2.717	0.999528327		2.7187	1.00019061921580
11	2.718	0.999896321		2.7188	1.00022738980240
12	2.719	1.000264160		2.7189	1.00022738980240
13	2.720	1.000631888		2.7190	1.00026416038900

The value of b is in fact e, Euler's number, which was introduced in previous work. Our results can be recorded:

> for $f(x) = e^x$, $f'(x) = e^x$

Consider $y = e^{kx}$ where $k \in R$. The chain rule is used to find the derivative.

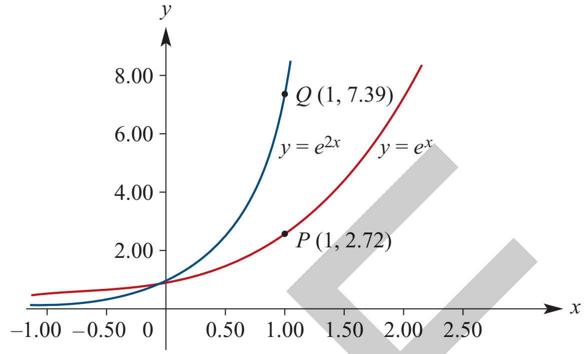
Let $u = kx$. Then $y = e^u$

The chain rule yields
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot k \\ &= ke^{kx}\end{aligned}$$

> For $f(x) = e^{kx}$, $k \in R$, $f'(x) = ke^{kx}$

The graph illustrates the effect of multiplying x by 2.

The gradient of $y = e^x$ at $P(1, e)$ is e and the gradient of $y = e^{2x}$ at $Q(1, e^2)$ is $2e^2$.



Example 1

Find the derivative of each of the following with respect to x:

a e^{3x} **b** e^{-2x} **c** e^{2x+1} **d** e^{x^2} **e** $\dfrac{1}{e^{2x}} + e^{3x}$

Solution

a Let $y = e^{3x}$ $\dfrac{dy}{dx} = 3e^{3x}$

b Let $y = e^{-2x}$ $\dfrac{dy}{dx} = -2e^{-2x}$

c Let $y = e^{2x+1}$

Then $y = e^{2x} \cdot e$

$= e \cdot e^{2x}$ (index laws)

$\therefore \dfrac{dy}{dx} = 2e \cdot e^{2x}$

$= 2e^{2x+1}$

d Let $y = e^{x^2}$

Let $u = x^2$

Then $y = e^u$ and the chain rule yields:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 2x$$

$$= 2xe^{x^2}$$

e Let $y = e^{-2x} + e^{3x}$

$\left(\textbf{Note: } \dfrac{1}{e^{2x}} = e^{-2x}\right)$

Then $\dfrac{dy}{dx} = -2e^{-2x} + 3e^{3x}$

Example 2

Differentiate each of the following with respect to x:

a $e^x(2x^2 + 1)$ **b** $\dfrac{e^x}{e^{2x} + 1}$ **c** $e^x\sqrt{x - 1}$

Solution

a We use the product rule:

Let $y = e^x(2x^2 + 1)$

Then $\dfrac{dy}{dx} = e^x(2x^2 + 1) + 4xe^x$

$= e^x(2x^2 + 4x + 1)$

b We use the quotient rule:

Let $y = \dfrac{e^x}{e^{2x} + 1}$

Then $\dfrac{dy}{dx} = \dfrac{(e^{2x} + 1)e^x - 2e^{2x} \cdot e^x}{(e^{2x} + 1)^2}$

$= \dfrac{e^{3x} + e^x - 2e^{3x}}{(e^{2x} + 1)^2}$

$= \dfrac{e^x - e^{3x}}{(e^{2x} + 1)^2}$

c The product rule and chain rule are used:

Let $y = e^x\sqrt{x-1}$

$$\text{Then } \frac{dy}{dx} = e^x\sqrt{x-1} + \frac{1}{2}e^x(x-1)^{-\frac{1}{2}}$$

$$= e^x\sqrt{x-1} + \frac{e^x}{2(x-1)^{\frac{1}{2}}}$$

$$= \frac{2e^x(x-1) + e^x}{2(x-1)^{\frac{1}{2}}}$$

$$= \frac{2xe^x - e^x}{2\sqrt{x-1}}$$

In general, the chain rule gives:

For $h(x) = e^{f(x)}$, $h'(x) = f'(x)e^{f(x)}$

Example 3

Find the gradient of the curve $y = e^{2x} + 4$ at the point:

a $(0, 5)$ **b** $(1, e^2 + 4)$

Solution

a $\frac{dy}{dx} = 2e^{2x}$

When $x = 0$, $\frac{dy}{dx} = 2$

$\therefore$ the gradient of $y = e^{2x} + 4$ is 2 at $(0, 5)$

b When $x = 1$, $\frac{dy}{dx} = 2e^2$

$\therefore$ the gradient of $y = e^{2x} + 4$ is $2e^2$ (14.78 to two decimal places)

Example 4

Find the derivative, with respect to x, of:

a $e^{f(x)}$ **b** $f(e^x)$

and evaluate the derivative of each when $x = 2$ if $f(2) = 0$ and $f'(2) = 4$

Solution

a Let $y = e^{f(x)}$ and $u = f(x)$

Therefore $y = e^u$

$$\text{Then } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \text{ (chain rule)}$$

$$= e^u f'(x)$$

$$= e^{f(x)} f'(x)$$

When $x = 2$, $\frac{dy}{dx} = e^0 \times 4 = 4$

b Let $y = f(e^x)$ and $u = e^x$

Therefore $y = f(u)$

$$\text{Then } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \text{ (chain rule)}$$

$$= f'(u)\cdot e^x$$

$$= f'(e^x)\cdot e^x$$

$$= f'(e^2)\cdot e^2$$

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Exercise 11A

1 Find the derivative of each of the following with respect to x:

a e^{5x} **b** $7e^{-3x}$ **c** $3e^{-4x} + e^x - x^2$

d $\dfrac{e^{2x} - e^x + 1}{e^x}$ **e** e^{x^2+3x+1} **f** e^{3x^2-x}

g $e^{2x} + e^4 + e^{-2x}$

2 Find $f'(x)$ for each of the following:

a $f(x) = e^x(x^2 + 1)$ **b** $f(x) = e^{2x}(x^3 + 3x + 1)$

c $f(x) = e^{4x+1}(x + 1)^2$ **d** $f(x) = e^{-4x}\sqrt{(x + 1)}, x \geq -1$

3 Find $f'(x)$ for each of the following:

a $f(x) = \dfrac{e^x}{e^{3x} + 3}$ **b** $f(x) = \dfrac{e^x + 1}{e^x - 1}$ **c** $f(x) = \dfrac{e^{2x} + 2}{e^{2x} - 2}$

4 Differentiate each of the following with respect to x:

a $x^4 e^{-2x}$ **b** e^{2x+3} **c** $(e^{2x} + x)^{\frac{3}{2}}$

d $\dfrac{1}{x}e^x$ **e** $e^{\frac{1}{2}x^2}$ **f** $(x^2 + 2x + 2)e^{-x}$

5 Find each of the following:

a $\dfrac{d(e^x f(x))}{dx}$ **b** the derivative of $\dfrac{e^x}{f(x)}$ with respect to x

c $\dfrac{d\left(e^{f(x)}\right)}{dx}$ **d** $\dfrac{d(e^x[f(x)]^2)}{dx}$

11.2 Differentiation of the natural logarithm function

For the function with rule $f(x) = e^{kx}$, $f'(x) = ke^{kx}$

This will be used to find the derivative of the function $g: R^+ \to R, g(x) = \log_e x$

Let $y = \log_e kx$

Then $e^y = kx$

and with x the subject $x = \dfrac{1}{k}e^y$

From our observation above $\dfrac{dx}{dy} = \dfrac{1}{k}e^y$

But $e^y = kx$. Thus $\dfrac{dx}{dy} = \dfrac{kx}{k} = x$

$$\therefore \frac{dy}{dx} = \frac{1}{x}$$

For $f: R^+ \to R, f(x) = \log_e(kx)$ $f': R^+ \to R, f'(x) = \dfrac{1}{x}$

Example 5

Find the derivative of each of the following with respect to x:

a $\log_e(5x) \quad x > 0$ **b** $\log_e(5x+3) \quad x > \frac{-3}{5}$

Solution

a Let $y = \log_e(5x), x > 0$

Thus $\frac{dy}{dx} = \frac{1}{x}$

or let $u = 5x$. Then $y = \log_e u$

The chain rule gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{u} \times 5 \\ &= \frac{5}{u} \\ &= \frac{1}{x}\end{aligned}$$

b Let $y = \log_e(5x+3), x > \frac{-3}{5}$

Let $u = 5x + 3$. Then $y = \log_e u$

The chain rule gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{u} \times 5 \\ &= \frac{5}{u} \\ &= \frac{5}{5x+3}\end{aligned}$$

In general if $y = \log_e(ax+b)$ with $x > -\frac{b}{a}$ then $\frac{dy}{dx} = \frac{a}{ax+b}$

Example 6

Find: **a** $\frac{d(\log_e|x|)}{dx}, x \neq 0$ **b** $\frac{d(\log_e|ax+b|)}{dx}, x \neq \frac{-b}{a}$

Solution

a Let $y = \log_e|x|$

If $x > 0$,

$y = \log_e x$ and $\frac{dy}{dx} = \frac{1}{x}$

If $x < 0$,

then $y = \log_e(-x)$ and using the chain rule:

$$\frac{dy}{dx} = -1 \times \frac{1}{-x}$$

Hence $\frac{d(\log_e|x|)}{dx} = \frac{1}{x}$

b Let $y = \log_e|ax+b|$

Let $u = ax + b$

Then $y = \log_e|u|$

and the chain rule gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{u}(a) \\ &= \frac{1}{ax+b}(a) \\ &= \frac{a}{ax+b}\end{aligned}$$

In general if $y = \log_e|ax+b|$ with $x \neq -\frac{b}{a}$ then $\frac{dy}{dx} = \frac{a}{ax+b}$

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Example 7

Differentiate each of the following with respect to x:

a $\log_e(x^2+2)$ **b** $x^2\log_e x$, $x>0$ **c** $\dfrac{\log_e x}{x^2}$, $x>0$

Solution

a We use the chain rule.

Let $y=\log_e(x^2+2)$ and $u=x^2+2$

Then $y=\log_e u$, $\dfrac{dy}{du}=\dfrac{1}{u}$ and $\dfrac{du}{dx}=2x$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{u}\cdot 2x=\frac{2x}{x^2+2}$$

b We use the product rule.

Let $y=x^2\log_e x$

$$\text{Then } \frac{dy}{dx}=2x\log_e x+x^2\times\frac{1}{x}=2x\log_e x+x$$

c We use the quotient rule.

Let $y=\dfrac{\log_e x}{x^2}$

$$\text{Then } \frac{dy}{dx}=\frac{x^2\times\frac{1}{x}-2x\log_e x}{x^4}=\frac{x-2x\log_e x}{x^4}=\frac{1-2\log_e x}{x^3}$$

In general, the chain rule gives:

for $h(x)=\log_e(f(x))$, $h'(x)=\dfrac{f'(x)}{f(x)}$

Exercise 11B

1 Find the derivative of each of the following with respect to x:

a $y=2\log_e x$ **b** $y=2\log_e 2x$ **c** $y=x^2+3\log_e 2x$

d $y=3\log_e x+\dfrac{1}{x}$ **e** $y=3\log_e(4x)+x$ **f** $y=\log_e(x+1)$

g $y=\log_e(|2x+3|)$ **h** $y=\log_e(|3-2x|)$ **i** $y=3\log_e(|2x-3|)$

j $y=-3\log_e\left(\left|\dfrac{x}{5}-3\right|\right)$ **k** $y=4x-3\log_e\left(\left|\dfrac{x}{2}-3\right|\right)$

2 For each of the following find $f'(x)$:

a $f(x) = \log_e (x^2 + 1)$
b $f(x) = x \log_e x, x > 0$
c $f(x) = \dfrac{\log_e x}{x}, x > 0$
d $f(x) = 2x^2 \log_e x, x > 0$
e $f(x) = e^x \log_e x, x > 0$
f $f(x) = x \log_e (-x), x < 0$
g $f(x) = \log_e (e^x)$
h $f(x) = \dfrac{\log_e x}{x^2 + 1}$

3 Find the y-coordinate and the gradient at the points corresponding to a given value of x on the following curves:

a $y = \log_e x, x > 0; x = e$
b $y = x \log_e x, x > 0; x = e$
c $y = \log_e (x^2 + 1); x = e$
d $y = \log_e (-x), x < 0; x = -e$
e $y = x + \log_e x, x > 0; x = 1$
f $y = \log_e |x - 2|; x = 1$
g $y = \log_e |2x - 1|; x = 0$
h $y = \log_e |x^2 - 1|; x = 0$

4 Find $f'(1)$ if $f(x) = \log_e \sqrt{x^2 + 1}$

5 Differentiate $\log_e (1 + x + x^2)$.

6 If $f(x) = \log_e (x^2 + 1)$, find $f'(3)$.

7 It is known that $f(0) = 2$ and $f'(0) = 4$. Find $\dfrac{d(\log_e (f(x)))}{dx}$ when $x = 0$.

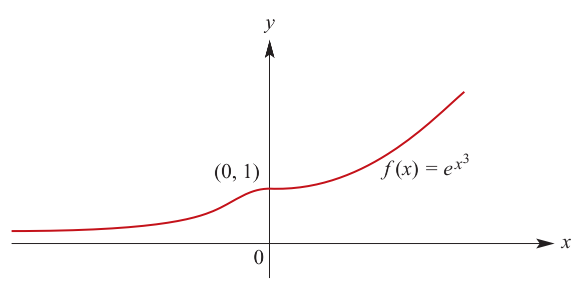
8 It is known that $f(1) = 2$ and $f'(1) = 4$. Find the derivative of $f(x) \log_e (x)$ when $x = 1$.

11.3 Applications of differentiation of exponential and logarithmic functions

Example 8

Sketch the graph of $f: R \to R, f(x) = e^{x^3}$

Solution



As $x \to -\infty, f(x) \to 0$

Axis intercepts

When $x = 0, f(x) = 1$

Turning points

$f'(x) = 3x^2 e^{x^3}$

and $f'(x) = 0$ implies $x = 0$

The gradient of f is always greater than or equal to 0 which means that (0, 1) is a stationary point of inflexion.

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Example 9

The growth of a population is modelled by the following alternatives:

a 24% a year **b** 12% every 6 months **c** 2% a month

If initially the population is 1000, find the population at the end of a year for each of the three alternatives.

Solution

a Population at the end of 1 year

$= 1.24 \times 1000$

$= 1240$

b Population at the end of 1 year

$= (1.12)^2 \times 1000$

$= 1.2544 \times 1000$

$= 1254.4$ (= 1254 as integer values are required)

c Population at the end of one year

$= (1.02)^{12} \times 1000$

$= 1.268\,242 \times 1000$

$= 1268.24$ (= 1268 as integer values are required)

Previously it was seen that the limiting process, indicated in Example 9, leads to the population growth model for a year. If the population is considered to be increasing continuously at a rate of 24% a year then the population growth for a year can be written as the limit as n approaches infinity of $\lim_{n \to \infty} 1000 \left(1 + \frac{24}{100n}\right)^n$. In Chapter 5 it was stated that $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. The similarity is clear and a little algebraic manipulation gives

$$\lim_{n \to \infty} 1000 \left(1 + \frac{24}{100n}\right)^n$$

$$= \lim_{n \to \infty} 1000 \left(1 + \frac{1}{\frac{100n}{24}}\right)^{\left(\frac{100n}{24}\right)0.24}$$

$$= 1000e^{0.24}$$

$$= 1.271\,25 \times 1000$$

$$= 1271.25 \text{ (1271 as integer values are required)}$$

After x years the population would be given by

$$P(x) = 1000e^{0.24x}$$

$$\text{and } P'(x) = 0.24 \times 1000e^{0.24x}$$

$$= 0.24\,P(x)$$

i.e. the population is growing continuously at a rate of 24% of its population at any particular time.

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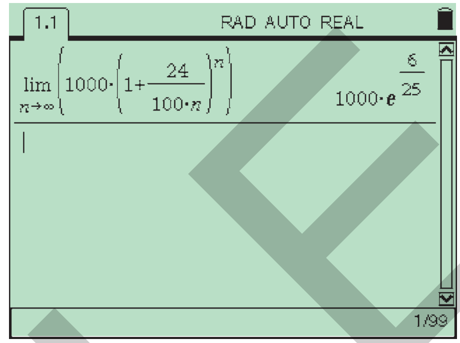
Using the TI-Nspire

Use the **Limit** template from the **Calculus** menu (menu 4 3) to find the limit $\lim_{n\to\infty} 1000\left(1+\frac{24}{100n}\right)^n$ ∞, the symbol for infinity, can be found in the catalog (catalog 4), or by typing ctrl i.

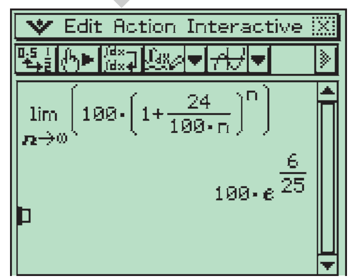


Using the Casio ClassPad

A CAS calculator can be used to evaluate this limit.

Find the limit $\lim_{n\to\infty} 1000\left(1+\frac{24}{100n}\right)^n$.

Enter and highlight the expression $1000\left(1+\frac{24}{100n}\right)^n$ then tap **Interactive** > **Calculation** > **lim** and set the variable as n and the point as ∞.



Example 10

Given that $f(x) = x - e^{2x}$ find in terms of p the approximate increase in $f(x)$ as x increases from 0 to $0 + p$, where p is small.

Solution

$$f'(x) = 1 - 2e^{2x} \text{ and } f'(0) = 1 - 2 = -1$$

$$\therefore f(0+p) \approx pf'(0) + f(0)$$

$$= -p - 1$$

$$\text{Therefore } f(0+p) - f(0) \approx -p - 1 - (-1)$$

$$= -p$$

Example 11

Given that $f(x) = e^{\frac{x}{10}}$ find in terms of h the approximate value of $f(10 + h)$, given that h is small.

Solution

$$f'(x) = \frac{1}{10}e^{\frac{x}{10}} \text{ and } f'(10) = \frac{1}{10}e^{\frac{10}{10}} = \frac{e}{10}$$

$$\therefore f(10 + h) \approx h \times \frac{e}{10} + e$$

$$= e\left(\frac{h}{10} + 1\right)$$

Example 12

For $f: (0, \infty) \to R,\ f(x) = x \log_e(x)$

a Find $f'(x)$.

b Solve the equation $f(x) = 0$

c Solve the equation $f'(x) = 0$

d Sketch the graph of $y = f(x)$

Solution

a $f'(x) = x \times \frac{1}{x} + \log_e(x)$ (product rule)

$= 1 + \log_e(x)$

b $f(x) = x\log_e(x)$ and $f(x) = 0$ implies

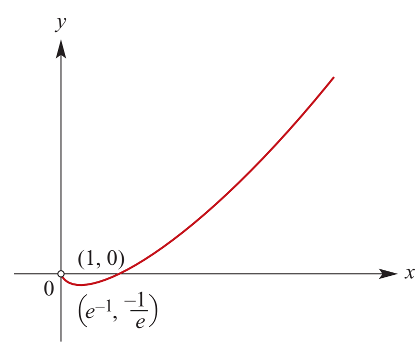
$x = 0$ or $\log_e(x) = 0$

Thus as $x \in (0, \infty)$, $x = 1$

c $f'(x) = 0$ implies $1 + \log_e(x) = 0$

Therefore $\log_e(x) = -1$ and $x = e^{-1}$

d When $x = e^{-1}$, $y = e^{-1}\log_e e^{-1} = \frac{-1}{e}$



Example 13

A particle moves along a curve with equation $y = \frac{1}{2}(e^{2x} + e^{-2x})$ where $x > 0$. The particle moves so that at time t seconds, its velocity in the positive y-axis direction is 2 units/second, i.e. $\frac{dy}{dt} = 2$

Find $\frac{dx}{dt}$ when $x = 1$.

Solution

For $y = \frac{1}{2}(e^{2x} + e^{-2x})$, $\frac{dy}{dx} = e^{2x} - e^{-2x}$

Using the chain rule
$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{dy}\frac{dy}{dt}\\ &= \frac{1}{e^{2x} - e^{-2x}} \times 2\\ &= \frac{1}{e^{2x} - \frac{1}{e^{2x}}} \times 2\\ &= \frac{1}{\frac{e^{4x} - 1}{e^{2x}}} \times 2\\ &= \frac{2e^{2x}}{e^{4x} - 1}\end{aligned}$$

The velocity in the positive direction of the x-axis when $x = 1$ is $\frac{2e^2}{e^4 - 1}$ units per second.

Exercise 11C

Skillsheet TEACHER

1 Sketch the graph of $f(x) = e^{-\frac{x^2}{2}}$

2 Let $f(x) = x^2e^x$. Find $\{x: f'(x) < 0\}$.

3 Find the values of x for which $100e^{-x^2+2x-5}$ increases as x increases and hence find the maximum value of $100e^{-x^2+2x-5}$.

4 Let $f(x) = e^x - 1 - x$

- **a** Find the minimum value of $f(x)$.
- **b** Hence show $e^x \geq 1 + x$ for all real x.

5 Find an equation of the tangent to the graph for each function at the given value of x:

- **a** $f(x) = e^x + e^{-x}; x = 0$
- **b** $f(x) = \frac{e^x - e^{-x}}{2}; x = 0$
- **c** $f(x) = x^2e^{2x}; x = 1$
- **d** $f(x) = e^{\sqrt{x}}; x = 1$
- **e** $f(x) = xe^{x^2}; x = 1$
- **f** $f(x) = x^2e^{-x}; x = 2$

6 For $f(x) = x + e^{-x}$:

- **a** Find the position and nature of any stationary points.
- **b** Find, if they exist, the equations of any asymptotes.
- **c** Sketch the graph of $y = f(x)$

7 A vehicle is travelling in a straight line from point O. Its displacement after t seconds is $0.4e^t$ metres. Find the velocity of the vehicle when $t = 0, t = 1, t = 2$.

8 A manufacturing company has a daily output y on day t of a production run given by $y = 600(1 - e^{-0.5t})$.

a Sketch the graph of y against t. (Assume a continuous model.)

b Find the instantaneous rate of change of output y with respect to t on the 10th day.

9 Assume that the number of bacteria present in a culture at time t is given by $N(t)$ where $N(t) = 24te^{-0.2t}$. At what time will the population be at a maximum? Find the maximum population.

10 Find $\frac{dy}{dx}$ for:

a $y = e^{-2x}$ **b** $y = Ae^{kx}$

In each case, express your answer in terms of y.

11 The mass m kg of radioactive lead remaining in a sample t hours after observations began is given by $m = 2e^{-0.2t}$.

a Find the mass left after 12 hours.

b Find how long it takes to fall to half of its value at $t = 0$.

c Find how long it takes for the mass to fall to **i** one-quarter and **ii** one-eighth of its value at $t = 0$.

d Express the rate of decay as a function of m.

12 Given that $y = e^{2x}$, find in terms of q the approximate increase in y as x increases from 0 to q.

13 For $f: R \to R$, $f(x) = e^{ax}$

a Find $f'(x)$.

b Find an approximation for $f(h)$, where h is small, in terms of h and a.

c Find an approximation for $f(b + h)$, where h is small, in terms of b, h and a.

14 For each of the following write down an expression for the approximate change, δy, in y when x changes from a to $a + p$ where p is small:

a $y = 2e^{\frac{x}{2}}$ **b** $y = 3 - 2e^x$ **c** $y = xe^x$ **d** $y = \frac{x}{e^x}$

15 Given that $y = e^{2t} + 1$ and $x = e^t + 1$, find:

a $\frac{dy}{dt}$ and $\frac{dx}{dt}$ **b** $\frac{dy}{dx}$ when $t = 0$

16 $y = e^x(px^2 + qx + r)$ is such that the tangents at $x = 1$ and $x = 3$ are parallel to the x-axis. The point with coordinates (0, 9) is on the curve. Find the values of p, q and r.

17 The volume, V cm^3, of water in a dish when the depth is h cm is given by the rule $V = \frac{\pi}{2}(e^{2h} - 1)$. The depth of the dish is 2.5 cm. If water is being poured in at 5 cm^3/s, find:

a the rate at which the depth of the water is increasing when the depth is 2 cm

b the rate at which the depth of the water is increasing when the depth is 2.5 cm

18 A particle moves along a curve with equation $y = \frac{1}{2}(e^{2x} + e^{-2x})$ where $x > 0$. The particle moves so that at time t seconds, its velocity in the positive y-axis direction is 2 units/second, i.e. $\frac{dy}{dt} = 2$

Find $\frac{dx}{dt}$ when $x = 1$.

19 **a** Let $y = e^{4x^2-8x}$. Find $\frac{dy}{dx}$

b Find the coordinates of the stationary point on the curve of $y = e^{4x^2-8x}$ and state its nature.

c Sketch the graph of $y = e^{4x^2-8x}$

d Find the equation of the normal to the curve of $y = e^{4x^2-8x}$ at the point where $x = 2$

20 **a** Find the equation of the tangent and normal of the graph of $f(x) = \log_e x$ at the point $(1, 0)$.

b Find the equation of the tangent and normal of the graph of $f(x) = \log_e |x|$ at the point $(-1, 0)$.

21 **a** Find the equation of the tangent of the graph of $f(x) = \log_e 2x$ at the point $\left(\frac{1}{2}, 0\right)$.

b Find the equation of the tangent of the graph of $f(x) = \log_e kx$ at the point $\left(\frac{1}{k}, 0\right)$, $(k \in R^+)$.

c Find the equation of the tangent of the graph of $f(x) = \log_e |kx|$ at the point $\left(\frac{1}{k}, 0\right)$, $(k \in R\backslash\{0\})$.

22 On the same set of axes sketch the graphs of $y = \log_e x$ and $y = \log_e 5x$ and use them to explain why $\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\log_e 5x)$

23 **a** For $y = \log_e x$, find the increase in y as x increases from 1 to $1 + p$ where p is small.

b For $y = \log_e x$, find the increase in y as x increases from a to $a + p$ where p is small and $a > 1$.

c For $f(x) = \log_e (x + 1)$ show that $f(h) \approx h$ for h 'close' to 0.

24 For $y = \log_e (1 + x^2)$ find the increase in y as x increases from 0 to p where p is a small positive number.

25 For $y = \log_e \sqrt{1 + x + x^2}$, find the increase in y as x increases from 0 to p where p is a small positive number.

26 Find an approximation for $\log_e (1.01)$.

27 If $y = \log_e (t)$ and $x = \log_e (t^2 + 1)$ for $t > 0$, find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and hence find $\frac{dy}{dx}$ when $t = 1$.

28 The volume V cm^3 of water in a dish when the depth is h cm is given by the rule $V = \pi(h+1)[(\log_e(h+1))^2 + h]$. The depth of the dish is 15 cm. If water is being poured in at 5 cm^3/s, find:

a the rate at which the depth of the water is increasing when the depth is 2 cm

b the rate at which the depth of the water is increasing when the depth is 10 cm

29 For the function $f: (0, \infty) \to R, f(x) = x^2 \log_e(x)$:

a Find $f'(x)$.

b Solve the equation $f(x) = 0$

c Solve the equation $f'(x) = 0$

d Sketch the graph of $y = f(x)$

11.4 Derivatives of circular functions

The following results, which were established in Chapter 6, will be utilised:

- $\cos^2\theta + \sin^2\theta = 1$
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

The derivative of sin $k\theta$

We first consider the function $f: R \to R, f(\theta) = \sin\theta$

Let $P(\theta, \sin\theta)$ and $Q(\theta + h, \sin(\theta + h))$ be points on the graph $f(\theta) = \sin\theta$

$$\begin{aligned}\text{The gradient of the chord } PQ &= \frac{\sin(\theta+h) - \sin\theta}{h}\\ &= \frac{\sin\theta\cos h + \cos\theta\sin h - \sin\theta}{h}\\ &= \frac{\sin\theta(\cos h - 1) + \cos\theta\sin h}{h}\\ &= \frac{\sin\theta(\cos h - 1)}{h} + \frac{\cos\theta\sin h}{h}\end{aligned}$$

We consider what happens when $h \to 0$.

Use your calculator to check the following tables:

h	$\cos h$
0.1	0.995 004
0.05	0.998 750
0.01	0.999 950
0.001	0.999 999

h	$\sin h$	$\frac{\sin h}{h}$
0.1	0.099 834	0.998 334
0.05	0.049 979	0.999 583
0.01	0.009 999	0.999 983
0.001	0.000 999	0.999 999

These results suggest that:

$$\lim_{h\to 0}(\cos h - 1) = 0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Therefore
$$\begin{aligned} f'(\theta) &= \lim_{h\to 0}\left(\frac{\sin\theta(\cos h - 1)}{h} + \frac{\cos\theta\sin h}{h}\right) \\ &= \sin\theta \times 0 + \cos\theta \times 1 \\ &= \cos\theta \end{aligned}$$

For $f: R \to R,\ f(\theta) = \sin\theta$
$f': R \to R,\ f'(\theta) = \cos\theta$

For $f: R \to R,\ f(\theta) = \sin k\theta$

We use the chain rule to determine the rule $f'(\theta)$.

Let $y = \sin k\theta$ and $u = k\theta$

Then $y = \sin u$ and
$$\begin{aligned} \frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta} \\ &= \cos u \cdot k \\ &= k\cos k\theta \end{aligned}$$

i.e. for $f: R \to R,\ f(\theta) = \sin(k\theta)$
$f': R \to R,\ f'(\theta) = k\cos(k\theta)$

The derivative of cos $k\theta$

We turn our attention to finding the derivative of $\cos k\theta$.

We first note the following:

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

These results will be used in the following way:

$$\begin{aligned} \text{let } y &= \cos\theta \\ &= \sin\left(\frac{\pi}{2} - \theta\right) \end{aligned}$$

We let $u = \frac{\pi}{2} - \theta$ and therefore $y = \sin u$

The chain rule gives
$$\begin{aligned} \frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta} \\ &= \cos u \cdot -1 \\ &= -\cos\left(\frac{\pi}{2} - \theta\right) \\ &= -\sin\theta \end{aligned}$$

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We have the following results:

$$f\colon R \to R,\ f(\theta) = \cos\theta$$
$$f'\colon R \to R,\ f'(\theta) = -\sin\theta$$
and
$$f\colon R \to R,\ f(\theta) = \cos(k\theta)$$
$$f'\colon R \to R,\ f'(\theta) = -k\sin(k\theta)$$

Notation: $\sin^n\theta = (\sin\theta)^n$ and $\cos^n\theta = (\cos\theta)^n$. For convenience a new function, secant, is introduced. The rule of secant is $\sec\theta = \dfrac{1}{\cos\theta}$

The derivative of tan $k\theta$

Let $y = \tan\theta$. We write $y = \dfrac{\sin\theta}{\cos\theta}$ and apply the quotient rule to find the derivative.

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{(\cos\theta)^2}\\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\\ &= \frac{1}{\cos^2\theta} \text{ (by the Pythagorean identity)}\\ &= \sec^2\theta\end{aligned}$$

From this we state the result:

For $f(\theta) = \tan k\theta$
$f'(\theta) = k\sec^2 k\theta$

The maximum domain for this function is $R\setminus\left\{(2n+1)\dfrac{\pi}{2} : n \in Z\right\}$.

Example 14

Find the derivative of each of the following with respect to θ:

a $\sin 2\theta$ **b** $\sin^2 2\theta$ **c** $\sin^2(2\theta + 1)$
d $\cos^3(4\theta + 1)$ **e** $\tan 3\theta$ **f** $\tan(3\theta^2 + 1)$

Solution

a $2\cos 2\theta$

b Let $y = \sin^2 2\theta$ and $u = \sin 2\theta$

Then $y = u^2$ and applying the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du} \cdot \frac{du}{d\theta} \\ &= 2u \cdot 2\cos 2\theta \\ &= 4u\cos 2\theta \\ &= 4\sin 2\theta \cos 2\theta \\ &= 2\sin 4\theta, \text{ as } \sin 4\theta = 2\sin 2\theta\cos 2\theta\end{aligned}$$

c Let $y = \sin^2(2\theta + 1)$ and $u = \sin(2\theta + 1)$

Then $y = u^2$ and applying the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{du}{du} \cdot \frac{du}{d\theta} \\ &= 2u \cdot 2\cos(2\theta + 1) \\ &= 4\sin(2\theta + 1)\cos(2\theta + 1) \\ &= 2\sin 2(2\theta + 1), \text{ as } \sin 2(2\theta + 1) = 2\sin(2\theta + 1)\cos(2\theta + 1) \\ &= 2\sin(4\theta + 2)\end{aligned}$$

d Let $y = \cos^3(4\theta + 1)$ and $u = \cos(4\theta + 1)$

Then $y = u^3$ and applying the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du} \cdot \frac{du}{d\theta} \\ &= 3u^2 \cdot -4\sin(4\theta + 1) \\ &= -12\cos^2(4\theta + 1)\sin(4\theta + 1) \\ &= -6\sin(8\theta + 2)\cos(4\theta + 1)\end{aligned}$$

e $3\sec^2 3\theta$

f Let $y = \tan(3\theta^2 + 1)$ and $u = 3\theta^2 + 1$

Then $y = \tan u$ and applying the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du} \cdot \frac{du}{d\theta} \\ &= \sec^2 u \cdot 6\theta \\ &= 6\theta\sec^2(3\theta^2 + 1)\end{aligned}$$

Example 15

Find the y-coordinate and the gradient at the points on the following curves corresponding to the given values of θ:

a $y = \sin\theta$, $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$

b $y = \cos\theta$, $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$

c $y = \tan\theta$, $\theta = 0$ and $\theta = \frac{\pi}{4}$

Solution

a For $\theta = \frac{\pi}{4}$, $y = \sin\frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}}$$

$$\frac{dy}{d\theta} = \cos\theta$$

$\therefore$ the gradient at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is $\frac{1}{\sqrt{2}}$

For $\theta = \frac{\pi}{2}$, $y = \sin\frac{\pi}{2} = 1$ and the gradient at $\left(\frac{\pi}{2}, 1\right)$ is 0

b For $\theta = \frac{\pi}{4}$, $y = \cos\frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}}$$

$$\frac{dy}{d\theta} = -\sin\theta$$

$\therefore$ the gradient at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is $-\frac{1}{\sqrt{2}}$

For $\theta = \frac{\pi}{2}$, $y = \cos\frac{\pi}{2} = 0$ and the gradient at $\left(\frac{\pi}{2}, 0\right)$ is -1

c For $\theta = 0$, $y = \tan 0 = 0$

$$\frac{dy}{d\theta} = \sec^2\theta$$

$\therefore$ the gradient at (0, 0) is 1

For $\theta = \frac{\pi}{4}$, $y = \tan\frac{\pi}{4} = 1$ and the gradient at $\left(\frac{\pi}{4}, 1\right)$ is 2

Example 16

Find the derivative of each of the following with respect to x:

a $2x^2 \sin 2x$ **b** $\frac{\sin x}{x+1}, x \neq -1$ **c** $e^{2x}\sin(2x+1)$ **d** $\cos 4x \sin 2x$

Solution

a Let $y = 2x^2 \sin 2x$

Applying the product rule:

$$\frac{dy}{dx} = 4x\sin 2x + 4x^2\cos 2x$$

b Let $y = \frac{\sin x}{x+1}, x \neq -1$

Applying the quotient rule:

$$\frac{dy}{dx} = \frac{(x+1)\cos x - \sin x}{(x+1)^2}$$

c Let $y = e^{2x}\sin(2x+1)$

Applying the product rule:

$$\frac{dy}{dx} = 2e^{2x}\sin(2x+1) + 2e^{2x}\cos(2x+1)$$
$$= 2e^{2x}[\sin(2x+1) + \cos(2x+1)]$$

d Let $y = \cos 4x \sin 2x$

Applying the product rule:

$$\frac{dy}{dx} = -4\sin 4x \sin 2x + 2\cos 2x \cos 4x$$

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Exercise 11D

1 Find the derivative with respect to x of each of the following:

a $\sin 5x$ **b** $\cos 5x$ **c** $\tan 5x$ **d** $\sin^2 x$ **e** $\tan(3x+1)$
f $\cos(x^2+1)$ **g** $\sin^2\left(x-\frac{\pi}{4}\right)$ **h** $2\cos x^\circ$ **i** $3\sin x^\circ$ **j** $\tan(3x)^\circ$

2 Find the y-coordinate and the gradient at the points corresponding to the given value of x on the following curves:

a $y=\sin 2x,\ x=\frac{\pi}{8}$ **b** $y=\sin 3x,\ x=\frac{\pi}{6}$ **c** $y=1+\sin 3x,\ x=\frac{\pi}{6}$
d $y=\cos^2 2x,\ x=\frac{\pi}{4}$ **e** $y=\sin^2 2x,\ x=\frac{\pi}{4}$ **f** $y=\tan 2x,\ x=\frac{\pi}{8}$

3 For each of the following find $f'(x)$:

a $f(x)=5\cos x-2\sin 3x$ **b** $f(x)=\cos x+\sin x$
c $f(x)=\sin x+\tan x$ **d** $f(x)=\tan^2 x$

4 Differentiate each of the following with respect to x:

a $x^3\cos x$ **b** $\frac{\cos x}{1+x}$ **c** $e^{-x}\sin x$ **d** $3x+2\cos x$ **e** $\sin 3x\cos 4x$
f $\tan 2x\sin 2x$ **g** $12x\sin x$ **h** $x^2e^{\sin x}$ **i** $x^2\cos^2 x$ **j** $e^x\tan x$

5 For each of the following find $f'(\pi)$:

a $f(x)=\frac{2x}{\cos x}$ **b** $f(x)=\frac{3x^2+1}{\cos x}$ **c** $f(x)=\frac{e^x}{\cos x}$
d $f(x)=e^x\sin x$ **e** $f(x)=\frac{\sin x}{x}$ **f** $f(x)=\cos^2 2x$

6 **a** If $y=-\log_e(\cos x)$, find $\frac{dy}{dx}$ **b** If $y=-\log_e|\cos x|$, find $\frac{dy}{dx}$

7 Find the derivative of each of the following functions:

a $\frac{\sin x+\cos x}{\sin x-\cos x}$ **b** $e^{\tan x}$ **c** $x^{\frac{7}{2}}\sin 3x$

11.5 Applications of derivatives of circular functions

Example 17

Find the equation of the tangent of the curve with equation $y=\sin x$ at the point where $x=\frac{\pi}{3}$.

Solution

$\frac{dy}{dx}=\cos x$. When $x=\frac{\pi}{3}$, $y=\frac{\sqrt{3}}{2}$ and $\frac{dy}{dx}=\frac{1}{2}$

Therefore the equation of tangent is:

$$y-\frac{\sqrt{3}}{2}=\frac{1}{2}\left(x-\frac{\pi}{3}\right),$$ which can be written in the form

$$y=\frac{x}{2}-\frac{\pi}{6}+\frac{\sqrt{3}}{2}$$

Example 18

Find the equation of the tangent and normal of the graph of $y = -\cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

Solution

We first find the gradient of the curve at this point.

$\frac{dy}{dx} = \sin x$ and when $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 1$

$\therefore$ the equation of the tangent is $y - 0 = 1\left(x - \frac{\pi}{2}\right)$

i.e. $y = x - \frac{\pi}{2}$

The gradient of the normal is -1 and therefore the equation of the normal is:

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

$$\text{i.e. } y = -x + \frac{\pi}{2}$$

Example 19

Let $f(\theta) = \sin(2\theta)$. If θ is increased by a small amount h find:

a an approximation for $f(\theta + h)$ where $\theta = \frac{\pi}{6}$

b an expression for the percentage change for $\theta = \frac{\pi}{6}$

Solution

a $f'(\theta) = 2\cos(2\theta)$ and $f'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) = 1$

Therefore $f\left(\frac{\pi}{6} + h\right) \approx h + f\left(\frac{\pi}{6}\right)$

$$= h + \frac{\sqrt{3}}{2}$$

b Percentage change $= \dfrac{100hf'\left(\frac{\pi}{6}\right)}{f\left(\frac{\pi}{6}\right)}$

$$= \frac{100 \times h \times 1}{\frac{\sqrt{3}}{2}}$$

$$= \frac{200h}{\sqrt{3}}$$

$$= \frac{200h\sqrt{3}}{3}$$

The percentage change is $\frac{200h\sqrt{3}}{3}$.

Example 20

The curves with equation $y = \cos x$ and $y = -\sin 2x$ intersect at the point where $x = \frac{7\pi}{6}$. Find the acute angle between the curves at this point.

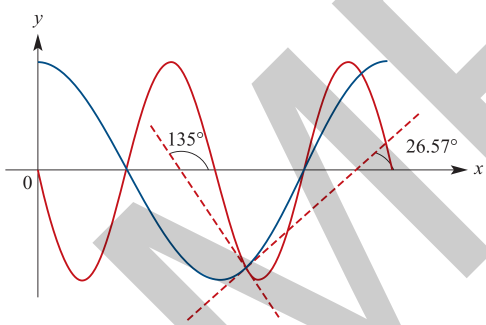
Solution

For $y = \cos x$, $\frac{dy}{dx} = \frac{1}{2}$ where $x = \frac{7\pi}{6}$.

For $y = -\sin 2x$, $\frac{dy}{dx} = -1$ where $x = \frac{7\pi}{6}$.

For the curve of $y = -\sin 2x$ at the point where $x = \frac{7\pi}{6}$, the angle of inclination to the positive direction of the x-axis is $135°$.

For the curve of $y = \cos x$ at the point where $x = \frac{7\pi}{6}$, the angle of inclination to the positive direction of the x-axis is $\left[\tan^{-1}\left(\frac{1}{2}\right)\right]^{\circ} \approx 26.57°$. Therefore the angle between the two curves $\approx 135° - 26.57° \approx 108.43°$. The acute angle is $71.57°$.



Example 21

Find the local maximum and minimum values of $f(x) = 2\sin x + 1 - 2\sin^2 x$ where $0 \le x \le 2\pi$.

Solution

Find $f'(x)$ and solve $f'(x) = 0$.

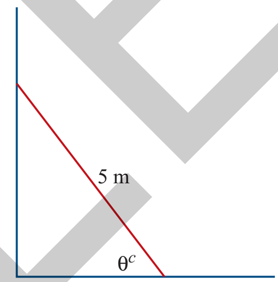
$$f(x) = 2\sin x + 1 - 2\sin^2 x$$
$$\therefore f'(x) = 2\cos x - 4\sin x\cos x$$
$$= 2\cos x(1 - 2\sin x)$$
$$f'(x) = 0 \text{ when } \cos x = 0 \text{ or } 1 - 2\sin x = 0$$
$$\text{i.e. when } \cos x = 0 \text{ or } \sin x = \frac{1}{2}$$
$$\cos x = 0 \text{ implies } x = \frac{\pi}{2}, \frac{3\pi}{2}$$
$$\sin x = \frac{1}{2} \text{ implies } x = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$f\left(\frac{\pi}{2}\right) = 1,\ f\left(\frac{3\pi}{2}\right) = -3,\ f\left(\frac{\pi}{6}\right) = \frac{3}{2},\ f\left(\frac{5\pi}{6}\right) = \frac{3}{2}$$

		$\frac{\pi}{6}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$		$\frac{3\pi}{2}$	
$f'(x)$	+	0	−	0	+	0	−	0	+
shape	/	—	\	—	/	—	\	—	/

$\therefore$ max. at $\left(\frac{\pi}{6}, \frac{3}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{3}{2}\right)$

and min. at $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -3\right)$

Example 22



The diagram shows a 5-metre pole leaning against a vertical wall. The lower end of the pole moves away from the wall at a constant speed of a m/s.

Find for the instant that the foot of the pole is 3 m from the wall

a the speed at which the top of the pole is descending

b the rate at which the radian measure of the angle defined by the pole and the horizontal between the bottom of the pole and the wall is changing.

Solution

a Let h metres be the distance, at time t seconds, from the top of the pole to the horizontal.

Let x metres be the distance, at time t seconds, from the bottom of the pole to the wall.

$$h = \sqrt{25 - x^2}$$

Therefore
$$\frac{dh}{dx} = -2x \times \frac{1}{2\sqrt{25 - x^2}} = \frac{-x}{\sqrt{25 - x^2}}$$

and
$$\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt} = \frac{-x}{\sqrt{25 - x^2}} \times a$$

Furthermore $x = 3$, $\frac{dh}{dt} = \frac{-3a}{4}$

The pole is sliding down the wall at a rate of $\frac{3a}{4}$ m/s.

b First it is noted $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$x = 5\cos\theta$. Therefore $\frac{dx}{d\theta} = -5\sin\theta$ and $\frac{d\theta}{dx} = \frac{-1}{5\sin\theta}$

Therefore $\frac{d\theta}{dt} = \frac{-1}{5\sin\theta} \times a$

When $x = 3$, $\sin\theta = \frac{4}{5}$ and $\frac{d\theta}{dt} = \frac{-a}{4}$ radians/second

Exercise 11E

Skillsheet TEACHER

1 Find the equation of the tangent to each of the following curves at the point corresponding to the given x-value:

a $y = \sin 2x,\ x = 0$ **b** $y = \cos x,\ x = \frac{\pi}{2}$ **c** $y = \tan x,\ x = \frac{\pi}{4}$

d $y = \tan 2x,\ x = 0$ **e** $y = \sin x + \sin 2x,\ x = 0$ **f** $y = x - \tan x,\ x = \frac{\pi}{4}$

2 If $y = 3x + 2\cos x$, find $\frac{dy}{dx}$ and hence show that y increases as x increases.

3 The tangent to the curve with equation $y = \tan 2x$ at the point where $x = \frac{\pi}{8}$ meets the y-axis at the point A. Find the distance OA where O is the origin.

4 Find the acute angle of intersection of the curves $y = \cos x$ and $y = e^{-x}$ at the point $(0, 1)$.

5 Find the acute angle of intersection of the curves $y = \sin 2x$ and $y = \cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

6 The volume, $V(t)$, of water in a reservoir at time t is given by $V(t) = 3 + 2\sin\frac{t}{4}$

a Find the volume in the reservoir at time $t = 10$.

b Find the rate of change of the volume of water in the reservoir at time $t = 10$.

7 Let $y = \tan\theta$.

a **i** If θ is increased by a small amount p find an approximation for the increase in y in terms of p and θ.

ii If $\theta = \frac{\pi}{4}$, find the increase in y in terms of p.

b Find an approximate value of $\tan(46°)$ using the results of **a**.

8 **a** Given $f\colon R \to R$, where $f(x) = \cos x$, find $f\left(\frac{\pi}{4}\right)$ and $f'\left(\frac{\pi}{4}\right)$

b Use the results of **a** to find an approximate value of:

i $\cos\left(\frac{\pi}{4} + h\right)$ where h is small **ii** $\cos 0.8$

9 For each of the following write down an expression for the approximate change, δy, in y when x changes from a to $a + p$ where p is small:

a $y = \cos(2x)$ **b** $y = \sin\left(\frac{x}{2}\right)$ **c** $y = \tan(2x)$

d $y = 1 - \tan\left(\frac{x}{2}\right)$ **e** $y = \cos\left(\frac{\pi}{4} - x\right)$ **f** $y = \sin\left(\frac{-x}{2}\right)$

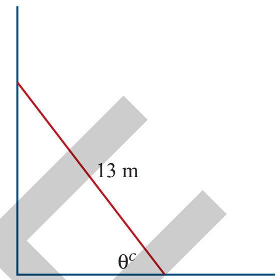
10 Find the local maximum and minimum values of $f(x)$ for each of the following and state the corresponding x-value of each. (Consider $0 \le x \le 2\pi$ only.)

a $f(x) = 2\cos x - (2\cos^2 x - 1)$ **b** $f(x) = 2\cos x + 2\sin x\cos x$

c $f(x) = 2\sin x - (2\cos^2 x - 1)$ **d** $f(x) = 2\sin x + 2\sin x\cos x$

11 The diagram shows a 13-metre pole leaning against a vertical wall. The lower end of the pole moves away from the wall at a constant speed of 1 m/s. For the instant that the foot of the pole is 12 m from the wall, find:

a the speed at which the top of the pole is descending
b the rate at which the radian measure of the angle defined by the pole and the horizontal between the bottom of the pole and the wall is changing



11.6 Miscellaneous exercises

In the following exercise a CAS calculator is to be used for all questions.

Exercise 11F

1 Find the derivative of each of the following for the given x-value. Give the answer correct to two decimal places.

a $y = e^{-\left(\frac{x}{10}\right)^2} \sin x,\ x = \frac{\pi}{4}$ **b** $y = \log_e(\sin x),\ x = \frac{\pi}{4}$
c $y = \sin x + \cos 3x,\ x = 0.7$ **d** $y = \log_e(x) + \sin x,\ x = 1$

2 Find the coordinates of the local minima and maxima for $x \in [0, 2\pi]$ for each of the following. (Values are to be given correct to two decimal places.)

a $y = \log_e(x) + \sin(x)$ **b** $y = \log_e(\sin x)$ **c** $y = \log_e(2x) + \cos(2x)$
d $y = x^2 + \cos(x)$ **e** $y = \tan(2x) + \frac{1}{\cos x}$ **f** $y = \cos^2(2x)\sin x$

3 For $y = \tan(2x) + \frac{1}{\cos x}$ find $\frac{dy}{dx}$ and plot the graph of $\frac{dy}{dx}$ against x for $x \in [4, 5.5]$. Use this to confirm the results of **2e**.

4 For each of the following, plot the graph of the derivative function for the stated domain:

a $f(x) = \cos^3(2x)\sin(x),\ [0, 2\pi]$ **b** $f(x) = e^{\frac{x}{10}}\log_e(x),\ (0, 10]$
c $f(x) = \sin^3(x)\cos^5(x),\ [0, 2\pi]$

5 **a** With a CAS calculator plot the graph of:
i $y = e^x - 2$ **ii** $y = x$
b Find the coordinates of the points of intersection of $y = e^x - 2$ and $y = x$ from the calculator, correct to two decimal places.
c Find the inverse function of $y = e^x - 2$ and plot this inverse on the same screen as **a**.
d Find $\{x: e^x - 2 > x\}$
e **i** The graph of $y = e^x + a$ passes through the origin. Find the value of a.
ii Find the equation of the tangent to $y = e^x - 1$ at the origin.
iii Find the point of intersection of $y = e^x - 1$ and $y = x$
iv Find the inverse of $y = e^x - 1$
v Using a calculator plot the graph of $y = e^x - 1$, the inverse of this function, and $y = x$

6 **a** With a CAS calculator plot the graph of:

i $y = 2e^x - 2$ **ii** $y = \log_e\left(\frac{x+2}{2}\right)$ **iii** $y = x$

b Find the gradient of $y = 2e^x - 2$ and $y = \log_e\left(\frac{x+2}{2}\right)$ at the origin.

c Find the equations of the tangents of $y = 2e^x - 2$ and $y = \log_e\left(\frac{x+2}{2}\right)$ at the origin.

d Consider the function with rule $f(x) = ae^x - a, a > 0$

i Find f^{-1}

ii Find the equations of the tangents to the graphs of $y = \log_e\left(\frac{x+a}{a}\right)$ and $y = ae^x - a$ at the origin.

iii Plot the graphs of $y = \log_e\left(\frac{x+a}{a}\right)$ and $y = ae^x - a$ for $a = 3$ and $a = \frac{1}{3}$.

iv Comment on the points of intersection of the two graphs.

11.7 Applications of transcendental functions

Example 23

The number of bacteria, N, in a culture increases at a rate proportional to the number present according to the law $N = N_0e^{kt}$, where t is the number of hours of growth and k and N_0 are constants.

a Prove that $\frac{dN}{dt}$ is proportional to N.

b If it takes 48 hours for the colony to double in number, find k and hence the rate at which the colony is increasing when $N = 10^4$.

Solution

a
$$\begin{aligned}\frac{dN}{dt} &= kN_0e^{kt}\\ &= kN\end{aligned}$$

i.e. $\frac{dN}{dt}$ is proportional to N.

b When $t = 0$, $N = N_0$

When $N = 2N_0$, $t = 48$

$$\begin{aligned}2N_0 &= N_0e^{48k}\\ 2 &= e^{48k}\\ \therefore k &= \frac{1}{48}\log_e 2\\ &\approx 0.0144\end{aligned}$$

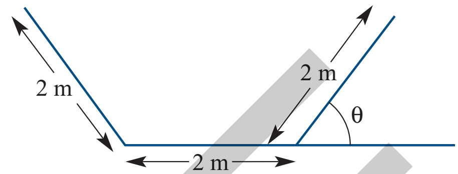
When $N = 10^4$,

$$\begin{aligned}\frac{dN}{dt} &= kN\\ &= \tfrac{1}{48}\log_e 2 \times 10^4\\ &\approx 144.41\end{aligned}$$

The colony is increasing at a rate of ≈ 144 per hour.

Example 24

The cross-section of a drain is to be an isosceles trapezoid, three of whose sides are 2 metres long, as shown. Find θ so that the cross-sectional area will be as great as possible and find this maximum.



Solution

$$\text{Area of the trapezium} = \frac{1}{2} \times 2\sin\theta(2 + 2 + 4 \times \cos\theta)$$
$$= \sin\theta(4 + 4\cos\theta)$$

Let A m^2 be the area of the trapezium.

Then $A = \sin\theta(4 + 4\cos\theta)$

and
$$\frac{dA}{d\theta} = \cos\theta(4 + 4\cos\theta) - 4\sin^2\theta$$
$$= 4\cos\theta + 4\cos^2\theta - 4(1 - \cos^2\theta)$$
$$= 4\cos\theta + 8\cos^2\theta - 4$$

The maximum will occur when $\frac{dA}{d\theta} = 0$.

Consider:

$$8\cos^2\theta + 4\cos\theta - 4 = 0$$
$$\therefore 2\cos^2\theta + \cos\theta - 1 = 0$$
$$\therefore (2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$\therefore \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$

The practical restriction on θ is that $0 < \theta \le \frac{\pi}{2}$

Therefore the only possible solution is that $\theta = \frac{\pi}{3}$ and a gradient table confirms that $\frac{\pi}{3}$ gives a maximum.

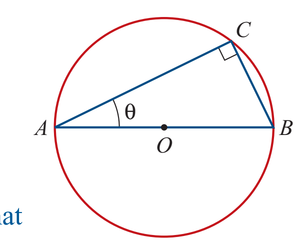
		$\frac{\pi}{3}$	
$A'(\theta)$	+	0	−
shape	/	—	\

When $\theta = \frac{\pi}{3}$, $A = \frac{\sqrt{3}}{2}(4 + 2) = 3\sqrt{3}$

i.e. the maximum cross-sectional area is $3\sqrt{3}$ m^2.

Example 25

The figure shows a circular lake, centre O, of radius 2 km. A man swims across the lake from A to C at 3 km/h and then walks around the edge of the lake from C to B at 4 km/h.



a If $\angle BAC = \theta$ radians and the total time taken is T hours, show that

$$T = \frac{1}{3}(4\cos\theta + 3\theta)$$

b Find the value of θ for which $\frac{dT}{d\theta} = 0$ and determine whether this gives a maximum or minimum value of T $(0° < \theta° < 90°)$.

Solution

a The time taken $= \frac{\text{distance travelled}}{\text{speed}}$

Therefore for the swim the time taken $= \frac{4\cos\theta}{3}$ hours and the walk takes $\frac{4\theta}{4}$ hours.

$\therefore$ the total time taken $= \frac{1}{3}(4\cos\theta + 3\theta)$

b $\frac{dT}{d\theta} = \frac{1}{3}(-4\sin\theta + 3)$

The stationary point occurs where $\frac{dT}{d\theta} = 0$ and $\frac{dT}{d\theta} = \frac{1}{3}(-4\sin\theta + 3)$

implies $\sin\theta = \frac{3}{4}$

$\therefore \theta = 48°35'$

If $\theta < 48°35'$, $T'(\theta) > 0$

if $\theta > 48°35'$, $T'(\theta) < 0$

$\therefore$ maximum when $\theta = 48°35'$

When $\sin\theta = \frac{3}{4}$, $T = 1.73$ hours.

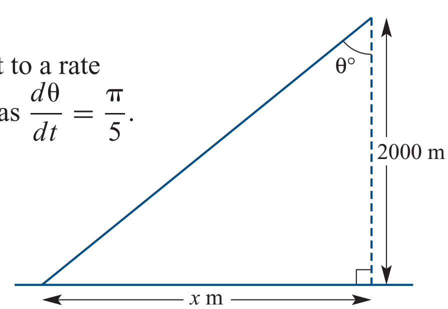
If the man swims straight across it takes $1\frac{1}{3}$ hours.

If he goes all the way around the edge it takes approximately 1.57 hours.

Example 26

A beacon that makes one revolution every 10 seconds is located on a ship 2 km from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shoreline?

Solution



One revolution every 10 seconds is equivalent to a rate of $\frac{\pi}{5}$ radians per second. This can be written as $\frac{d\theta}{dt} = \frac{\pi}{5}$.

Also $x = 2000\tan\theta$ and $\frac{dx}{d\theta} = 2000\sec^2\theta$

$$\begin{aligned}\text{By the chain rule } \frac{dx}{dt} &= \frac{dx}{d\theta} \times \frac{d\theta}{dt}\\ &= 2000\sec^2\theta \times \frac{\pi}{5}\\ &= 400\pi\sec^2\theta\end{aligned}$$

When the angle with the shoreline is 45°, $\theta = \frac{\pi}{4}$

$$\text{and } \frac{dx}{dt} = 400\pi \sec^2 \frac{\pi}{4}$$
$$= 800\pi$$

The beam is moving along the shoreline at a speed of 800π metres/second.

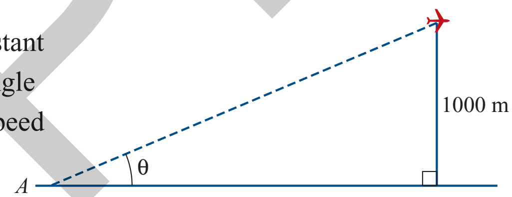
Exercise 11G

1 A car tyre is inflated to a pressure of 30 units. Eight hours later it is found to have deflated to 10 units. The pressure P at time t hours is given by:

$$P = P_0 e^{-\lambda t}$$

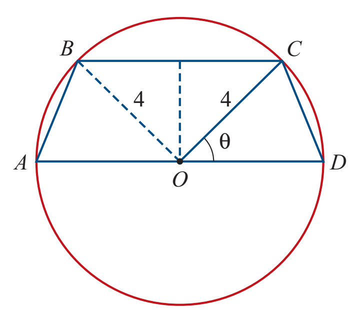
a Find the values of P_0 and λ. **b** At what time would the pressure be 8 units?
c Find the rate of loss of pressure at:
i time $t = 0$ **ii** time $t = 8$

2 An aeroplane is flying horizontally at a constant height of 1000 m. At a certain instant the angle of elevation is 30° and decreasing and the speed of the aeroplane is 480 km/h.



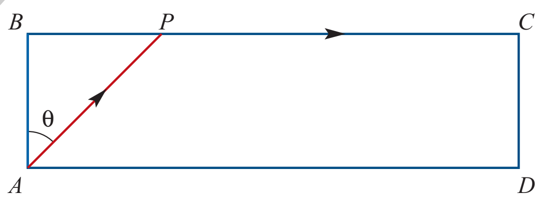
a How fast is θ decreasing at this instant? (Answer in degrees/s.)
b How fast is the distance between the aeroplane and the observation point changing at this instant?

3 $ABCD$ is a trapezium with $AB = CD$, with vertices on the circle and with centre O. AD is a diameter of the circle. The radius of the circle is 4 units.



a Find BC in terms of θ.
b Find the area of the trapezium in terms of θ and hence find the maximum area.

4 The figure shows a rectangular field in which $AB = 300$ m and $BC = 1100$ m.



$AB = 300$ m
$BC = 1100$ m

a An athlete runs across the field from A to P at 4 m/s. Find the time taken to run from A to P in terms of θ.

b The athlete, on reaching P, immediately runs to C at 5 m/s. Find the time taken to run from P to C in terms of θ.

c Show, using the results from **a** and **b**, that the total time taken, T seconds, is given by

$$T = 220 + \frac{75 - 60\sin\theta}{\cos\theta}$$

d Find $\frac{dT}{d\theta}$.

e Find the value of θ for which $\frac{dT}{d\theta} = 0$ and show that this is the value of θ for which T is a minimum.

f Find the minimum value of T and the distance of point P from B that will minimise her running time.

MC TEST D&D TEST

Review

Chapter summary

- For $f(x) = e^{kx}$, $f'(x) = ke^{kx}$
- For $f(x) = \log_e kx$, with $kx > 0$, $f'(x) = \frac{1}{x}$
- For $f(x) = \log_e |kx|$, $f'(x) = \frac{1}{x}$
- For $f(x) = \sin kx$, $f'(x) = k \cos kx$
- For $f(x) = \cos kx$, $f'(x) = -k \sin kx$
- For $f(x) = \tan(kx)$, $f'(x) = k \sec^2(kx)$

$$= k\frac{1}{\cos^2(kx)}$$

$$= \frac{k}{\cos^2(kx)}$$

Multiple-choice questions

1 The derivative of $e^{-2ax}\cos(ax)$ with respect to x is:

A $-ae^{-2ax}\cos(ax) - 2ae^{-2ax}\sin(ax)$ **B** $ae^{-2ax}\cos(ax) - 2ae^{-2ax}\sin(ax)$

C $-2ae^{-2ax}\cos(ax) - ae^{-2ax}\sin(ax)$ **D** $2ae^{-2ax}\cos(ax) + 2ae^{-2ax}\sin(ax)$

E $-ae^{-2ax}\cos(ax) - 2ae^{-2ax}\sin(ax)$

2 For $f(x) = \frac{\cos x}{x - a}$, where a is a constant, find $f'(x)$.

A $\frac{\sin x}{x-a} + \frac{\cos x}{(x-a)^2}$ **B** $-\frac{\sin x}{x-a} - \frac{\cos x}{(x-a)^2}$ **C** $\frac{\sin x}{x-a} - \frac{\cos x}{(x-a)^2}$

D $\frac{x \sin x}{x-a} - \frac{x \cos x}{(x-a)^2}$ **E** $\frac{\sin x}{x} - \frac{\cos x}{x}$

3 For $f: R \to R$, $f(x) = e^x - ex$, the coordinates of the turning point of the graph of $y = f(x)$ are:

A $\left(1, \frac{1}{e}\right)$ **B** $(1, e)$ **C** $(0, 1)$ **D** $(1, 0)$ **E** $(e, 1)$

4 The equation of the tangent of $y = e^{ax}$ at the point $\left(\frac{1}{a}, e\right)$ is:

A $y = e^{ax-1} + 1$ **B** $y = ae^{ax}x$ **C** $y = 1 - ae^{ax}$

D $y = \frac{e^2 x}{a}$ **E** $y = aex$

5 If $z = \log_e(x)$ then δz is approximately equal to:

A $\log_e(x + \delta x)$ **B** $\log_e(\delta x)$ **C** $\frac{1}{\delta x}$ **D** $\frac{\delta x}{x}$ **E** $\frac{1}{x}$

6 If $z = \sin x$ and $\sin 1 = a$ then using the linear approximation the value of $\sin(1.1)$ is equal to:

A $0.1\sqrt{1 - a^2}$ **B** $0.1 \cos 1$ **C** $0.1\sqrt{1 - a^2} + a$ **D** $a + 0.1$ **E** $0.1a$

7 Under certain conditions, the number of bacteria, N, in a sample increases with time, t hours, according to the rule $N = 4000e^{0.2t}$. The rate, to the nearest whole number of bacteria per hour, that the bacteria are growing 3 hours from the start is:

A 1458 **B** 7288 **C** 16 068 **D** 80 342 **E** 109 731

8 The gradient of the tangent to the curve $y = x^2 \cos 5x$ at the point where $x = \pi$ is:

A $5\pi^2$ **B** $-5\pi^2$ **C** 5π **D** -5π **E** -2π

9 The equation of the tangent to the curve with equation $y = e^{-x} - 1$ at the point where the curve crosses the y-axis is:

A $y = x$ **B** $y = -x$ **C** $y = \frac{1}{2}x$ **D** $y = -\frac{1}{2}x$ **E** $y = -2x$

10 For $f: R \to R,\ f(x) = e^{ax} - \frac{ax}{e}$, the coordinates of the turning point of the graph of $y = f(x)$ are:

A $\left(-\frac{1}{a}, 0\right)$ **B** $\left(\frac{1}{a}, \frac{1}{e}\right)$ **C** $\left(-\frac{1}{a}, \frac{2}{e}\right)$ **D** $\left(-1, \frac{1}{e}\right)$ **E** $(1, 0)$

Short-answer questions (technology-free)

1 Differentiate each of the following with respect to x:

a $\log_e(x^2 + 2)$ **b** $\sin(3x + 2)$ **c** $\cos\left(\frac{x}{2}\right)$ **d** e^{x^2-2x}

e $\log_e(3 - x)$ **f** $\sin(2\pi x)$ **g** $\sin^2(3x + 1)$ **h** $\sqrt{\log_e x},\ x > 1$

i $\frac{2\log_e 2x}{x}$ **j** $x^2 \sin(2\pi x)$

2 Differentiate each of the following with respect to x:

a $e^x \sin 2x$ **b** $2x^2 \log_e x$ **c** $\frac{\log_e x}{x^3}$ **d** $\sin 2x \cos 3x$

e $\frac{\sin 2x}{\cos 2x}$ **f** $\cos^3(3x + 2)$ **g** $x^2 \sin^2(3x)$

3 Find the gradient of each of the following curves for the stated value of x:

a $y = e^{2x} + 1,\ x = 1$ **b** $y = e^{x^2+1},\ x = 0$

c $y = 5e^{3x} + x^2,\ x = 1$ **d** $y = 5 - e^{-x},\ x = 0$

4 Differentiate each of the following with respect to x:

a e^{ax} **b** e^{ax+b} **c** e^{a-bx} **d** $be^{ax} - ae^{bx}$ **e** $\frac{e^{ax}}{e^{bx}}$

5 A vehicle is travelling in a straight line from a point O. Its displacement after t seconds is $0.25e^t$ metres. Find the velocity of the vehicle at $t = 0$, $t = 1$, $t = 2$ and $t = 4$ (seconds).

6 The temperature, θ°C, of material inside a nuclear power station at time t seconds after a reaction begins is given by $\theta = \frac{1}{4}e^{100t}$

a Find the rate of temperature increase at time t.

b Find the rate of increase of temperature when $t = \frac{1}{20}$.

7 Find the equation of the tangent of $y = e^x$ at $(1, e)$.

8 The diameter of a tree (D cm) t years after 1 January 1990 is given by $D = 50e^{kt}$

a Prove that $\frac{dD}{dt} = cD$ for some constant c.

b If $k = 0.2$, find the rate of increase of D where $D = 100$.

9 If $x = 2e^{-2t} + 3e^{-t}$, show that $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$

10 Find the minimum value of $e^{3x} + e^{-3x}$.

11 $y = e^{ax}$ is a solution of the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$. Find two values of a.

12 **a** Find the equation of the tangent of $y = \log_e x$ at the point $(e, 1)$.

b Find the equation of the tangent to $y = 2\sin\left(\frac{x}{2}\right)$ at the point $\left(\frac{\pi}{2}, \sqrt{2}\right)$.

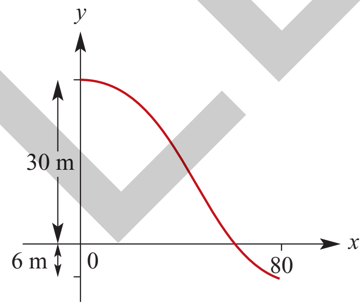
c Find the equation to the tangent to $y = \cos x$ at the point $\left(\frac{3\pi}{2}, 0\right)$.

d Find the equation of the tangent of $y = \log_e |x|$ at the point $(-e, 1)$.

Extended-response questions

1 A section of a rollercoaster can be described by the rule:

$y = 18\cos\left(\frac{\pi x}{80}\right) + 12, 0 \le x \le 80$



a Find the gradient function, $\frac{dy}{dx}$.

b Sketch the graph of $\frac{dy}{dx}$ against x.

c State the coordinates of the point on the track for which the magnitude of the gradient is maximum.

2 The kangaroo population in a certain confined region is given by $f(x) = \frac{100\,000}{1 + 100e^{-0.3x}}$ where x is the time in years.

a Find $f'(x)$

b Find the rate of growth of the kangaroo population when:

i $x = 0$ and **ii** $x = 4$

3 Consider the function $f: \{x: x < a\} \to R$, $f(x) = 8\log_e(6 - 0.2x)$ where a is the largest value for which f is defined.

a What is the value of a?

b Find the exact values for the coordinates of the points where the graph of $y = f(x)$ crosses each axis.

c Find the gradient of the tangent to the graph of $y = f(x)$ at the point where $x = 20$.

d Find the rule of the inverse function f^{-1}.

e State the domain of the inverse function f^{-1}.

f Sketch the graph of $y = f(x)$

CAS CALCULATOR

4 **a** Using a calculator plot the graphs of $f(x) = \sin x$ and $g(x) = e^{\sin x}$ on the one screen.

b Find $g'(x)$ and hence find the coordinates of the stationary points of $y = g(x)$ for $x \in [0, 2\pi]$.

c Give the range of g.

d State the period of g.

5 **a** Show that the tangent to the graph of $y = e^x$ for $x = 0$ has equation $y = x + 1$

b Plot the graphs of $y = e^x$ and $y = x + 1$ on a calculator.

c Let $f(x) = e^x$ and $g(x) = x + 1$. Use a calculator to investigate functions of the form:

$$h(x) = af(x - b) + c \quad \text{and} \quad k(x) = ag(x - b) + c$$

Comment on your observations.

d Use the chain rule and properties of transformations to prove that if the tangent of the curve with equation $y = f(x)$ at the point (x_1, y_1) is $y = mx + c$, then the equation of the tangent of the curve with equation $y = af(bx)$ at the point $\left(\frac{x_1}{b}, y_1 a\right)$ is $y = a(mbx + c)$

6 A certain chemical starts to dissolve in water at time $t \geq 0$. It is known that if x is the number of grams not dissolved after t hours then:

$$x = \frac{60}{5e^{\lambda t} - 3}, \quad \text{where } \lambda = \frac{1}{2}\log_e \frac{6}{5}$$

a Find the amount of chemical present when:

i $t = 0$ **ii** $t = 5$

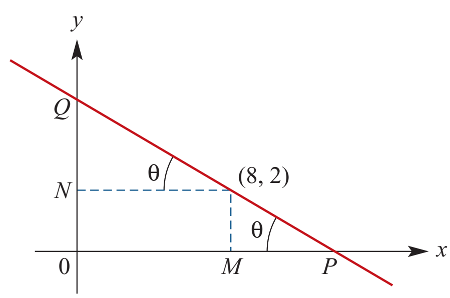
b Find $\frac{dx}{dt}$ in terms of t.

c **i** Show that $\frac{dx}{dt} = -\lambda x - \frac{\lambda x^2}{20}$

ii Sketch the graph of $\frac{dx}{dt}$ against x, for $x \geq 0$.

iii Write a short explanation of your result.

7 A straight line is drawn through the point (8, 2) to intersect the positive y-axis at Q and the positive x-axis at P. (In this problem we will determine the minimum value of $OP + OQ$.)



a Show that the derivative of $\frac{1}{\tan\theta}$ is $-\text{cosec}^2\theta$.

b Find MP in terms of θ.

c Find NQ in terms of θ.

d Hence find $OP + OQ$ in terms of θ. Denote $OP + OQ$ by x.

e Find $\frac{dx}{d\theta}$.

f Find the minimum value for x and the value of θ for which this occurs.

8 Let $f: R \to R$, $f(x) = e^x - e^{-x}$

a Find $f'(x)$. **b** Find $\{x: f(x) = 0\}$

c Show that $f'(x) > 0$ for all x. **d** Sketch the graph of f.

9 **a** Find all values of x for which $(\log_e x)^2 = 2\log_e x$

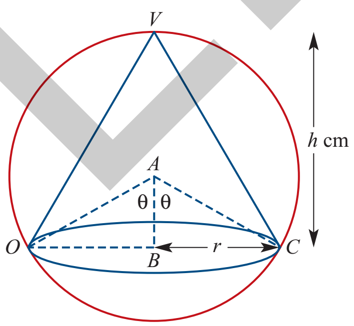
b Find the gradient of each of the curves $y = 2\log_e x$ and $y = (\log_e x)^2$ at the point $(1, 0)$.

c Use these results to sketch on one set of axes the graphs of $y = 2\log_e x$ and $y = (\log_e x)^2$

d Find $\{x: 2\log_e x > (\log_e x)^2\}$

10 For what value of x is $\dfrac{\log_e x}{x}$ a maximum? That is, when is the ratio of the logarithm of a number to the number a maximum?

11 A cone is inscribed inside a sphere as illustrated. The radius of the sphere is a cm, and the magnitude of $\angle OAB$ = magnitude of $\angle CAB = \theta$. The height of the cone is denoted by h. The radius of the cone is denoted by r.



a Find h, the height of the cone in terms of a and θ.

b Find r, the radius of the cone in terms of a and θ.

The volume, V cm^3, of the cone is given by $V = \frac{1}{3}\pi r^2 h$.

c Use the results from **a** and **b** to show that:

$$V = \frac{1}{3}\pi a^3 \sin^2\theta(1 + \cos\theta)$$

d Find $\dfrac{dV}{d\theta}$ (a is a constant) and hence find the value of θ for which the volume is a maximum.

e Find the maximum volume of the cone in terms of a.

12 A psychologist hypothesised that the ability of a mouse to memorise during the first 6 months of its life can be modelled by the function f, given by $f: (0, 6] \to R$, $f(x) = x\log_e x + 1$, i.e. the ability to memorise at age x years is $f(x)$.

a Find $f'(x)$

b Find the value of x for which $f'(x) = 0$ and hence find when the mouse's ability to memorise is a minimum.

c Sketch the graph of f.

d When is the mouse's ability to memorise a maximum in this period?

13 Some bacteria are introduced into a supply of fresh milk. After t hours there are y grams of bacteria present where $y = \dfrac{Ae^{bt}}{1 + Ae^{bt}}$ (1) and A and b are positive constants.

a Show that $0 < y < 1$ for all values of t.

b Find $\dfrac{dy}{dt}$ in terms of t.

c From (1) show that $Ae^{bt} = \dfrac{y}{1-y}$

d **i** Show that $\dfrac{dy}{dt} = by(1-y)$

ii Hence, or otherwise, show that the maximum value of $\dfrac{dy}{dt}$ occurs when $y = 0.5$.

e If $A = 0.01$ and $b = 0.7$ find when, to the nearest hour, the bacteria will be increasing at the fastest rate.

14 A searchlight is located at ground level vertically below the path of an approaching aircraft, which is flying at a constant speed of 400 m/s, at a height of 10 000 metres. If the light is continuously directed at the aircraft, find the rate, in degrees per second, at which the searchlight is turning at the instant when the aircraft is at a horizontal distance of 5000 m from the searchlight. Give your answer correct to three decimal places.

15 A rocket, R, rises vertically from level ground at a point A. It is observed from another point B on the ground where B is 10 km from A. When the angle of elevation ABR has the value $\frac{\pi}{4}$ radians, this angle is increasing at the rate of 0.005 radians per second. Find in, km/s, the velocity of the rocket at that instant.

16 **a** Find the equation of the tangent to the curve $y = e^x$ at the point $(1, e)$.

b Find the equation of the tangent to the curve $y = e^{2x}$ at the point $\left(\frac{1}{2}, e\right)$.

c Find the equation of the tangent to the curve $y = e^{kx}$ at the point $\left(\frac{1}{k}, e\right)$.

d Show that $y = xke$ is the only tangent to the curve $y = e^{kx}$ which passes through the origin.

e Hence determine for what values of k the equation $e^{kx} = x$ has:

i a unique real root **ii** no real roots

17 Let $f: R^+ \to R, f(x) = \frac{e^x}{x}$

a Find $f'(x)$ **b** Find $\{x: f'(x) = 0\}$

c Find the coordinates of the one stationary point and state its nature.

d **i** Find $\frac{f'(x)}{f(x)}$ **ii** Find $\lim_{x\to\infty} \frac{f'(x)}{f(x)}$ and comment.

e Sketch the graph of f.

f The number of birds (n) in an island colony decreased and increased with time (t) years according to the approximate formula

$$n = \frac{ae^{kt}}{t}$$

over some interval of years, where t is measured from 1900 and a and k are constant. If during this period the population was the same in 1965 as it was in 1930, when was it least?

18 A culture contains 1000 bacteria and 5 hours later the number has increased to 10 000. The number, N, of bacteria present at any time, t hours, is given by $N = Ae^{kt}$

a Find the values of A and k.

b Find the rate of growth at any time t.

c Show that the rate of growth is proportional to the number of bacteria present at any time.

d Find this rate of growth when:

i $t = 4$ **ii** $t = 50$

19 The populations of two ant colonies, A and B, are increasing according to the rules:

- A: population $= 2 \times 10^4 e^{0.03t}$
- B: population $= 10^4 e^{0.05t}$

After how many years will their populations:

a be equal? **b** be increasing at the same rate?

20 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by:

$$D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 24$$

a Sketch the graph of $D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

c Find the rate at which the depth is changing when:

i $t = 3$ **ii** $t = 6$ **iii** $t = 12$

d **i** At what times is the depth increasing most rapidly?

ii At what times is the depth decreasing most rapidly?

21 A particle on the end of a spring, which is hanging vertically, is oscillating so that its length h metres above the floor after t seconds is given by:

$$y = 0.5 + 0.2\sin(3\pi t), t \ge 0$$

a Find the greatest height above the floor and the time at which this height is first reached.

b Find the period of oscillation. **c** Find the speed of the particle when $t = \frac{1}{3}, \frac{2}{3}, \frac{1}{6}$.

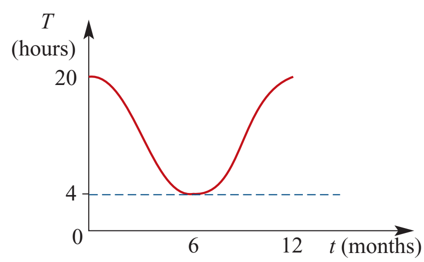
22 The length of night on Seal Island varies between 20 hours in midwinter and 4 hours in midsummer. The relationship between T, the number of hours of night, and t, the number of months past the longest night in 1991, is given by

$$T(t) = p + q\cos \pi r t$$

where p, q and r are constants.

Assume that the year consists of 12 months of equal length.

The graph of T against t is illustrated.



a Find the value of:

i r and **ii** p and q

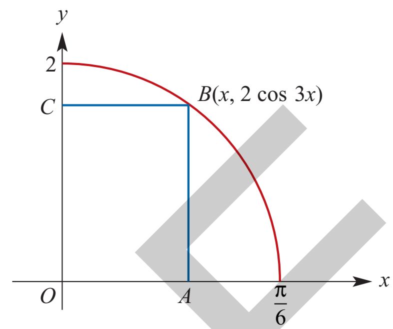
b Find $T'(3)$ and $T'(9)$ and find the rate of change of hours of night with respect to the number of months.

c Find the average rate of change of hours of night from $t = 0$ to $t = 6$.

d After how many months is the rate of change of hours of night a maximum?

Review

23 A section of the graph of $y = 2\cos 3x$ is shown in the diagram.



CAS CALCULATOR

a Show that the area, A, of the rectangle $OABC$ in terms of x is $2x\cos 3x$

b **i** Find $\frac{dA}{dx}$.

ii Find $\frac{dA}{dx}$ when $x = 0$ and $x = \frac{\pi}{6}$.

c **i** On a calculator plot the graph of $A = 2x\cos 3x$ for $x \in \left[0, \frac{\pi}{6}\right]$.

ii Find the two values of x for which the area of the rectangle is 0.2 square units.

iii Find the maximum area of the rectangle and the value of x for which this occurs.

d **i** Show that $\frac{dA}{dx} = 0$ is equivalent to $\tan 3x = \frac{1}{3x}$

ii Using a calculator plot the graphs of $y = \tan 3x$ and $y = \frac{1}{3x}$ for $x \in \left(0, \frac{\pi}{6}\right)$ and find the coordinates of the point of intersection.

24 **a** A population of insects grows according to the model

$$N(t) = 1000 - t + 2e^{\frac{t}{20}} \text{ for } t \geq 0$$

where t is the number of days after 1 January 2000.

i Find the rate of growth of the population as a function of t.

ii Find the minimum population size and value of t for which this occurs.

iii Find $N(0)$. **iv** Find $N(100)$.

v Sketch the graph of N against t for $0 \leq t \leq 100$.

b It is found that the change in population of another species is given by:

$$N_2(t) = 1000 - t^{\frac{1}{2}} + 2e^{\frac{t^{\frac{1}{2}}}{20}}$$

i Find $N_2(0)$. **ii** Find $N_2(100)$.

iii Plot the graph of $y = N_2(t)$ for $t \in [0, 5000]$ on a calculator.

iv Solve the equation $N_2'(t) = 0$ and hence give the minimum population of this species of insects.

CAS CALCULATOR

c A third model is:

$$N_3(t) = 1000 - t^{\frac{3}{2}} + 2e^{\frac{t}{20}}$$

Use a calculator to:

i plot a graph for $0 \leq t \leq 200$

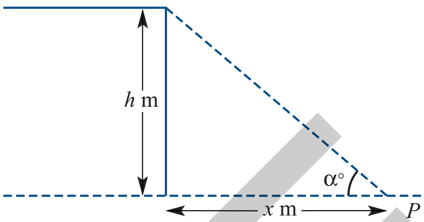
ii find the minimum population and the time at which this occurs

d **i** For N_3, find $N_3'(t)$.

ii Show $N_3'(t) = 0$ is equivalent to $t = 20\log_e(15\sqrt{t})$

Review

25 The height h m of a cliff is determined by measuring the angle of elevation of the top of the cliff from a point P level with the base of the cliff and at a distance x m from it.



a Show that if an error is made in the measurement of the angle by $(\delta\alpha)^\circ$, then the calculation of the height of the cliff yields an error of approximately $\frac{\pi x}{180}\sec^2(\alpha^\circ)\delta\alpha$

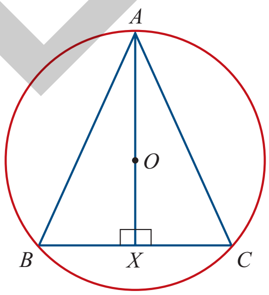
CAS CALCULATOR

b Let $x = 50$. Using a calculator plot the graph of:

$$y = \frac{\pi}{18} \times 5 \times \sec^2(\alpha^\circ) \text{ for } 0 < \alpha < 90$$

c If the greatest error in the measurement of the angle is $(0.02)^\circ$, what is the greatest possible error for the calculation of the height of the cliff in terms of α?

26 Triangle ABC is isosceles with $AB = AC$. O is the centre of the circumcircle of the triangle. The radius of the circle is R cm. Let $AX = h$ cm and angle $BAX = \alpha^\circ$ where X is the midpoint of BC.



a **i** Find BX in terms of h and α.

ii Apply Pythagoras' theorem to triangle OBX to find R in terms of h and α.

iii Use the result that $1 + \tan^2\alpha = \sec^2\alpha$ to show:

$$R = \frac{h}{2}\sec^2\alpha$$

b If the measure of h is exact but that of the semi-vertex angle α° is to have an error of $(\delta\alpha)^\circ$.

i Find the error in R, i.e. find δR.

ii Find δR if $h = 0$ and the semi-vertex angle of 30° is measured to be 29.1°.

27 **a** Consider the curve with equation $y = (2x^2 - 5x)e^{ax}$. If the curve passes through the point with coordinates (3, 10), find the value of a.

b **i** For the curve with equation $y = (2x^2 - 5x)e^{ax}$, find the x-axis intercepts.

ii Use calculus to find the x-values for which there is a turning point, in terms of a.