CHAPTER 2

Linear Relations

Objectives

- To calculate the gradient of a straight line.
- To interpret and use the general equation of a straight line \( y = mx + c \).
- To solve simultaneous linear equations graphically.
- To calculate the product of the gradients of two perpendicular straight lines.
- To find the distance between two points.
- To find the midpoint of a straight line.
- To calculate the angle between two intersecting straight lines.
- To apply a knowledge of linear relations to solving problems.

A relation is defined as a set of ordered pairs in the form \((x, y)\).

A rule relating the \(x\)-value to the \(y\)-value of each ordered pair sometimes exists, such as \(y = 2x + 1\), and this is a more convenient way of describing the relation.

A relation may also be represented graphically on a set of axes. If the resulting graph is a straight line, then the relation is called a linear relation.

2.1 The gradient of a straight line

Through any two points it is only possible to draw a single straight line. Therefore a straight line is defined by any two points on the line.

From previous work, you should be familiar with the concept of the gradient or slope of a line. The symbol used for gradient is \(m\). This may be defined as:

\[
\text{Gradient} = \frac{\text{rise}}{\text{run}}
\]
Hence given any two points on the line, \((x_1, y_1)\) and \((x_2, y_2)\), the gradient of the line can be found.

\[
\text{Gradient } m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}
\]

**Example 1**

Find the gradient of the given line.

**Solution**

Let \((x_2, y_2) = (0, 2)\) and \((x_1, y_1) = (-2, 0)\)

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{0 - (-2)} = \frac{2}{2} = 1
\]

**Example 2**

Find the gradient of the given line.

**Solution**

Let \((x_1, y_1) = (0, 3)\) and \((x_2, y_2) = (2, 0)\)

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3}{2 - 0} = \frac{-3}{2}
\]
Chapter 2 — Linear Relations

It should be noted that the gradient of a line that slopes upwards from left to right is **positive**, as illustrated in Example 1, and the gradient of a line that slopes downwards from left to right is **negative**, as illustrated in Example 2.

The gradient of a horizontal (parallel to the *x*-axis) line is zero, since \( y_2 - y_1 = 0 \).

The gradient of a vertical (parallel to the *y*-axis) line is undefined, since \( x_2 - x_1 = 0 \).

**Example 3**

Find the gradient of the line that passes through the points \((1, 6)\) and \((-3, 7)\).

**Solution**

The gradient \( m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 6}{-3 - 1} = \frac{1}{-4} = -\frac{1}{4} \).

**Exercise 2A**

1. Calculate the gradient of each of the following lines:
   a. ![Graph](image1.png)
   b. ![Graph](image2.png)
   c. ![Graph](image3.png)
   d. ![Graph](image4.png)
   e. ![Graph](image5.png)
   f. ![Graph](image6.png)
   g. ![Graph](image7.png)
   h. ![Graph](image8.png)
   i. ![Graph](image9.png)
2 Sketch a graph of a line with gradient 1.

3 Sketch a graph of a line with gradient 0 which passes through the point (1, 6).

4 For each of the following find the gradient of the line that passes through the two points with the given coordinates:
   - a (6, 3) (2, 4)
   - b (−3, 4) (1, −6)
   - c (6, 7) (11, −3)
   - d (5, 8) (6, 0)
   - e (6, 0) (−6, 0)
   - f (0, −6) (−6, 0)
   - g (3, 9) (4, 16)
   - h (5, 25) (6, 36)
   - i (−5, 25) (−8, 64)
   - j (1, 1) (10, 100)
   - k (1, 1) (10, 1000)
   - l (5, 125) (4, 64)

5 a Find the gradient of the straight line that passes through the points with coordinates (5a, 2a) and (3a, 6a).
   b Find the gradient of the straight line that passes through the points with coordinates (5a, 2a) and (5b, 2b).

6 a A line has gradient 6 and passes through the points with coordinates (−1, 6) and (7, a). Find the value of a.
   b A line has gradient −6 and passes through the points with coordinates (1, 6) and (b, 7). Find the value of b.

7 a Find the equation of the line that is parallel to the y-axis and passes through the point with coordinates (4, 7).
   b Find the equation of the line that is parallel to the x-axis and passes through the point with coordinates (−4, 11).

2.2 The general equation of a straight line

The general equation of a straight line is \( y = mx + c \), where \( m \) is the gradient of the line. This form, expressing the relation in terms of \( y \), is called the \textbf{gradient form}.

Let \( x = 0 \), then \( y = m(0) + c \)
and thus \( y = c \)
i.e. the y-axis intercept is equal to \( c \).
Example 4

Find the gradient and y-axis intercept of the graph of \( y = 3x - 4 \).

Solution

The value of \( m \) is 3 and the value of \( c \) is \(-4\).
Therefore the gradient of the above line is 3 and the y-axis intercept is \(-4\).

If the rule of a straight line is given, the graph can be sketched using the gradient and the y-axis intercept.

Example 5

Sketch the graph of \( y = 3x - 1 \).

Solution

Gradient = 3, i.e. \( \frac{\text{rise}}{\text{run}} = \frac{3}{1} \)

y-axis intercept = \(-1\)

Plot the point (0, \(-1\)), the y-axis intercept.

From there move across 1 (run) and up 3 (rise) to plot the point (1, 2).

If the equation for the straight line is not written in gradient form, to use the above method for sketching a graph, the equation must first be transposed into gradient form.

Example 6

Sketch the graph of \( 3y + 6x = 9 \).

Solution

First rearrange the equation into gradient form:

\[
3y + 6x = 9 \\
3y = 9 - 6x \\
y = \frac{9 - 6x}{3} \\
y = 3 - 2x \\
\text{i.e. } y = -2x + 3
\]

Therefore \( m = -2 \) and \( c = 3 \).
Using the TI-Nspire

First obtain the gradient form of the equation. To do this enter
\[ \text{solve}(3y + 6x = 9, y) \] to make \( y \) the subject.

Open a Graphs & Geometry application and define
\[ f(x) = 3 - 2x. \]

Note that the Window Settings will have to be changed if the axis intercepts do not appear on the screen.

The Entry Line can be hidden by pressing \( \text{ [/ G} \).

The axis intercepts can be found using the Intersection Point(s) menu. Select the \( x \)-axis and the graph to display the \( x \)-intercept point and select the \( y \)-axis and the graph to display the \( y \)-intercept point.

To show the coordinates of these points select the Coordinates and Equations menu and double-click on each of the points.

Press \( \text{ to exit the Coordinates and Equations menu.} \)
Using the Casio ClassPad

First obtain the gradient form of the equation. To do this in enter and highlight the equation. Then click Interactive—Equation/Inequality—solve and set the variable to y.

Copy the part after the = sign and paste it in as equation y1.

Check the tickbox and click on the button.

The window setting may be altered by using the button (if this button does not appear, click on the graph window first to select it).

The axis intercepts can be found in the graph window. Make sure the graph window is selected and that the intercepts are visible on the graph. Click Analysis—G-solve and select y-Intercept for the y-axis intercept and Root for the x-axis intercept.
36 Essential Mathematical Methods 1 & 2 CAS

Parallel lines

If the value of \( m \) is the same for two rules, then the lines are parallel.

For example, consider the lines with the following rules:

\[
\begin{align*}
  y &= 2x + 3 \\
  y &= 2x - 4
\end{align*}
\]

Exercise 2B

1 Sketch the graphs of each of the following:
   a) \( y = x + 2 \)  
   b) \( y = -x + 2 \)  
   c) \( y = 2x + 1 \)  
   d) \( y = -2x + 1 \)

2 Sketch the graphs of each of the following using the gradient form, \( y = mx + c \):
   a) \( x + y = 1 \)  
   b) \( x - y = 1 \)  
   c) \( y - x = 1 \)  
   d) \( -x - y = 1 \)

3 Sketch the graphs of each of the following using \( y = mx + c \):
   a) \( y = x + 3 \)  
   b) \( y = 3x + 1 \)  
   c) \( y = 4 - \frac{1}{2}x \)  
   d) \( y = 3x - 2 \)
   e) \( 4y + 2x = 12 \)  
   f) \( 3x + 6y = 12 \)  
   g) \( 4y - 6x = 24 \)  
   h) \( 8x - 3y = 24 \)

4 For which of the following pairs of equations are the corresponding lines parallel to each other? Sketch graphs to show the non-parallel lines.
   a) \( 2y = 6x + 4 \); \( y = 3x + 4 \)  
   b) \( x = 4 - y \); \( 2x + 2y = 6 \)
   c) \( 3y - 2x = 12 \); \( y + \frac{1}{3}x = \frac{2}{3} \)  
   d) \( 4y - 3x = 4 \); \( 3y = 4x - 3 \)

5 For which of the following do the lines pass through the origin?
   a) \( y + x = 1 \)  
   b) \( y + 2x = 2(x + 1) \)  
   c) \( x + y = 0 \)  
   d) \( x - y = 1 \)

6 Give the gradient for each of the lines in Question 5.

2.3 Finding the equation of a straight line

The equation of a straight line may be found if the gradient and \( y \)-axis intercept are known.

Example 7

Find the equation if \( m = -3 \) and \( c = 10 \).

Solution

Equation is \( y = -3x + 10 \).
Example 8

Find the equation of the straight line with gradient $-3$ which passes through the point with coordinates $(-5, 10)$.

Solution

The general equation of lines with gradient $-3$ is $y = -3x + c$. If a point on the line is given the value of $c$ can be determined.

When $x = -5$, $y = 10$

Thus $10 = -3 \times -5 + c$.

Solving for $c$: $10 - 15 = c$

and therefore $c = -5$. The equation of the line is $y = -3x - 5$.

Example 9

Find the equation of the straight line with $y$-axis intercept $-3$ which passes through the point with coordinates $(1, 10)$.

Solution

The general equation of lines with $y$-axis intercept $-3$ is $y = mx - 3$.

The line passes through the points with coordinates $(0, -3)$ and $(1, 10)$.

Therefore the gradient $m = \frac{10 - (-3)}{1 - 0} = 13$.

The equation is $y = 13x - 3$.

In general the equation of a straight line can be determined by two ‘independent pieces of information’. Two cases are considered below.

Case 1  Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$

Using these two points, the gradient of the line $AB$ can be determined:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Using the general point $P(x, y)$, also on the line,

$$m = \frac{y - y_1}{x - x_1}$$

Therefore the equation of the line is

$$y - y_1 = m(x - x_1)$$

where $m = \frac{y_2 - y_1}{x_2 - x_1}$
Example 10

Find the equation of the straight line passing through the points (1, –2) and (3, 2).

**Solution**

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-2)}{3 - 1} = \frac{4}{2} = 2
\]

\[
\therefore 2 = \frac{y - (-2)}{x - 1} \quad \therefore 2x - 2 = y + 2
\]

\[
\therefore y = 2x - 4
\]

**Case 2**  Given the gradient \( m \) and one other point, \( A(x_1, y_1) \)

As the gradient \( m \) is already known, the rule can be found using \( y - y_1 = m(x - x_1) \).

Example 11

Find the equation of the line that passes through the point (3, 2) and has a gradient of \(-2\).

**Solution**

\[
y - 2 = -2(x - 3)
\]

\[
y - 2 = -2x + 6
\]

\[
y = -2x + 8
\]

\[
y = -2x + 8 \text{ is the equation}
\]

which could also be expressed as

\[
y + 2x - 8 = 0
\]

The equation of a straight line can also be found from the graph by reading off two points and using them to find the equation as outlined above.
Example 12

Find the equation of the line shown in the graph.

Solution

From the graph it can be seen that the $y$-axis intercept is $(0, 4)$, i.e. $c = 4$.

As the coordinates of $A$ and $B$ are $(0, 4)$ and $(2, 0)$:

Gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0 - 4}{2 - 0}$

$= -\frac{4}{2}$

$= -2$

The equation of the line is $y = -2x + 4$.

Vertical and horizontal lines

If $m = 0$, then the line is horizontal and the equation is simply $y = c$, where $c$ is the $y$-axis intercept.

If the line is vertical, the gradient is undefined and its rule is given as $x = a$, where $a$ is the $x$-axis intercept.

Exercise 2C

1. a Find the equation of the straight line with gradient $3$ and $y$-axis intercept $5$.
   b Find the equation of the straight line with gradient $-4$ and $y$-axis intercept $6$.
   c Find the equation of the straight line with gradient $3$ and $y$-axis intercept $-4$.

2. a Find the equation of the straight line with gradient $3$ and which passes through the point with coordinates $(6, 7)$.
   b Find the equation of the straight line with gradient $-2$ and which passes through the point with coordinates $(1, 7)$.
3. For the straight line with y-axis intercept 6 and passing through the point with coordinates (1, 8) find:
   a. the gradient
   b. the equation

4. For the straight line with y-axis intercept 6 and passing through the point with coordinates (1, 4) find:
   a. the gradient
   b. the equation

5. Find the equation of the straight line that passes through the point with (1, 6) and has gradient:
   a. 2
   b. -2

6. Find the equation of each of the following lines:
   a. 
   ![Graph A](image1)
   b. 
   ![Graph B](image2)
   c. 
   ![Graph C](image3)
   d. 
   ![Graph D](image4)
   e. 
   ![Graph E](image5)
   f. 
   ![Graph F](image6)

7. Write another equation that would give a parallel line for each of those shown in Question 6. Check your answers by sketching graphs.

8. Find the equations of the following straight lines. (Hint: It may help to sketch the graphs.)
   a. Gradient \( \frac{3}{4} \), passing through (-6, 5)
   b. Gradient \( -\frac{1}{2} \), passing through (4, -3)
   c. Gradient 0, passing through (0, 3)
   d. Gradient 0, passing through (0, -3)

9. Write, in the form \( y = mx + c \), the equations of lines which have the given gradient and pass through the given point:
   a. \( \frac{1}{3} \), (0, 3)
   b. \( -\frac{1}{2} \), (0, 3)
   c. -0.7, (1, 6)
   d. \( \frac{1}{2} \), (4, 3)
   e. \( \frac{3}{4} \), (4, 3)
   f. -1, (0, 0)

10. Find equations defining the lines which pass through the following pairs of points:
    a. (0, 4), (6, 0)
    b. (-3, 0), (0, -6)
    c. (4, 0), (4, 2)
    d. (2, 6), (5, 3)
11 Find the equations, in the form $y = mx + c$, of the lines which pass through the following pairs of points:

**a** $(0, 4), (3, 6)$

**b** $(1, 0), (4, 2)$

**c** $(-3, 0), (3, 3)$

**d** $(-2, 3), (4, 0)$

**e** $(-1.5, 2), (4.5, 8)$

**f** $(-3, 1.75), (4.5, -2)$

12 Do the points $P(1, -3), Q(2, 1)$ and $R\left(\frac{1}{2}, 3\right)$ lie on the same straight line?

13 Find the equations defining each of the three sides of the triangle $ABC$ for which the coordinates of the vertices are $A(-2, -1), B(4, 3)$ and $C(6, 0)$.

### 2.4 Equation of a straight line in intercept form and sketching graphs

Often we encounter a linear relation that is not expressed in the form $y = mx + c$. An alternative standard notation is

$$ax + by = c$$

This is sometimes referred to as the **intercept form**.

While it is necessary to transpose the equation into gradient form if you wish to find the gradient, it is often convenient to work with linear relations in the intercept form.

### Sketching graphs in intercept form

A convenient way to sketch graphs of straight lines is to plot the two axes intercepts.

#### Example 15

Sketch the graph of $2x + 4y = 10$.

**Solution**

$x$-axis intercept ($y = 0$): $2x + 4(0) = 10$

$x = 5$

$y$-axis intercept ($x = 0$): $2(0) + 4y = 10$

$y = 2.5$

When finding the equation of a straight line, it is also sometimes more convenient to express it in intercept form.

#### Example 16

Find the equation, in intercept form, of the line passing through the points $A(2, 5)$ and $B(6, 8)$.

**Solution**

$$m = \frac{8 - 5}{6 - 2} = \frac{3}{4}$$
Therefore, using the gradient and the point $A(2, 5)$, we have the equation:

$$y - 5 = \frac{3}{4}(x - 2)$$
$$4(y - 5) = 3(x - 2)$$
$$4y - 20 = 3x - 6$$
$$4y - 3x = 14$$
$$-3x + 4y = 14$$

Straight lines may be sketched by finding the axes intercepts even when the equation is given in the form $y = mx + c$.

**Example 17**

Sketch the graph of $y = 2x - 6$, by first finding the intercepts.

**Solution**

When $x = 0$, $y = -6$.

Hence the $y$-axis intercept is $-6$.

When $y = 0$, $2x - 6 = 0$

$\therefore 2x = 6$ and $x = 3$

The $x$-axis intercept is 3.

**Exercise 2D**

1. For each of the following give the coordinates of the axes intercepts:
   - a $x + y = 4$
   - b $x - y = 4$
   - c $-x - y = 6$
   - d $y - x = 8$

2. For each of the following find the equation of the straight line graph passing through the points $A$ and $B$:
   - a $A(0, 6)$ and $B(3, 0)$
   - b $A(0, -2)$ and $B(4, 0)$
   - c $A(2, 2)$ and $B(6, 6)$
   - d $A(2, -2)$ and $B(-6, 6)$

3. For each of the following sketch the graph by first finding the axes intercepts:
   - a $y = x - 1$
   - b $y = x + 2$
   - c $y = 2x - 4$

4. Sketch the graphs of each of the following linear relations:
   - a $2x - 3y = 12$
   - b $x - 4y = 8$
   - c $-3x + 4y = 24$
   - d $-5x + 2y = 20$
   - e $4x - 3y = 15$
   - f $7x - 2y = 15$
5  Find the equations of the straight lines passing through the following pairs of points.  
(Express your answer in intercept form.)

a  \((-1, 4), (2, 3)\)  b  \((0, 4), (5, -3)\)  c  \((3, -2), (4, -4)\)  d  \((5, -2), (8, 9)\)

6  Transpose from the intercept form to the gradient form and hence state the gradient of each
of the following linear relations:

a  \(2x - y = 9\)  b  \(3x + 4y = 10\)  c  \(-x - 3y = 6\)  d  \(5x - 2y = 4\)

7  Sketch the graphs of each of the following by first determining the axes intercepts:

a  \(y = 2x - 10\)  b  \(y = 3x - 9\)  c  \(y = 5x + 10\)  d  \(y = -2x + 10\)

8  A straight line has equation \(y = 3x - 4\). The points with coordinates \((0, a), (b, 0), (1, d)\)
and \((e, 10)\) lie on the line. Find the values of \(a, b, d\) and \(e\).

2.5  Linear models

In many practical situations a linear function can be used.

**Example 18**

Austcom’s rates for local calls from private telephones consist of a quarterly rental fee of $40
plus 25c for every call. Construct a cost function that describes the quarterly telephone bill.

**Solution**

Let \(C = \) cost ($) of quarterly telephone bill  
\(n = \) number of calls  
then:  \(C = 0.25n + 40\)

As the number of calls is counted in whole
numbers only, the domain of this function is \(N \cup \{0\}\).

Draw the graph of the function  
\(C = 0.25n + 40, n \in N \cup \{0\}\)

**Note:** The graph should be a series of discrete points rather than a continuous line
because \(n \in N \cup \{0\}\). With the scale used it is not practical to show it correctly.

**Example 19**

The tyres on a racing car had lost 3 mm of tread after completing 250 km of a race and 4 mm
of tread after completing 1000 km. Assuming that the loss of tread was proportional to the
distance covered, find the total loss of tread, \(d\) mm, after \(s\) km from the start of the race. What
would be the tread loss by the end of a 2000 km race?
### Solution

Gradient \( m = \frac{d \text{ increase}}{s \text{ increase}} = \frac{4 - 3}{1000 - 250} = \frac{1}{750} \)

\[ d = \frac{1}{750} s + c \]

When \( s = 250, d = 3 \)

\[ 3 = \frac{1}{3} + c \]

\[ c = \frac{2}{3} \]

Total loss of tread after \( s \) km, \( d = \frac{1}{750} s + \frac{2}{3} \)

When \( s = 2000, d = \frac{1}{750} \times 2000 + \frac{2}{3} \)

\[ = \frac{2}{3} + \frac{2}{3} \]

\[ = \frac{5}{3} \]

\[ \therefore \text{the loss of tread at the end of a 2000 km race is } 5\frac{1}{3} \text{ mm.} \]

An important linear relation is the relation between distance travelled and time taken when an object is travelling with constant speed. If a car travels at 40 km/h, the relationship between distance travelled (\( s \) kilometres) and time taken (\( t \) hours) is \( s = 40t, t \geq 0 \). The graph of \( s \) against \( t \) is a straight line graph through the origin. The gradient of the graph is 40.

### Example 20

A car starts from point \( A \) on a highway 10 kilometres past the Wangaratta post office. The car travels at an average speed of 90 km/h towards picnic stop \( B \), which is 120 kilometres further on from \( A \). Let \( t \) hours be the time after the car leaves point \( A \).

a Find an expression for the distance \( d_1 \) of the car from the post office at time \( t \) hours.

b Find an expression for the distance \( d_2 \) of the car from point \( B \) at time \( t \) hours.

c On separate sets of axes sketch the graphs of \( d_1 \) against \( t \) and \( d_2 \) against \( t \) and state the gradient of each graph.

### Solution

a At time \( t \) the distance of the car from the post office is \( 10 + 90t \) kilometres.

b At time \( t \) the distance of the car from \( B \) is \( 120 - 90t \) kilometres.
Chapter 2 — Linear Relations

Exercise 2E

1. a A train moves at 50 km/h in a straight line away from town. Give a rule for the distance, \( d \) km, from the town at time \( t \) hours after leaving the town.
b A train has stopped at a siding 5 km from the town and then moves at 40 km/h in a straight line away from the siding. Give a rule for the distance, \( d \) km, from the town at time \( t \) hours after leaving the siding.

2. a An initially empty container is being filled with water at a rate of 5 litres per minute. Give a rule for the volume, \( V \) litres, of water in the container at time \( t \) minutes after the filling of the container starts.
b A container contains 10 litres of water. Water is then poured in at a rate of 5 litres per minute. Give a rule for the volume, \( V \) litres, of water in the container at time \( t \) minutes after the pouring starts.

3. The weekly wage, \( w \), of a vacuum cleaner salesperson consists of a fixed sum of $350 plus $20 for each cleaner sold. If \( n \) cleaners are sold per week, construct a rule that describes the weekly wage of the salesperson.

4. The reservoir feeding an intravenous drip contains 500 mL of a saline solution. The drip releases the solution into a patient at the rate of 2.5 mL/minute.
a Construct a rule which relates \( v \), the amount of solution left in the reservoir, to time, \( t \) minutes.
b State the possible values of \( t \) and \( v \). c Sketch the graph of the relation.

5. The cost (\( C \)) of hiring a taxi consists of two elements, a fixed flagfall and an amount that varies with the number (\( n \)) of kilometres travelled. If the flagfall is $2.60 and the cost per kilometre is $1.50, determine a rule which gives \( C \) in terms of \( n \).

6. A car rental company charges $85, plus an additional amount of 24c per kilometre.
a Write a rule to determine the total charge \( C \) for hiring a car and travelling \( x \) kilometres.
b What would be the cost to travel 250 kilometres?

7. Two towns \( A \) and \( B \) are 200 km apart. A man leaves town \( A \) and drives at a speed of 5 km/h towards town \( B \). Find the distance of the man from town \( B \) at time \( t \) hours after leaving town \( A \).
8 The following table shows the extension of a spring when weights are attached to it.

<table>
<thead>
<tr>
<th>$x$, extension (cm)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$, weight (g)</td>
<td>50</td>
<td>50.2</td>
<td>50.4</td>
<td>50.6</td>
<td>50.8</td>
<td>51.0</td>
<td>51.2</td>
</tr>
</tbody>
</table>

a Sketch a graph to show the relationship between $x$ and $w$.
b Write a rule that describes the graph.
c What will be the extension if $w = 52.5$ g?

9 A printing firm charges $35 for printing 600 sheets of headed notepaper and $47 for printing 800 sheets.
a Find a formula, assuming the relationship is linear, for the charge, $C$, in terms of number of sheets printed, $n$.
b How much would they charge for printing 1000 sheets?

10 An electronic bankteller registered $775 after it had counted 120 notes and $975 after it had counted 160 notes.
a Find a formula for the sum registered ($C$) in terms of the number of notes ($n$) counted.
b Was there a sum already on the register when counting began?
c If so, how much?

2.6 Problems involving simultaneous linear models

Example 21

There are two possible methods for paying gas bills.
Method $A$: A fixed charge of $25 per quarter + 50c per unit of gas used
Method $B$: A fixed charge of $50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method $B$ becomes cheaper than method $A$.

Solution

Let $C_1 =$ charge in $\$ using method $A$
$C_2 =$ charge in $\$ using method $B$
$x =$ number of units of gas used

Now $C_1 = 25 + 0.5x$
$C_2 = 50 + 0.25x$

It can be seen from the graph that if the number of units exceeds 100 method $B$ is cheaper.
Chapter 2 — Linear Relations

The solution can be obtained by solving simultaneous linear equations:

\[
\begin{align*}
C_1 &= C_2 \\
25 + 0.5x &= 50 + 0.25x \\
0.25x &= 25 \\
x &= 100
\end{align*}
\]

**Example 22**

Robyn and Cheryl race over 100 metres. Robyn runs so that it takes \(a\) seconds to run 1 metre and Cheryl runs so that it takes \(b\) seconds to run 1 metre. Cheryl wins the race by 1 second. The next day they again race over 100 metres but Cheryl gives Robyn a 5 metre start so that Robyn runs 95 metres. Cheryl wins this race by 0.4 second. Find the values of \(a\) and \(b\) and the speed at which Robyn runs.

**Solution**

For the first race: \(100a - 100b = 1\) \hspace{1cm} (1)

For the second race: \(95a - 100b = 0.4\) \hspace{1cm} (2)

Subtract (2) from (1). Hence \(5a = 0.6\) and \(a = 0.12\).

Substitute in (1) to find \(b = 0.11\).

Robyn’s speed = \(\frac{1}{0.12} = \frac{25}{3}\) metres per second.

**Exercise 2F**

1. Two bicycle hire companies have different charges. Company A charges $C$, according to the rule \(C = 10t + 20\), where \(t\) is the time in hours for which a bicycle is hired. Company B charges $C$, according to the rule \(C = 8t + 30\).
   a. Sketch each of the graphs on the same set of axes.
   b. Find the time, \(t\), for which the charge of both companies is the same.

2. The distances, \(d_A\) km and \(d_B\) km, of cyclists A and B travelling along a straight road from a town hall step are given respectively by \(d_A = 10t + 15\) and \(d_B = 20t + 5\), where \(t\) is the time in hours after 1.00 pm.
   a. Sketch each of the graphs on the one set of axes.
   b. Find the time in hours at which the two cyclists are at the same distance from the town hall step.

3. A helicopter can be hired for $210 per day plus a distance charge of $1.60 per km or, alternatively, at a fixed charge of $330 per day for an unlimited distance.
For each of the methods of hiring, find an expression for cost, $C$, in terms of $x$ km, the distance travelled.

On one set of axes, draw the graph of cost versus distance travelled for each of the methods.

Determine for what distances the fixed-charge method is cheaper.

Three power boats in a 500 km handicap race leave at 5 hourly intervals. Boat A leaves first and has a speed for the race of 20 km/h. Boat B leaves 5 hours later and travels at an average speed of 25 km/h. Boat C leaves last, 5 hours after B, and completes the race at a speed of 40 km/h.

Draw a graph of each boat’s journey on the same set of axes.

Use your graphs to find the winner of the race.

Check your answer algebraically.

Write a short description of what happened to each boat in the race.

If the line $OT$ has the equation $y = \frac{3}{4}x$ and the line $HT$ has the equation $y = \frac{3}{2}x - 12$, determine the point over which both craft would pass.

A school wishes to take some of its students on an excursion. If they travel by tram it will cost the school $2.80 per student. Alternatively, the school can hire a bus at a cost of $54 for the day plus a charge of $1 per student.

For each mode of transport, write an expression for the cost ($C$) of transport in terms of the number of students ($x$).

On one set of axes, draw the graph of cost, $C$, versus number of students, $x$, for each mode of transport.

Determine for how many students it will be more economical to hire the bus.

Anne and Maureen live in towns that are 57 km apart. Anne sets out at 9.00 am one day to ride her bike to Maureen’s town at a constant speed of 20 km/h. At the same time Maureen sets out to ride to Anne’s town at a constant speed of 18 km/h.

Write down a rule for the distance, $d$ km, that each of them is from Anne’s place at a time $t$ minutes after 9.00 am.

On the same set of axes, draw graphs of the distance, $d$ km, versus time, $t$ minutes after 9.00 am, for each cyclist.
c Find the time at which they will meet.
d How far has each of them travelled when they meet?

**Example 22**

John and Michael race over 50 metres. John runs so that it takes $a$ seconds to run 1 metre and Michael runs so that it takes $b$ seconds to run 1 metre. Michael wins the race by 1 second. The next day they again race over 50 metres but Michael gives John a 3 metre start so that John runs 47 metres. Michael wins this race by 0.1 second. Find the values of $a$ and $b$ and the speed at which Michael runs.

**2.7 The tangent of the angle of slope and perpendicular lines**

From year 10 you will be familiar with the trigonometric ratio

\[ \tan \theta = \frac{\text{opposite}}{\text{adjacent}} \]

The gradient, $m$, of a straight line is given by

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_2 \neq x_1) \]

It therefore follows that $m = \tan \theta$, where $\theta$ is the angle that the line makes with the positive direction of the $x$-axis.

If $m$ is positive, $\theta$ will be an acute angle. If $m$ is negative, $\theta$ will be an obtuse angle.

It follows that the value of $m$ in the general equation of a straight line $y = mx + c$ gives the value of the tangent of the angle made by the line and the $x$-axis.

**Example 23**

Determine the gradient of the line passing through the given points and the angle the line makes with the positive direction of the $x$-axis for:

**a** (3, 2) and (5, 7)

**b** (5, −3) and (−1, 5)
Essential Mathematical Methods 1 & 2 CAS

Solution

\[ a \quad m = \frac{7 - 2}{5 - 3} = 2.5 \]

Tangent of angle = 2.5

\[ \therefore \text{Angle} = 68.20^\circ \]

\[ b \quad m = \frac{5 - (-3)}{-1 - 5} = -\frac{8}{6} \]

Tangent of angle = \(-\frac{4}{3}\)

This implies the angle is obtuse

\[ \therefore \text{angle} = 180^\circ - (53.130\ldots)^\circ = 126.87^\circ \]

Example 24

Find the magnitude of the angle each of the following make with the positive direction of the x-axis:

\[ a \quad y = 2x + 3 \quad b \quad 3y = 3x - 6 \quad c \quad y = -0.3x + 1.5 \]

Solution

\[ a \quad y = 2x + 3 \quad \text{Gradient} = 2 \]

\[ \therefore \text{Tangent of angle} = 2 \]

\[ \text{giving an angle of } 63.43^\circ (63^\circ 26') \]

\[ b \quad 3y = 3x - 6 \quad y = x - 2 \]

\[ \therefore \text{Tangent of angle} = 1 \]

\[ \text{giving an angle of } 45^\circ \]

\[ c \quad y = -0.3x + 1.5 \quad \text{Gradient} = -0.3 \]

\[ \therefore \text{Tangent of angle} = -0.3 \]

\[ \text{giving an angle of } (180 - 16.7)^\circ \text{ with the positive direction of the } x\text{-axis} \]

\[ = 163.3^\circ (163^\circ 18') \]

Perpendicular lines

Multiplying the gradients of two perpendicular straight lines leads to a useful result.

Gradient of (1) = \(\tan \theta\)

\[ a \quad b \quad m_1 \]

Gradient of (2) = \(-\frac{a}{c}\)

\[ = -\tan \alpha \]

\[ = \frac{b}{a} \]

\[ = \frac{1}{m_1} \]

\[ \therefore \quad m_1m_2 = -1 \]
If two straight lines are perpendicular, the product of their gradients is $-1$.
Conversely, if the product of the gradients of two lines is $-1$, then the two lines are perpendicular.

**Note:** This result holds if one of the two lines is not parallel to an axis.

### Example 25

Find the equation of the straight line which passes through $(1, 2)$ and is:

- **a** parallel to the line with equation $2x - y = 4$
- **b** perpendicular to the line with equation $2x - y = 4$.

#### Solution

The equation $2x - y = 4$ can be rearranged to $y = 2x - 4$. Hence the gradient of the line can be seen to be $2$.

**a** Therefore the gradient of any line parallel to this line is $2$.

The equation of the straight line with this gradient and passing through the point with coordinates $(1, 2)$ is:

$$y - 2 = 2(x - 1)$$

Therefore $y = 2x$ is the equation of the line which passes through $(1, 2)$ and is parallel to the line with equation $2x - y = 4$.

**b** The gradient of any line perpendicular to the line with equation $y = 2x - 4$ is $\frac{1}{2}$.

The equation of the straight line with this gradient and passing through the point with coordinates $(1, 2)$ is:

$$y - 2 = \frac{1}{2}(x - 1)$$

Therefore $2y - 4 = -x + 1$ and equivalently $2y + x = 5$.

Therefore $2y + x = 5$ is the equation of the line which passes through $(1, 2)$ and is perpendicular to the line with equation $2x - y = 4$.

### Example 26

The coordinates of the vertices of a triangle $ABC$ are $A(0, -1), B(2, 3)$ and $C\left(3, -\frac{3}{2}\right)$. Show that the side $AB$ is perpendicular to side $AC$.

#### Solution

**Note:** $m_{AB}$ is the gradient of the line $AB$.

$$m_{AB} = \frac{3 - (-1)}{2 - 0}$$

$$= \frac{4}{2}$$

$$= 2$$
\[ m_{AC} = \frac{-2 - (-1)}{2 - 0} \]
\[ = \frac{-1}{2} \]
\[ = -\frac{1}{2} \]

Since \( m_{AB} \times m_{AC} = 2 \times -\frac{1}{2} = -1 \), the lines \( AB \) and \( AC \) are perpendicular to each other.

### Exercise 2G

1. Find the angle that the lines joining the given points make with the positive direction of the \( x \)-axis:
   a. (0, 3), (3, 0)
   b. (0, -4), (4, 0)
   c. (0, 2), (-4, 0)
   d. (0, -5), (-5, 0)

2. Find the magnitude of the angle made by each of the following with the positive direction of the \( x \)-axis:
   a. \( y = x \)
   b. \( y = -x \)
   c. \( y = x + 1 \)
   d. \( x + y = 1 \)
   e. \( y = 2x \)
   f. \( y = -2x \)

3. Find the angle that the lines joining the given points make with the positive direction of the \( x \)-axis:
   a. (-4, -2), (6, 8)
   b. (2, 6), (-2, 4)
   c. (-3, 4), (6, 1)
   d. (-4, -3), (2, 4)
   e. (3b, a), (3a, b)
   f. (c, b), (b, c)

4. Find the magnitude of the angle made by each of the following with the positive direction of the \( x \)-axis:
   a. \( y = 3x + 2 \)
   b. \( 2y = -2x + 1 \)
   c. \( 2y - 2x = 6 \)
   d. \( 3y + x = 7 \)

5. If the points \( A, B \) and \( C \) have the coordinates \( A(5, 2), B(2, -3) \) and \( C(-8, 3) \), show that the triangle \( ABC \) is a right-angled triangle.

6. Show that \( RSTU \) is a rectangle if the coordinates of the vertices are respectively \( R(2, 6), S(6, 4), T(2, -4) \) and \( U(-2, -2) \).

7. Find the equation of the straight line which passes through the point (1, 4) and is perpendicular to the line with equation \( y = -\frac{1}{2}x + 6 \).

8. Find the equation of the straight line which passes through (4, -2) and is:
   a. parallel to the line with equation \( 2x - 3y = 4 \)
   b. perpendicular to the line with equation \( 2x - 3y = 4 \).
Given that the lines $4x - 3y = 10$ and $4x - ly = m$ are perpendicular and intersect at the point $(4, 2)$, find the values of $l$ and $m$.

2.8 The distance between two points

The distance between the given points $A(x_1, y_1)$ and $B(x_2, y_2)$ can be found by applying the Theorem of Pythagoras to triangle $ABC$:

$$AB^2 = AC^2 + BC^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

∴ The distance between the two points $A$ and $B$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 27

Calculate the distance $EF$ if $E$ is $(-3, 2)$ and $F$ is $(4, -2)$.

Solution

$$EF = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4 - (-3))^2 + (-2 - 2)^2}$$
$$= \sqrt{7^2 + (-4)^2}$$
$$= \sqrt{65}$$
$$= 8.06 \text{ (to 2 decimal places)}$$

Exercise 2H

1. Find the distance between each of the following (correct to 2 decimal places):
   - a (3, 6) and (-4, 5)
   - b (4, 1) and (5, -3)
   - c (-2, -3) and (-5, -8)
   - d (6, 4) and (-7, 4)

2. Calculate the perimeter of a triangle with vertices $(-3, -4)$, $(1, 5)$ and $(7, -2)$.

3. There is an off-shore oil drilling platform in Bass Strait situated at $D(0, 6)$, where 1 unit = 5 km. Pipes for this oil drill come ashore at $M(-6, 1)$ and $N(3, -1)$. Assuming the pipelines are straight, which is the shorter $DM$ or $DN$?
2.9 **Midpoint of a line segment**

Let \( P(x, y) \) be the midpoint of the line segment joining \( A(x_1, y_1) \) and \( B(x_2, y_2) \). The triangles \( APC \) and \( PBD \) are congruent.

\[
\begin{align*}
AC &= PD & \text{and} & \quad PC &= BD \\
\therefore \quad x - x_1 &= x_2 - x & \quad y - y_1 &= y_2 - y \\
2x &= x_1 + x_2 & \quad 2y &= y_1 + y_2 \\
x &= \frac{x_1 + x_2}{2} & \quad y &= \frac{y_1 + y_2}{2}
\end{align*}
\]

To find the midpoint \((x, y)\)

\[
\begin{align*}
x &= \frac{x_1 + x_2}{2} & \quad y &= \frac{y_1 + y_2}{2}
\end{align*}
\]

**Example 28**

Find the midpoint of the line segment joining \( A(2, 6) \) with \( B(-3, -4) \).

**Solution**

Midpoint of line segment \( AB \) has coordinates \(( \frac{2 + (-3)}{2}, \frac{6 + (-4)}{2} ) = \left( \frac{-1}{2}, 1 \right) \).

**Exercise 2I**

1. Find the coordinates of \( M \), the midpoint of \( AB \), where \( A \) and \( B \) have the following coordinates:
   - \( A(2, 12), B(8, 4) \)
   - \( A(-3, 5), B(4, -4) \)
   - \( A(-1.6, 3.4), B(4.8, -2) \)
   - \( A(3.6, -2.8), B(-5, 4.5) \)

2. Find the midpoints of each of the sides of a triangle \( ABC \), where \( A \) is \((1, 1)\), \( B \) is \((5, 5)\) and \( C \) is \((11, 2)\).

3. The secretary of a motorcross club wants to organise two meetings on the same weekend. One is a hill climb starting from point \( A(3.1, 7.1) \) and the other is a circuit event with the start at \( B(8.9, 10.5) \), as shown on the map. Only one ambulance can be provided. The ambulance can be called up by radio, so it is decided to keep it at \( C \), halfway between \( A \) and \( B \). What are the coordinates of \( C \)?
4 The diagram shows the four points \( A(6, 6) \), \( B(10, 2) \), \( C(-1, 5) \) and \( D(-7, 1) \).
   a If the midpoint of \( AB \) is \( P \) and the midpoint of \( CD \) is \( M \), calculate the distance \( PM \).
   b Does the line joining these midpoints pass through \( \left(0, 3\frac{1}{4}\right) \)?

5 If \( M \) is the midpoint of \( XY \), find the coordinates of \( Y \) when \( X \) and \( M \) have the following values:
   a \( X(-4, 2), M(0, 3) \)
   b \( X(-1, -3), M(0.5, -1.6) \)
   c \( X(6, -3), M(2, 1) \)
   d \( X(4, -3), M(0, -3) \)

6 Find the coordinates of the midpoint of the line joining \( (1, 4) \) and \( (a, b) \), in terms of \( a \) and \( b \). If \( (5, -1) \) is the midpoint find the values of \( a \) and \( b \).

2.10 **Angle between intersecting lines**

It is possible to use coordinate geometry to find the angle between two intersecting lines.

Let \( \alpha = \text{angle between the intersecting lines } AB \text{ and } CD \).

\[ \theta_1 = \text{angle between the positive direction of the } x \text{-axis and the line } AB. \]

\[ \theta_2 = \text{angle between the positive direction of the } x \text{-axis and the line } CD. \]

Then \( \theta_1 + \alpha = \theta_2 \). (The exterior angle of a triangle equals the sum of the two opposite interior angles.)

\[ \therefore \alpha = \theta_2 - \theta_1 \]
Example 29

Find the two angles between the intersecting lines $3y + 2x = 6$ and $y = x + 1$. Sketch each line and label the angles.

Solution

To find $\theta_1$

\[
y = x + 1
\]

\[
m = 1
\]

\[
\tan \theta_1 = 1
\]

\[
\theta_1 = 45^\circ
\]

To find $\theta_2$

\[
3y + 2x = 6
\]

\[
3y = -2x + 6
\]

\[
y = \frac{-2}{3}x + 2
\]

\[
m = \frac{-2}{3}
\]

\[
= -0.6667
\]

\[
\tan \theta_2 = -0.6667
\]

\[
\theta_2 = 180^\circ - 33^\circ 41' = 146^\circ 19'
\]

\[
\alpha = \theta_2 - \theta_1
\]

\[
= 146^\circ 19' - 45^\circ
\]

\[
= 101^\circ 19'
\]

\[
\beta = 180 - 101^\circ 19'
\]

\[
= 78^\circ 41'
\]

Exercise 2J

1 Find the acute angle between each of the following pairs of straight lines:

a With gradients 3 and $\frac{3}{4}$  
b With gradients $-2$ and 3  
c With gradients $\frac{2}{3}$ and $-\frac{3}{2}$  
d With equations $2y = 8x + 10$ and $3x - 6y = 22$  
e With equations $4x - 3y = 5$ and $2x - 4y = 9$
Chapter summary

- Gradient of a straight line joining two points:
  \[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

- The general equation of a straight line is
  \[ y = mx + c \]
  where \( m \) is the gradient and \( c \) is the value of the intercept on the \( y \)-axis.

- Equation of a line passing through a given point \((x_1, y_1)\) and having a gradient \( m \) is
  \[ y - y_1 = m(x - x_1) \]

- Equation of a line passing through two given points \((x_1, y_1)\) and \((x_2, y_2)\) is
  \[ y - y_1 = m(x - x_1) \]
  where \( m = \frac{y_2 - y_1}{x_2 - x_1} \).

- Simultaneous equations can be solved graphically by drawing the graph of each equation on the same set of axes. The point at which the lines intersect, \((x, y)\), gives the \( x \)- and \( y \)-values that satisfy both equations.

- The tangent of the angle of slope \( \theta \) can be found with
  \[ \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} \]
  where \( \theta \) is the angle the line makes with the positive direction of the \( x \)-axis.

- If two straight lines are perpendicular to each other the product of their gradients is \(-1\).
  \[ m_1 m_2 = -1 \]

- Distance between 2 points \( A \) and \( B \) is
  \[ AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

- The midpoint of a straight line joining \((x_1, y_1)\) and \((x_2, y_2)\) is the point \(\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)\).

- The angle, \( \alpha \), between intersecting straight lines:
  \[ \alpha = \theta_2 - \theta_1 \]
Multiple-choice questions

1 The gradient of the line passing through the points (5, –8) and (6, –10) is
   A $-2$  B $\frac{1}{2}$  C $\frac{1}{2}$  D $-\frac{1}{18}$  E $\frac{3}{2}$

2 The gradient of the line passing through points (4a, 2a) and (9a, –3a) is
   A $a$  B $-5a$  C $1$  D $-5$  E $-1$

3 The equation of a straight line with gradient 3 and passing through the point (1, 9) is
   A $y = x + 9$  B $y = 3x + 9$  C $y = 3x + 6$
   D $y = \frac{1}{2}x + 1$  E $y = -\frac{1}{3}x + 6$

4 A straight line passes through the points (2, –6) and (–2, –14). The equation of the line is
   A $y = x - 8$  B $y = \frac{1}{2}x - 7$  C $y = \frac{1}{2}x - 10$
   D $y = 2x - 10$  E $y = -\frac{1}{2}x - 8$

5 The line with equation $y = 2x - 6$ passes through the point $(a, 2)$. The value of $a$ is
   A 2  B 4  C 5  D –4  E –2

6 The relation with graph as shown has rule
   A $y = -3x - 3$  B $y = -\frac{1}{2}x - 3$
   C $y = \frac{1}{3}x - 3$  D $y = 3x + 3$
   E $y = 3x - 3$

7 The coordinates of the midpoint of $AB$, where
   $A$ has coordinates (4, 12) and $B$ has coordinates (6, 2), are
   A (4, 8)  B (4.5, 8)  C (5, 8)  D (5, 7)  E (1, 5)

8 If two lines $5x - y + 7 = 0$ and $ax + 2y - 11 = 0$ are parallel then $a$ equals
   A $-5$  B 5  C $-10$  D 10  E $-\frac{1}{2}$

9 (6, 3) is the midpoint of the line joining the points (–4, $y$) and ($x$, –6). The value of $x + y$ is
   A 0  B 16  C 20  D –10  E 28

10 The cost (SC) of hiring a car is given by the formula $C = 2.5x + 65$, where $x$ is the number of kilometres travelled. A person is charged $750 for the hire of the car. The number of kilometres travelled was
   A 65  B 145  C 160  D 200  E 274

Short-answer questions (technology-free)

1 Find the gradients of the lines joining each of the following pairs of points:
   a (4, 3) and (8, 12)  b (–3, 4) and (8, –6)  c (2, 1) and (2, 9)
   d (0, $a$) and ($a$, 0)  e (0, 0) and ($a$, $b$)  f (0, $b$) and ($a$, 0)
Chapter 2 — Linear Relations

2 Find the equation of the straight line of gradient 4 which passes through the point with coordinates:
\[ a (0, 0) \quad b (0, 5) \quad c (1, 6) \quad d (3, 7) \]

3 a The point \((1, a)\) lies on the line with equation \(y = 3x - 5\). Find the value of \(a\).

b The point \((b, 15)\) lies on the line with equation \(y = 3x - 5\). Find the value of \(b\).

4 Find the equation of the straight line joining the points \((-5, 2)\) and \((3, -4)\).

5 Find the equation of the straight line of gradient \(-\frac{2}{3}\), which passes through \((–4, 1)\).

6 The straight line with equation \(ax + by = c\) passes through the points \((2, 4)\) and \((-3, 1)\). Find the values of \(a\), \(b\) and \(c\).

7 Write down the equation of the straight line that:
\[ a \] passes through \((5, 11)\) and is parallel to the \(x\)-axis
\[ b \] passes through \((0, –10)\) and is parallel to the line with equation \(y = 6x + 3\)
\[ c \] passes through the point \((0, –1)\) and is perpendicular to the line with equation \(3x - 2y + 5 = 0\)

8 Find the length and the coordinates of the midpoint of the line segment joining each of the following pairs of points:
\[ a \] \(A(1, 2)\) and \(B(5, 2)\)
\[ b \] \(A(–4, –2)\) and \(B(3, –7)\)
\[ c \] \(A(3, 4)\) and \(B(7, 1)\)

9 Find the equation of a straight line which passes through the point \((2, 3)\) and is inclined at \(30°\) to the positive direction of the \(x\)-axis.

10 Find the equation of the straight line which passes through the point \((-2, 3)\) and makes an angle of \(135°\) with the positive direction of the \(x\)-axis.

11 Find the angle between the straight line \(3x - 2y = 4\) and the line joining the points \((-2, –1)\) and \((4, 1)\).

**Extended-response questions**

1 The table below shows different shoe sizes and their corresponding lengths in millimetres, to the nearest millimetre.

<table>
<thead>
<tr>
<th>Shoe size, (S)</th>
<th>1</th>
<th>(1 \frac{1}{2})</th>
<th>2</th>
<th>(2 \frac{1}{2})</th>
<th>3</th>
<th>(3 \frac{1}{2})</th>
<th>4</th>
<th>(4 \frac{1}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length in mm, (l)</td>
<td>220</td>
<td>224</td>
<td>229</td>
<td>233</td>
<td>237</td>
<td>241</td>
<td>246</td>
<td>250</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shoe size, (S)</th>
<th>5</th>
<th>(5 \frac{1}{2})</th>
<th>6</th>
<th>(6 \frac{1}{2})</th>
<th>7</th>
<th>(7 \frac{1}{2})</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length in mm, (l)</td>
<td>254</td>
<td>258</td>
<td>263</td>
<td>267</td>
<td>271</td>
<td>275</td>
<td>279</td>
</tr>
</tbody>
</table>
60 Essential Mathematical Methods 1 & 2 CAS

Review

a Plot a graph to show the approximate linear relationship between \( S \) and \( l \).
b Write the formula which describes this linear relationship.

c European ‘continental’ shoe sizes are different and are shown in the new table below.

<table>
<thead>
<tr>
<th>Continental shoe size, ( C )</th>
<th>33</th>
<th>34</th>
<th>35</th>
<th>36</th>
<th>37</th>
<th>38</th>
<th>39</th>
<th>40</th>
<th>41</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length in mm, ( l )</td>
<td>220</td>
<td>227</td>
<td>233</td>
<td>240</td>
<td>247</td>
<td>253</td>
<td>260</td>
<td>267</td>
<td>273</td>
</tr>
</tbody>
</table>

d Plot a graph to show the approximate linear relationship of \( C \) against \( l \).
e Write a rule which describes this linear relationship.

2 The cost of hiring a motor cruiser consists of a down payment of $100 and a running charge of $20 per day, or part of a day. The cost of fuel is $5.50 per day. There is also a charge of $10 for filling the freshwater tanks and charging the batteries. Food for a cruise of \( n \) days costs $12.50 per day.
a Express \( C \), the total cost in $, of hiring the cruiser for \( n \) days (all costs to be included).
b For how many days can a cruiser be hired if the cost of a cruise is to be no more than $600?
c A rival company has a fixed rate of $60 per day. For how many days would it be cheaper to hire from this company?

3 The cost of fitting a new plug and cable for an electric drill is $\( C \), when the length of the cable is \( x \) metres and \( C = 4.5 + 1.8x \).
a What meaning could be given for the constant term 4.5?
b What could be the meaning of the coefficient 1.8?
c What would be the gradient of the graph of \( C \) against \( x \)?
d What length of cable would give a total cost of $24.50?

4 The profit made on a single journey of an Easyride bus tour is $\( P \), when there are \( x \) empty seats and \( P = 1020 - 24x \).
a What do you think is the meaning of the constant term 1020?
b What is the least number of empty seats which would result in a loss on a single journey?
c Suggest a meaning for the coefficient 24.

5 A quarterly electricity bill shows the following charges:

- For the first 50 kWh (kilowatt hours): 9.10c per kWh
- For the next 150 kWh: 5.80c per kWh
- Thereafter: 3.56c per kWh

a Write down a formula relating cost, $\( C \), to \( n \), the number of kWh of electricity used:
   i for the first 50 kWh
   ii for the next 150 kWh
   iii for more than 200 kWh

b Draw a graph of \( C \) against \( n \). Use the graph, or otherwise, to determine the charges for
   i 30 kWh
   ii 90 kWh
   iii 300 kWh

c How much electricity could be used for a cost of $20?
6. O is the position of the air traffic control tower at an airport. An aircraft travelling in a straight line is identified at A(2, 10) and again at B(8, –4).
   a. What is the equation that describes the flight path of the aircraft?
   b. How far south of O is the aircraft when x = 15 km?

7. A construction company estimates that for every 1% of air left in concrete as it is being laid, the strength of the hardened concrete decreases by 7%. Let x represent the percentage of air in the concrete (by volume), and the strength of the concrete be s units, where s = 100 when x = 0.
   a. Write a linear model for s in terms of x.
   b. Sketch a graph of s against x.
   c. Calculate how much air can be allowed to remain in the concrete for a strength of at least 95%.
   d. Estimate how much air the concrete will contain at 0% strength.
   e. Is the model sensible at 0% strength?
   f. State the possible values of x.

8. The diagram shows a plan view of a paddock over which a cartesian framework has been superimposed. From an observation point O a rabbit has been spotted, first at A(0, 2) and then at B(4, 6). A fox is seen at C(3, 0) and later at D(5, 4).
   a. Find the equations defining AB and CD.
   b. Assuming that both the rabbit and the fox were running along straight lines, calculate whether the fox’s path would cross the rabbit’s track before the irrigation channel.

9. The diagram shows the side view of a rough, uncut diamond fixed in position on a computer-controlled cutting machine. The diamond is held at the points A(–4.5, 2), B(0.25, 7), C(5, 1.5) and D(1.5, 0). The units are in millimetres.
   a. If a straight cut is made joining A and B, find the y-coordinate of the point V at which the cut will cross the vertical axis.
   b. Find the equation of the line joining V and C.
   c. Would the cuts AB and VC be equally inclined to the vertical axis? Explain your answer.
10 A new light beacon is proposed at \( P(4, -75) \) for air traffic flying into an airport located at \( O(0, 0) \). It is intended that the aircraft should follow a course over beacons at \( P \) and \( Q(36, -4) \), turning at \( Q \) towards the runway at \( O \).

a Would a direct line from \( P \) to \( Q \) pass directly over a hospital located at \( H(20, -36) \)?

b If not, state how far east or west of \( H \) the aircraft would be when the \( y \)-coordinate of an aircraft’s flight path is \(-36\).

11 The map shows an area where it is proposed to construct a new airport. It is thought that the main runway of the airport will have one end of its centre line at \( A(48, 10) \), but the position of the other end of this line, \( B \), has not been decided. There is a light aircraft airport at \( E(68, 35) \) and a radio beacon at \( C(88, -10) \).

a What is the equation that will define the new runway if aircraft coming in to land from the east must be on the extended central line of the new runway when they are 5 km due south of \( E \)?

b If \( B \) is to be 8 km to the east of \( A \), what will be its coordinates?

c A marker beacon is to be built at \( D(68, 30) \) and it is proposed that several auxiliary beacons should be placed on the line \( CD \). What is the equation defining the line \( CD \)?

d If one of the auxiliary beacons is to be placed due east of \( A \), what are the coordinates of its position?

12 A silversmith is making a piece of jewellery consisting of a set of hollow tubular pieces of silver that can slide in and out over each other. Each tube is a circular cylinder made by soldering together the ends of a rectangular strip of silver. The diameters of the cylinders increase in equal steps.
The length of the strip of silver used for the smallest innermost cylinder is 10 mm. The largest is the eighteenth cylinder and requires a strip of silver 61 mm long.

a Find a formula for the length $L$ of the $n$th strip.

b Draw a graph to enable the lengths of the intermediate strips to be read off.

13 Wheelrite, a small company that manufactures garden wheelbarrows, has overhead expenses of $30,000 per year. In addition, it costs $40 to manufacture each wheelbarrow.

a Write a rule which determines the total cost, $C$, of manufacturing $x$ wheelbarrows per year.

b If the annual production is 6000 wheelbarrows, what is the overall cost per wheelbarrow?

c How many wheelbarrows must be made so that the overall cost is $46 per wheelbarrow?

d Wheelrite sells wheelbarrows to retailers for $80 each. Write a rule which determines the revenue, $R$, from the sale of $x$ wheelbarrows to retailers.

e Plot the graphs for $C$ and $R$ against $x$ on the same axes.

f What is the minimum number of wheelbarrows that must be produced for Wheelrite to make a profit each year?

g Write a rule which determines the profit, $P$, from the manufacture and sale of $x$ number of wheelbarrows.

14 An electricity supply authority is offering customers a choice of two methods of paying electricity bills. Method 1 involves payment annually and Method 2 involves payment each quarter (that is every three months). The charges for each method are as follows:

<table>
<thead>
<tr>
<th>Method 1 – per year</th>
<th>Method 2 – per quarter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed charge</td>
<td>$100</td>
</tr>
<tr>
<td>Price per unit</td>
<td>$0.08125</td>
</tr>
<tr>
<td></td>
<td>Fixed charge</td>
</tr>
<tr>
<td></td>
<td>Price per unit</td>
</tr>
</tbody>
</table>

a Suppose Customer A used 1560 units of electricity in a year. Calculate which is the cheaper method of payment.

b Copy and then complete the following table:

<table>
<thead>
<tr>
<th>Number of units of electricity</th>
<th>0</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost ($) calculated by Method 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost ($) calculated by Method 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c Use these values to plot graphs of the costs for each method of paying for electricity. Clearly indicate the approximate number of units of electricity for which the cost is the same for each method of payment.
If \( C_1 \) is the cost by Method 1, \( C_2 \) is the cost by Method 2, and \( x \) is the number of units of electricity used in a year, write down the two formulae which show the cost of \( x \) units calculated by each method. Use these formulae to calculate the exact number of units for which the cost is the same by each method.

In a metal fabricating yard which has been flooded by overflow from a local river, a large steel frame (see diagram) has been partly submerged. The ends \( A, B, C \) and \( D \) are the only parts visible above the level of the flood water.

The coordinates of the ends relative to an overhead crane are \( A(10, 16), B(16, 20), C(24, 8) \) and \( D(18, 4) \). The overhead crane moves east–west along its rail, and the distance east from a point \( O(0, 0) \) is denoted by \( x \). The crane’s hook moves north–south across the frame and the distance to the north of the south rail is denoted by \( y \). Units are in metres.

The steel frame is to be raised out of the water by lifting it at the midpoint, \( M \), of its middle section.

a. Find the coordinates, \( x \) and \( y \), of the point to which the hook must be moved so that it will be directly above the midpoint, \( M \), of the steel frame.

b. In order to minimise the risk of the hook slipping, the hook will be moved slowly along a line parallel to \( AB \). Find the equation of the line along which the hook will be moved.

The diagram shows part of a micro-electronics circuit, as seen through a magnifying glass; the circuit has been etched onto a chip of plated silica. The four points \( A, B, C \) and \( D \) stand away from the chip itself. \( A \) is \((100, 60)\), \( B \) is \((200, 100)\), \( C \) is \((160, 200)\), \( D \) is \((60, 160)\). Units are in \( \frac{1}{25} \) mm.
The unit $S$ is a moveable micro-soldering unit, its tip being at $P(0, 120)$. It is desired to program the tip of the soldering iron, $P$, to solder wires to the points $A$, $B$, $C$ and $D$, moving along the broken lines as shown in the graph.

- Find equations for the lines defining each section of the path along which $P$ must be programmed to move.
- Will any of the turns be through right angles? Explain.

The diagram shows a quadrilateral. Angle $BAD$ is a right angle and $C$ lies on the perpendicular bisector of $AB$. The equation of the line through points $B$ and $C$ is $3y = 4x - 14$.

Find:
- the equation of the line through points $A$ and $D$
- the coordinates of $D$
- the equation of the perpendicular bisector of $AB$
- the coordinates of $C$
- the area of triangle $ADC$
- the area of the quadrilateral $ABCD$. 