CHAPTER 16

Circular Functions

Objectives

- To use radians and degrees for the measurement of angle.
- To convert radians to degrees and vice versa.
- To define the circular functions sine, cosine and tangent.
- To explore the symmetry properties of circular functions.
- To find standard exact values of circular functions.
- To understand and sketch the graphs of circular functions.

16.1 Measuring angles in degrees and radians

The diagram shows a unit circle, i.e. a circle of radius 1 unit.

The circumference of the unit circle $= 2\pi \times 1 = 2\pi$ units

∴ the distance in an anticlockwise direction around the circle from

- $A$ to $B = \frac{\pi}{2}$ units
- $A$ to $C = \pi$ units
- $A$ to $D = \frac{3\pi}{2}$ units

Definition of a radian

In moving around the circle a distance of 1 unit from $A$ to $P$, the angle $POA$ is defined. The measure of this angle is 1 radian.

One radian (written $1^c$) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.
**Note:** Angles formed by moving **anticlockwise** around the circumference of the unit circle are defined as **positive**. Those formed by moving in a **clockwise** direction are said to be **negative**.

### Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi$.

\[
\therefore 2\pi = 360^\circ
\]

\[
\therefore \pi = 180^\circ
\]

\[
\therefore 1^\circ = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi}{180}
\]

---

**Example 1**

Convert $30^\circ$ to radians.

**Solution**

\[
1^\circ = \frac{\pi}{180}
\]

\[
\therefore 30^\circ = \frac{30 \times \pi}{180} = \frac{\pi}{6}
\]

---

**Example 2**

Convert $\frac{\pi}{4}$ to degrees.

**Solution**

\[
1^\circ = \frac{180^\circ}{\pi}
\]

\[
\therefore \frac{\pi}{4} = \frac{\pi \times 180}{4 \times \pi} = 45^\circ
\]

**Note:** Often the symbol for radian, $^c$, is omitted. For example, angle $45^\circ$ is written as $\frac{\pi}{4}$ rather than $\frac{\pi^c}{4}$.  

Using the TI-Nspire

To change 32 degrees to radians, type $32^\circ \xrightarrow{\text{Rad}}$ as shown.

The degree symbol $^\circ$ is found in the catalog (\(k\)). The $\xrightarrow{\text{Rad}}$ command can be found in the catalog (\(k\)).

To change 2 radians to degrees, type $2 \xrightarrow{\text{DD}}$ as shown.

The radian symbol $\text{r}$ is found in the catalog (\(k\)). The $\xrightarrow{\text{DD}}$ command can be found in the catalog (\(k\)).

**Note:** If the calculator is already in radian mode you can change $32^\circ$ to radians by simply typing $32^\circ$ followed by enter. Likewise, if the calculator is already in degree mode, simply type $2^\text{r}$ followed by enter to obtain the result in degrees.

Using the Casio ClassPad

When using the CAS calculator to change $32^\circ$ to radians, ensure your calculator is in Radian mode and enter $32^\circ$, then EXE. (The degree symbol is found in the mth—TRIG menu on the keyboard.)

The answer can be displayed in exact notation, as shown, or highlight the solution and click $\text{EXE}$ to convert to decimal.

To change $0.5$ radians to degrees click Interactive—Transformation—to and enter $0.5$.

**Exercise 16A**

Express the following angles in radian measure in terms of $\pi$:

1. a $60^\circ$  
   b $144^\circ$  
   c $240^\circ$

2. d $330^\circ$  
   e $420^\circ$  
   f $480^\circ$
2 Express, in degrees, the angles with the following radian measures:

\[
\begin{align*}
\text{a} & \quad \frac{2\pi}{3} \\
\text{b} & \quad \frac{5\pi}{6} \\
\text{c} & \quad \frac{7\pi}{6} \\
\text{d} & \quad 0.9\pi \\
\text{e} & \quad \frac{5\pi}{9} \\
\text{f} & \quad \frac{9\pi}{5} \\
\text{g} & \quad \frac{11\pi}{9} \\
\text{h} & \quad 1.8\pi
\end{align*}
\]

3 Use a calculator to convert the following angles from radians to degrees:

\[
\begin{align*}
\text{a} & \quad 0.6 \\
\text{b} & \quad 1.89 \\
\text{c} & \quad 2.9 \\
\text{d} & \quad 4.31 \\
\text{e} & \quad 3.72 \\
\text{f} & \quad 5.18 \\
\text{g} & \quad 4.73 \\
\text{h} & \quad 6.00
\end{align*}
\]

4 Use a calculator to express the following in radian measure:

\[
\begin{align*}
\text{a} & \quad 38^\circ \\
\text{b} & \quad 73^\circ \\
\text{c} & \quad 107^\circ \\
\text{d} & \quad 161^\circ \\
\text{e} & \quad 84.1^\circ \\
\text{f} & \quad 228^\circ \\
\text{g} & \quad 136.4^\circ \\
\text{h} & \quad 329^\circ
\end{align*}
\]

5 Express, in degrees, the angle with the following radian measure:

\[
\begin{align*}
\text{a} & \quad -\frac{\pi}{3} \\
\text{b} & \quad -4\pi \\
\text{c} & \quad -3\pi \\
\text{d} & \quad -\pi \\
\text{e} & \quad \frac{5\pi}{3} \\
\text{f} & \quad \frac{-11\pi}{6} \\
\text{g} & \quad \frac{23\pi}{6} \\
\text{h} & \quad \frac{-23\pi}{6}
\end{align*}
\]

6 Express each of the following in radian measure, in terms of \(\pi\):

\[
\begin{align*}
\text{a} & \quad -360^\circ \\
\text{b} & \quad -540^\circ \\
\text{c} & \quad -240^\circ \\
\text{d} & \quad -720^\circ \\
\text{e} & \quad -330^\circ \\
\text{f} & \quad -210^\circ
\end{align*}
\]

16.2 Defining circular functions: sine and cosine

Consider the unit circle.

The position of point \(P\) on the circle can be described by relating the cartesian coordinates \(x\) and \(y\) and the angle \(\theta\). The point \(P\) on the circumference corresponding to an angle \(\theta\) is written \(P(\theta)\).

Many different angles will give the same point \(P\) on the circle, so the relation linking an angle to the coordinates is a many-to-one function. There are, in fact, two functions involved and they are called sine and cosine, and they are defined as follows:

- The \(x\)-coordinate of \(P\), \(x = \cos \theta\), \(\theta \in \mathbb{R}\)
- The \(y\)-coordinate of \(P\), \(y = \sin \theta\), \(\theta \in \mathbb{R}\)

Note: These functions are usually written in an abbreviated form as follows:

\[
\begin{align*}
\text{x} & = \cos \theta \\
\text{y} & = \sin \theta
\end{align*}
\]

Note: \(\cos (2\pi + \theta) = \cos \theta\) and \(\sin (2\pi + \theta) = \sin \theta\), as adding \(2\pi\) results in a return to the same point on the unit circle.
Example 3
Evaluate $\sin \pi$ and $\cos \pi$.

**Solution**
In moving through an angle of $\pi$, the position is $P(\pi)$, which is $(-1, 0)$.

$\therefore \cos \pi = -1$
$\sin \pi = 0$

Example 4
Evaluate $\sin \left(-\frac{3\pi}{2}\right)$ and $\cos \left(-\frac{\pi}{2}\right)$.

**Solution**
$\sin \left(-\frac{3\pi}{2}\right) = 1$
$\cos \left(-\frac{\pi}{2}\right) = 0$

Example 5
Evaluate $\sin \left(\frac{5\pi}{2}\right)$ and $\sin \left(\frac{7\pi}{2}\right)$.

**Solution**
$\sin \left(\frac{5\pi}{2}\right) = \sin \left(2\frac{1}{2}\pi\right) = \sin \left(2\pi + \frac{\pi}{2}\right) = \sin \left(\frac{\pi}{2}\right) = 1$
$\sin \left(\frac{7\pi}{2}\right) = \sin \left(3\frac{1}{2}\pi\right) = \sin \left(2\pi + \frac{3\pi}{2}\right) = \sin \left(\frac{3\pi}{2}\right) = -1$

Example 6
Evaluate $\sin \left(\frac{9\pi}{2}\right)$ and $\cos (2\pi)$.

**Solution**
$\sin \left(\frac{9\pi}{2}\right) = \sin \left(4\pi + \frac{\pi}{2}\right) = \sin \left(\frac{\pi}{2}\right) = 1$
$\cos (2\pi) = \cos (2\pi + \pi) = \cos \pi = -1$
Exercise 16B

1 For each of the following angles, \( t \), determine the values of \( \sin t \) and \( \cos t \):

- a \( t = 0 \)
- b \( t = \frac{3\pi}{2} \)
- c \( t = -\frac{3\pi}{2} \)
- d \( t = \frac{5\pi}{2} \)
- e \( t = -3\pi \)
- f \( t = \frac{9\pi}{2} \)
- g \( t = \frac{7\pi}{2} \)
- h \( t = 4\pi \)

2 Evaluate using your calculator (Check that your calculator is in Rad mode):

- a \( \sin 1.9 \)
- b \( \sin 2.3 \)
- c \( \sin 4.1 \)
- d \( \cos 0.3 \)
- e \( \cos 2.1 \)
- f \( \cos (−1.6) \)
- g \( \sin (−2.1) \)
- h \( \sin (−3.8) \)

3 For each of the following angles, \( \theta \), determine the values of \( \sin \theta \) and \( \cos \theta \):

- a \( \theta = \frac{27\pi}{2} \)
- b \( \theta = -\frac{5\pi}{2} \)
- c \( \theta = \frac{27\pi}{2} \)
- d \( \theta = -\frac{9\pi}{2} \)
- e \( \theta = \frac{11\pi}{2} \)
- f \( \theta = 57\pi \)
- g \( \theta = 211\pi \)
- h \( \theta = -53\pi \)

16.3 Another circular function: tangent

Again consider the unit circle.

If a tangent to the unit circle at \( A \) is drawn then the \( y \)-coordinate of \( C \), the point of intersection of the extension of \( OP \) and the tangent, is called \( \tan \theta \) (abbreviated to \( \tan \theta \)).

By considering the similar triangles \( OPD \) and \( OCA \):

\[
\tan \theta = \frac{\sin \theta}{\cos \theta}
\]

\[ \therefore \tan \theta = \frac{\sin \theta}{\cos \theta} \]

Now when \( \cos \theta = 0 \), \( \tan \theta \) is undefined.

Hence \( \tan \theta \) is undefined when \( \theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots \)

\[ \therefore \text{Domain of } \tan = \mathbb{R} \setminus \{ \theta : \cos \theta = 0 \} \]

Example 7

Evaluate, using a calculator:

- a \( \tan 1.3 \)
- b \( \tan 1.9 \)
- c \( \tan (−2.8) \)
- d \( \tan 59^\circ \)
- e \( \tan 138^\circ \)

Solution

- a \( \tan 1.3 = 3.6 \) (Don’t forget calculator must be in RAD mode)
- b \( \tan 1.9 = -2.93 \) (\( \cos 1.9 \) is negative)
- c \( \tan (−2.8) = 0.36 \) (\( \cos -2.8 \) and \( \sin -2.8 \) both negative \( \therefore \tan \) is positive)
- d \( \tan 59^\circ = 1.66 \) (Calculate in DEG mode)
- e \( \tan 138^\circ = -0.9 \)
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**Exercise 16C**

1. Evaluate:
   a. \( \tan \pi \)
   b. \( \tan (-\pi) \)
   c. \( \tan \left( \frac{7\pi}{2} \right) \)
   d. \( \tan (-2\pi) \)
   e. \( \tan \left( \frac{5\pi}{2} \right) \)
   f. \( \tan \left( -\frac{\pi}{2} \right) \)

2. Use a calculator to find correct to 2 decimal places:
   a. \( \tan 1.6 \)
   b. \( \tan (-1.2) \)
   c. \( \tan 136^\circ \)
   d. \( \tan (-54^\circ) \)
   e. \( \tan 3.9 \)
   f. \( \tan (-2.5) \)
   g. \( \tan 239^\circ \)

3. For each of the following values of \( \theta \) find \( \tan \theta \):
   a. \( \theta = 180^\circ \)
   b. \( \theta = 360^\circ \)
   c. \( \theta = 0 \)
   d. \( \theta = -180^\circ \)
   e. \( \theta = -540^\circ \)
   f. \( \theta = 720^\circ \)

16.4 **Reviewing trigonometric ratios**

For right-angled triangles:

\[
\sin \theta = \frac{O}{H}, \quad \cos \theta = \frac{A}{H}, \quad \tan \theta = \frac{O}{A}
\]

Applying these trigonometric ratios to the right-angled triangle, \( OAB \), in the unit circle:

\[
\sin \theta = \frac{O}{H} = y = y
\]

\[
\cos \theta = \frac{A}{H} = x = x
\]

\[
\tan \theta = \frac{O}{A} = \frac{\sin \theta}{\cos \theta
\]

For \( 0 < \theta < \frac{\pi}{2} \), the functions \( \sin, \cos \) and \( \tan \) are defined by the trigonometric ratios and are the same as the respective circular functions introduced earlier.
Exercise 16D

1. Find the value of the pronumerals for each of the following:

   a. $8$
   b. $5$
   c. $6$
   d. $10$
   e. $6$
   f. $10$
   g. $65^\circ$
   h. $70^\circ$
   i. $70^\circ$

2. a. Use your calculator to find $a$ and $b$ correct to 4 decimal places.
   b. Hence find the values of $c$ and $d$.
   c. i. Use your calculator to find $\cos 140^\circ$ and $\sin 140^\circ$.
      ii. Write $\cos 140^\circ$ in terms of $\cos 40^\circ$.

16.5 Symmetry properties of circular functions

The coordinate axes divide the unit circle into four quadrants. The quadrants can be numbered, anticlockwise from the positive direction of the $x$-axis, as shown.

Relationships, based on symmetry, between circular functions for angles in different quadrants can be determined.
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Quadrant 2
By symmetry:
\[
\sin(\pi - \theta) = b = \sin\theta \\
\cos(\pi - \theta) = -a = -\cos\theta \\
\tan(\pi - \theta) = \frac{b}{-a} = -\tan\theta
\]

Quadrant 3
\[
\sin(\pi + \theta) = -b = -\sin\theta \\
\cos(\pi + \theta) = -a = -\cos\theta \\
\tan(\pi + \theta) = \frac{-b}{-a} = \tan\theta
\]

Quadrant 4
\[
\sin(2\pi - \theta) = -b = -\sin\theta \\
\cos(2\pi - \theta) = a = \cos\theta \\
\tan(2\pi - \theta) = \frac{-b}{a} = -\tan\theta
\]

Note: These relationships are true for all values of \(\theta\).

Signs of circular functions
These symmetry properties can be summarised for the signs of \(\sin\), \(\cos\) and \(\tan\) for the four quadrants as follows:
1st quadrant: All are positive (A)
2nd quadrant: \(\sin\) is positive (S)
3rd quadrant: \(\tan\) is positive (T)
4th quadrant: \(\cos\) is positive (C).

Negative of angles
By symmetry:
\[
\cos(-\theta) = \cos \theta \\
\sin(-\theta) = -\sin \theta \\
\tan(-\theta) = \frac{-\sin \theta}{\cos \theta} = -\tan \theta
\]

Example 8
If \(\sin x = 0.6\), find the value of:
\(\text{a} \) \ \sin (\pi - x) \\
\(\text{b} \) \ \sin (\pi + x) \\
\(\text{c} \) \ \sin (2\pi - x) \\
\(\text{d} \) \ \sin (-x)

Example 8

Write down the values of:

\[
\begin{align*}
& \text{a} \quad \sin (\pi - x) = \sin x = 0.6 \\
& \text{b} \quad \sin (\pi + x) = -\sin x = -0.6 \\
& \text{c} \quad \sin (2\pi - x) = -\sin x = -0.6 \\
& \text{d} \quad \sin (-x) = -\sin x = -0.6
\end{align*}
\]

Example 9

If \(\cos x = 0.8\), find the value of:

\[
\begin{align*}
& \text{a} \quad \cos (180 - x)^\circ = -\cos x = -0.8 \\
& \text{b} \quad \cos (180 + x)^\circ = -\cos x = -0.8 \\
& \text{c} \quad \cos (360 - x)^\circ = \cos x = 0.8 \\
& \text{d} \quad \cos (-x)^\circ = \cos x = 0.8
\end{align*}
\]

Exercise 16E

1. If \(\sin \theta = 0.42\), \(\cos x = 0.7\) and \(\tan \alpha = 0.38\), write down the values of:

\[
\begin{align*}
& \text{a} \quad \sin (\pi + \theta) \\
& \text{b} \quad \cos (\pi - x) \\
& \text{c} \quad \sin (2\pi - \theta) \\
& \text{d} \quad \tan (\pi - \alpha) \\
& \text{e} \quad \sin (\pi - \theta) \\
& \text{f} \quad \tan (2\pi - \alpha) \\
& \text{g} \quad \cos (\pi + x) \\
& \text{h} \quad \cos (2\pi - x).
\end{align*}
\]

2. If \(\sin x = \sin 60^\circ\) and \(90^\circ < x < 180^\circ\), find the value of \(x\).

3. If \(\cos x = -\cos (\frac{\pi}{6})\) and \(\frac{\pi}{2} < x < \pi\), find the value of \(x\).

4. Write down the values of:

\[
\begin{align*}
& \text{a} \quad a = \cos (\pi - \theta) \\
& \text{b} \quad b = \sin (\pi - \theta) \\
& \text{c} \quad c = \cos (-\theta) \\
& \text{d} \quad d = \sin (-\theta) \\
& \text{e} \quad \tan (\pi - \theta) \\
& \text{f} \quad \tan (-\theta)
\end{align*}
\]
5 Write down the values of:
   a  \( d = \sin (\pi + \theta) \)
   b  \( c = \cos (\pi + \theta) \)
   c  \( \tan (\pi + \theta) \)
   d  \( \sin (2\pi - \theta) \)
   e  \( \cos (2\pi - \theta) \)

Example 6 If \( \sin \alpha = 0.7 \), \( \cos \beta = 0.6 \) and \( \tan \gamma = 0.4 \), write down the values of:
   a  \( \sin (180 + \alpha) \)
   b  \( \cos (180 + \beta) \)
   c  \( \tan (360 - \gamma) \)
   d  \( \cos (180 - \beta) \)
   e  \( \sin (360 - \alpha) \)
   f  \( \sin (-\gamma) \)
   g  \( \tan (360 + \gamma) \)
   h  \( \cos (-\beta) \)

16.6 Exact values of circular functions
A calculator can be used to find the values of the circular functions for different values of \( \theta \). For many values of \( \theta \) the calculator gives an approximation. We consider some values of \( \theta \) such that sin, cos and tan can be calculated exactly.

Exact values for \( 0(0^\circ) \) and \( \frac{\pi}{2} (90^\circ) \)

From the unit circle:
When \( \theta = 0 \),
   \( \sin \theta = 0 \)
   \( \cos \theta = 1 \)
   \( \tan \theta = 0 \)
When \( \theta = \frac{\pi}{2} \),
   \( \sin \left(\frac{\pi}{2}\right) = 1 \)
   \( \cos \left(\frac{\pi}{2}\right) = 0 \)
   \( \tan \left(\frac{\pi}{2}\right) \) is undefined.

Exact values for \( \frac{\pi}{6} (30^\circ) \) and \( \frac{\pi}{3} (60^\circ) \)

Consider an equilateral triangle \( ABC \) of side length 2 units.
In \( \triangle ACD \), by Pythagoras’ Theorem:
\[
DC = \sqrt{AC^2 - AD^2} = \sqrt{3}
\]
\[
\sin 30^\circ = \frac{AD}{AC} = \frac{1}{2}
\]
\[
\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}
\]
\[
\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}}
\]

\[\sin 60^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}\]
\[\cos 60^\circ = \frac{AD}{AC} = \frac{1}{2}\]
\[\tan 60^\circ = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}\]
Exact values for $\frac{\pi}{4}$ (45°)

\[
AC = \sqrt{1^2 + 1^2} = \sqrt{2}
\]

$$\sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = 1$$

As an aid to memory, the exact values for circular functions can be tabulated.

### Summary

<table>
<thead>
<tr>
<th>$\theta$ (θ°)</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$ (30°)</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$ (45°)</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$ (60°)</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$ (90°)</td>
<td>1</td>
<td>0</td>
<td>undefined</td>
</tr>
</tbody>
</table>

### Example 10

Evaluate:

a. $\cos 150^\circ$

b. $\sin 690^\circ$

**Solution**

a. $\cos 150^\circ = \cos (180 - 30)^\circ$

   $= -\cos 30^\circ$

   $= -\frac{\sqrt{3}}{2}$

b. $\sin 690^\circ = \sin (2 \times 360 - 30)^\circ$

   $= \sin (-30)^\circ$

   $= -\frac{1}{2}$
Example 11

Evaluate:

\( a \cos \left( \frac{5\pi}{4} \right) \quad b \sin \left( \frac{11\pi}{6} \right) \)

Solution

\[
\begin{align*}
 a \cos \left( \frac{5\pi}{4} \right) &= \cos \left( \pi + \frac{\pi}{4} \right) \\
&= -\cos \left( \frac{\pi}{4} \right) \text{ (by symmetry)} \\
&= -\frac{1}{\sqrt{2}}
\end{align*}
\]

\[
\begin{align*}
 b \sin \left( \frac{11\pi}{6} \right) &= \sin \left( 2\pi - \frac{\pi}{6} \right) \\
&= -\sin \left( \frac{\pi}{6} \right) \text{ (by symmetry)} \\
&= -\frac{1}{2}
\end{align*}
\]

Exercise 16F

Example 10

1 Without using a calculator, evaluate the \( \sin \), \( \cos \) and \( \tan \) of each of the following:

\[
\begin{align*}
a &\quad 120^\circ & b &\quad 135^\circ & c &\quad 210^\circ & d &\quad 240^\circ & e &\quad 315^\circ \\
f &\quad 390^\circ & g &\quad 420^\circ & h &\quad -135^\circ & i &\quad -300^\circ & j &\quad -60^\circ
\end{align*}
\]

Example 11

2 Write down the exact values of:

\[
\begin{align*}
a &\quad \sin \left( \frac{2\pi}{3} \right) & b &\quad \cos \left( \frac{3\pi}{4} \right) & c &\quad \tan \left( \frac{5\pi}{6} \right) \\
d &\quad \sin \left( \frac{7\pi}{6} \right) & e &\quad \cos \left( \frac{5\pi}{4} \right) & f &\quad \tan \left( \frac{4\pi}{3} \right) \\
g &\quad \sin \left( \frac{5\pi}{3} \right) & h &\quad \cos \left( \frac{7\pi}{4} \right) & i &\quad \tan \left( \frac{11\pi}{6} \right)
\end{align*}
\]

3 Write down the exact values of:

\[
\begin{align*}
a &\quad \sin \left( -\frac{2\pi}{3} \right) & b &\quad \cos \left( \frac{11\pi}{4} \right) & c &\quad \tan \left( \frac{13\pi}{6} \right) & d &\quad \tan \left( \frac{15\pi}{6} \right) \\
e &\quad \cos \left( \frac{14\pi}{4} \right) & f &\quad \cos \left( -\frac{3\pi}{4} \right) & g &\quad \sin \left( \frac{11\pi}{4} \right) & h &\quad \cos \left( -\frac{21\pi}{3} \right)
\end{align*}
\]
16.7 Graphs of sine and cosine

Graphs of sine functions

A table of values for $y = \sin x$ is given below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\pi$</th>
<th>$-\frac{3\pi}{4}$</th>
<th>$-\frac{\pi}{2}$</th>
<th>$0$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$2\pi$</th>
<th>$\frac{9\pi}{4}$</th>
<th>$\frac{5\pi}{2}$</th>
<th>$\frac{11\pi}{4}$</th>
<th>$3\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$-1$</td>
<td>0</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>1</td>
</tr>
</tbody>
</table>

A calculator can be used to plot the graph of $y = \sin x$, $(-\pi \leq x \leq 3\pi)$. Note that Radian mode must be selected.

Observations from the graph of $y = \sin x$

- The graph repeats itself after an interval of $2\pi$ units. A function which repeats itself regularly is called a periodic function and the interval between the repetitions is called the period of the function (also called the wavelength).
  
  Thus $\sin x$ has a period of $2\pi$ units.

- The maximum and minimum values of $\sin x$ are 1 and $-1$ respectively.
  
  The distance between the mean position and the maximum position is called the amplitude. The graph of $y = \sin x$ has an amplitude of 1.

Graphs of cosine functions

A table of exact values for $\cos x$, $-\pi \leq x \leq 3\pi$, is shown:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\pi$</th>
<th>$-\frac{3\pi}{4}$</th>
<th>$-\frac{\pi}{2}$</th>
<th>$0$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$2\pi$</th>
<th>$\frac{9\pi}{4}$</th>
<th>$\frac{5\pi}{2}$</th>
<th>$\frac{11\pi}{4}$</th>
<th>$3\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$-1$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$0$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>1</td>
<td>0</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Chapter 16 — Circular Functions

Using the TI-Nspire

A graph of \( y = \cos x \) for \(-\pi \leq x \leq 3\pi\) can be plotted in a **Graphs & Geometry** application by entering

\[ f_1(x) = \cos(x) \mid -\pi \leq x \leq 3\pi. \]

Using the Casio ClassPad

A graph of \( y = \cos x \) for \(-\pi \leq x \leq 3\pi\) can be plotted on the CAS calculator.

Enter \( y_1 \) as shown, tick to select and click \( \text{ GRAPH } \) to produce the graph. (The window shown is produced by clicking in the graph window and selecting **Zoom—Auto**.)

Observations from the graph of \( y = \cos x \)

- **Period** = \( 2\pi \)
- **Amplitude** = 1
- The graph of \( y = \cos x \) is the graph of \( y = \sin x \) translated \( \frac{\pi}{2} \) units to the left and parallel to the \( x \)-axis.
Sketch graphs of $y = a \sin (nt)$, $y = a \cos (nt)$

**Example 12**

On separate axes draw graphs of the functions:

**a**  $y = 3 \sin (2t)$, $0 \leq t \leq \pi$

**b**  $y = 2 \cos (3t)$, $0 \leq t \leq \frac{2\pi}{3}$

**Solution**

**a**

<table>
<thead>
<tr>
<th>$t$</th>
<th>$0$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 3 \sin (2t)$</td>
<td>$0$</td>
<td>$3$</td>
<td>$0$</td>
<td>$-3$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

**b**

<table>
<thead>
<tr>
<th>$t$</th>
<th>$0$</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{2\pi}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 2 \cos (3t)$</td>
<td>$2$</td>
<td>$0$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

**Observations**

<table>
<thead>
<tr>
<th>Function</th>
<th>Amplitude</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 3 \sin (2t)$</td>
<td>$3$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$y = 2 \cos (3t)$</td>
<td>$2$</td>
<td>$\frac{2\pi}{3}$</td>
</tr>
</tbody>
</table>
Comparing these results with those for \( y = \sin t \) and \( y = \cos t \), the following general rules can be stated for \( a \) and \( n \) positive:

\[
\begin{array}{|c|c|c|}
\hline
\text{Function} & \text{Amplitude} & \text{Period} \\
\hline
y = a \sin (nt) & a & \frac{2\pi}{n} \\
\hline
y = a \cos (nt) & a & \frac{2\pi}{n} \\
\hline
\end{array}
\]

From the above it can be seen the transformation which takes the graph of \( y = \sin t \) to the graph of \( y = 3 \sin (2t) \) has the following result on some important points of the graph of \( y = \sin t \):

\[
(0, 0) \rightarrow (0, 0); \left( \frac{\pi}{2}, 1 \right) \rightarrow \left( \frac{\pi}{4}, 3 \right); (\pi, 0) \rightarrow \left( \frac{\pi}{2}, 0 \right); \\
\left( \frac{3\pi}{2}, 1 \right) \rightarrow \left( \frac{3\pi}{4}, 3 \right); (2\pi, 0) \rightarrow (\pi, 0)
\]

Also, it can be seen that the transformation which takes the graph of \( y = \cos t \) to \( y = 2 \cos (3t) \) has the following result on some important points of the graph of \( y = \cos t \):

\[
(0, 1) \rightarrow (0, 2); \left( \frac{\pi}{2}, 0 \right) \rightarrow \left( \frac{\pi}{6}, 0 \right); (\pi, -1) \rightarrow \left( \frac{\pi}{3}, -2 \right); \\
\left( \frac{3\pi}{2}, 0 \right) \rightarrow \left( \frac{2\pi}{3}, 2 \right)
\]

Note: The graph of \( y = 3 \sin (2t) \) can be obtained from the graph of \( y = \sin t \) by applying two dilations. If \( f(t) = \sin t \), the graph of \( y = f(t) \) is transformed to the graph of \( y = 3f(2t) \). From this it can be recognised that the sequence of transformations is:

- dilation of factor \( \frac{1}{2} \) from the \( y \)-axis
- dilation of factor 3 from the \( t \)-axis

The point with coordinates \((t, y)\) is mapped to the point with coordinates \(\left( \frac{t}{2}, 3y \right)\).

In general, for \( a \) and \( n \) positive numbers, the following are important properties of the functions \( f(t) = a \sin (nt) \) and \( g(t) = a \cos (nt) \):

- The maximal domain of each of the functions is \( \mathbb{R} \).
- The amplitude of each of the functions is \( a \).
- The period of each of the functions is \( \frac{2\pi}{n} \).
- The graph of \( y = a \sin (nt) \) (\( y = a \cos (nt) \)) is obtained from the graph of \( y = \sin t \) (\( y = \cos t \)) by a dilation of factor \( a \) from the \( t \)-axis and a factor of \( \frac{1}{n} \) from the \( y \)-axis. The point with coordinates \((t, y)\) is mapped to the point with coordinates \(\left( \frac{t}{n}, ay \right)\).
- The range of each function is \([-a, a]\).
Example 13

For each of the following functions with domain \( R \) state the amplitude and period:

\[ a \ f(t) = 2 \sin (3t) \quad b \ f(t) = -\frac{1}{2} \sin \left( \frac{t}{2} \right) \quad c \ f(t) = 4 \cos (3\pi t) \]

Solution

\[ a \ \text{Amplitude is 2} \quad b \ \text{Amplitude is} \ \frac{1}{2} \quad c \ \text{Amplitude is 4} \]

\[ a \ \text{Period} = \frac{2\pi}{3} \quad b \ \text{Period} = 2\pi \div \frac{1}{2} = 4\pi \quad c \ \text{Period} = \frac{2\pi}{3\pi} = \frac{2}{3} \]

Example 14

Sketch the graphs of:

\[ a \ y = 2 \cos (2\theta) \quad b \ y = \frac{1}{2} \sin \left( \frac{x}{2} \right) \]

Show one complete cycle.

Solution

\[ a \ \text{The graph of} \ y = 2 \cos (2\theta) \ \text{is obtained from the graph of} \ y = \cos \theta \ \text{by a dilation of factor} \ 2 \ \text{from the \( \theta \)-axis and by a dilation of factor} \ \frac{1}{2} \ \text{from the} \ y\text{-axis.} \]

\[ \text{The period} = \frac{2\pi}{2} = \pi \ \text{and the amplitude is} \ 2. \]

\[ b \ \text{The graph of} \ y = \frac{1}{2} \sin \left( \frac{x}{2} \right) \ \text{is obtained from the graph of} \ y = \cos x \ \text{by a dilation of factor} \ \frac{1}{2} \ \text{from the} \ x\text{-axis and by a dilation of factor} \ 2 \ \text{from the} \ y\text{-axis.} \]

\[ \text{The period} = 2\pi \div \frac{1}{2} = 4\pi \ \text{and the amplitude is} \ \frac{1}{2}. \]

Example 15

Sketch the following graphs for \( x \in [0, 4\pi] \):

\[ a \ f(x) = -2 \sin \left( \frac{x}{2} \right) \quad b \ y = -\cos (2x) \]
Solution

a The graph of \( f(x) = -2 \sin \left( \frac{x}{2} \right) \)

is obtained from the graph of 
\( y = 2 \sin \left( \frac{x}{2} \right) \) by a reflection

in the \( x \)-axis.

The period is \( 4\pi \) and the
amplitude is 2.

b The graph of \( y = -\cos (2x) \) is

obtained from the graph of 
\( y = \cos (2x) \) by a reflection in

the \( x \)-axis.

The period is \( \pi \) and the
amplitude is 1.

In general, for \( a \) and \( n \) positive numbers, the following are important properties of the
functions \( f(t) = -a \sin (nt) \) and \( g(t) = -a \cos (nt) \):

- The amplitude of each of the functions is \( a \).
- The period of each of the functions is \( \frac{2\pi}{n} \).
- The graph of \( y = -a \sin (nt) \) (\( y = -a \cos (nt) \)) is obtained from the graph of
\( y = a \sin (nt) \) (\( a \cos (nt) \)) by a reflection in the \( t \)-axis.
- The range of each function is \([-a, a]\).

Remember that \( \sin (-x) = -\sin x \) and \( \cos (-x) = \cos x \). Hence when reflected in the \( y \)-axis the

graph of \( y = \cos x \) transforms onto itself and the graph of \( y = \sin x \) transforms onto the graph
of \( y = -\sin x \).

Example 16

Sketch the graph of \( f : [0, 2] \to \mathbb{R}, f(t) = 3 \sin (\pi t) \)

Solution

Exercise 16G

1 Write down i the period and ii the amplitude of each of the following:

a \( 2 \sin \theta \)  

b \( 3 \sin (2\theta) \)  

c \( \frac{1}{2} \cos (3\theta) \)
Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

\[ a \quad y = 3 \sin (2x) \]
\[ b \quad y = 2 \cos (3x) \]
\[ c \quad y = 4 \sin \left( \frac{x}{2} \right) \]
\[ d \quad y = \frac{1}{2} \cos (3x) \]
\[ e \quad y = 4 \sin (3x) \]
\[ f \quad y = 5 \cos (2x) \]
\[ g \quad y = -3 \cos \left( \frac{x}{2} \right) \]
\[ h \quad y = 2 \cos (40t) \]
\[ i \quad y = -2 \sin \left( \frac{x}{3} \right) \]

For each of the following give a sequence of transformations which takes the graph of \( y = \sin x \) to the graph of \( y = g(x) \), and state the amplitude and period of \( g(x) \):

\[ a \quad g(x) = 3 \sin x \]
\[ b \quad g(x) = \sin (5x) \]
\[ c \quad g(x) = \sin \left( \frac{x}{3} \right) \]
\[ d \quad g(x) = 2 \sin (5x) \]
\[ e \quad g(x) = -\sin (5x) \]
\[ f \quad g(x) = \sin (-x) \]
\[ g \quad g(x) = 2 \sin \left( \frac{x}{3} \right) \]
\[ h \quad g(x) = -4 \sin \left( \frac{x}{2} \right) \]
\[ i \quad g(x) = 2 \sin \left( \frac{-x}{3} \right) \]

Sketch the graph of:

\[ a \quad f(x) = \sin (2x) \text{ for } x \in [-2\pi, 2\pi] \]
\[ b \quad f(x) = 2 \sin \left( \frac{x}{3} \right) \text{ for } x \in [-6\pi, 6\pi] \]
\[ c \quad f(x) = 2 \cos (3x) \text{ for } x \in [0, 2\pi] \]
\[ d \quad f(x) = -2 \sin (3x) \text{ for } x \in [0, 2\pi] \]

Sketch the graph of \( f: [0, 2\pi] \to \mathbb{R}, f(x) = \frac{5}{2} \cos \left( \frac{2x}{3} \right) \). Hint: For endpoints find \( f(0) \) and \( f(2\pi) \).

16.8 **Sketch graphs of** \( y = a \sin (n(t \pm \varepsilon)) \) \( y = a \cos (n(t \pm \varepsilon)) \)**

In this section translations of graphs of functions of the form \( f(t) = a \sin (nt) \) and \( g(t) = a \cos (nt) \) in the direction of the \( t \)-axis are considered. When a translation of \( \frac{\pi}{4} \) units in the positive direction of the \( t \)-axis is applied to the graph of \( y = f(t) \), the resulting image has equation \( y = f \left( t - \frac{\pi}{4} \right) \). That is, the graph of \( f(t) = a \sin (nt) \) is mapped to the graph with equation \( y = a \sin n \left( t - \frac{\pi}{4} \right) \).
Example 17

On separate axes draw the graphs of the following functions. Use a calculator to help establish the shape. Set the window appropriately by noting the range and period.

\[ a \quad y = 3 \sin \left( t - \frac{\pi}{4} \right), \quad \frac{\pi}{4} \leq t \leq \frac{5\pi}{4} \]

\[ b \quad y = 2 \cos \left( t + \frac{\pi}{3} \right), \quad -\frac{\pi}{3} \leq t \leq \frac{\pi}{3} \]

Solution

\[ a \quad y = 3 \sin \left( t - \frac{\pi}{4} \right), \quad \frac{\pi}{4} \leq t \leq \frac{5\pi}{4} \]

Note that the range is \([-3, 3]\) and the period is \( \pi \).

\[ b \quad y = 2 \cos \left( t + \frac{\pi}{3} \right), \quad -\frac{\pi}{3} \leq t \leq \frac{\pi}{3} \]

Note that the range is \([-2, 2]\) and the period is \( \frac{2\pi}{3} \).

Observations

1. For \( y = 3 \sin \left( t - \frac{\pi}{4} \right) \), amplitude = 3, period = \( \pi \). The graph is the same shape as \( y = 3 \sin (2t) \) but is translated \( \frac{\pi}{4} \) units in the positive direction of the \( t \)-axis.

2. For \( y = 2 \cos \left( t + \frac{\pi}{3} \right) \), amplitude = 2, period = \( \frac{2\pi}{3} \). The graph is the same shape as \( y = 2 \cos (3t) \) but is translated \( \frac{\pi}{3} \) units in the negative direction of the \( t \)-axis.

In each case \( \varepsilon \) has the effect of translating the graph parallel to the \( t \)-axis (\( \varepsilon \) is called the phase).

Note: To determine the sequence of transformations needed, the techniques of Section 6.10 can also be used.

The graph of \( y = \sin t \) is transformed to the graph of \( y = 3 \sin \left( t - \frac{\pi}{4} \right) \).

Write the second equation as \( \frac{y'}{3} = \sin \left( t' - \frac{\pi}{4} \right) \). From this it can be seen that \( y = \frac{y'}{3} \) and \( t = 2 \left( t' - \frac{\pi}{4} \right) \).

Hence \( y' = 3y \) and \( t' = t + \frac{\pi}{4} \). Hence the sequence of transformations is:

- dilation of factor 3 from the \( t \)-axis
- dilation of factor \( \frac{1}{2} \) from the \( y \)-axis and
- translation of \( \frac{\pi}{4} \) units in the positive direction of the \( t \)-axis.

The observation that the graph of \( y = f(t) \) is transformed to the graph of \( y = 3f \left( 2 \left( t - \frac{\pi}{4} \right) \right) \), where \( f(t) = \sin t \), also yields this information.
Exercise 16H

1 Sketch the graph of each of the following, showing one complete cycle. State the period and amplitude, and the greatest and least values of \(y\).

- \(a\) \(y = 3 \sin \left( \theta - \frac{\pi}{2} \right)\)
- \(b\) \(y = \sin (2 \theta + \pi)\)
- \(c\) \(y = 2 \sin 3 \left( \theta + \frac{\pi}{4} \right)\)
- \(d\) \(y = \sqrt{2} \sin 2 \left( \theta - \frac{\pi}{3} \right)\)
- \(e\) \(y = 3 \sin (2 \theta)\)
- \(f\) \(y = 2 \cos \left( \theta + \frac{\pi}{4} \right)\)
- \(g\) \(y = \sqrt{2} \sin 2 \left( \theta - \frac{\pi}{3} \right)\)
- \(h\) \(y = -3 \sin (2 \theta)\)
- \(i\) \(y = -3 \cos 2 \left( \theta + \frac{\pi}{2} \right)\)

2 For the function \(f : [0, 2\pi] \rightarrow \mathbb{R}, f(x) = \cos \left( x - \frac{\pi}{3} \right)\): 
   - find \(f(0), f(2\pi)\)
   - sketch the graph of \(f\)

3 For the function \(f : [0, 2\pi] \rightarrow \mathbb{R}, f(x) = \sin 2 \left( x - \frac{\pi}{3} \right)\):
   - find \(f(0), f(2\pi)\)
   - sketch the graph of \(f\)

4 For the function \(f : [-\pi, \pi] \rightarrow \mathbb{R}, f(x) = \sin 3 \left( x + \frac{\pi}{4} \right)\):
   - find \(f(-\pi), f(\pi)\)
   - sketch the graph of \(f\)

5 Find the equation of the image of \(y = \sin x\) for each of the following transformations:
   - \(a\) dilation of factor 2 from the \(y\)-axis followed by dilation of factor 3 from the \(x\)-axis
   - \(b\) dilation of factor \(\frac{1}{2}\) from the \(y\)-axis followed by dilation of factor 3 from the \(x\)-axis
   - \(c\) dilation of factor 3 from the \(y\)-axis followed by dilation of factor 2 from the \(x\)-axis
   - \(d\) dilation of factor \(\frac{1}{2}\) from the \(y\)-axis followed by a translation of \(\frac{\pi}{3}\) units in the positive direction of the \(x\)-axis
   - \(e\) dilation of factor 2 from the \(y\)-axis followed by a translation of \(\frac{\pi}{3}\) units in the negative direction of the \(x\)-axis

16.9 Solution of trigonometric equations

Example 18

Find all solutions to the equation \(\sin \theta = \frac{1}{2}\) for \(\theta \in [0, 4\pi]\).

Solution

It is clear from the graph that there are four solutions in the interval \([0, 4\pi]\).

The solution for \(x \in \left[0, \frac{\pi}{2}\right]\) is

\[x = \frac{\pi}{6}.\]
This solution can be obtained from a knowledge of exact values or by using \( \sin^{-1} \) on your calculator.

The second solution is obtained by symmetry. The function is positive in the second quadrant and \( \sin (\pi - \theta) = \sin \theta \).

Therefore \( x = \frac{5\pi}{6} \) is the second solution.

It can be seen that further solutions can be achieved by adding \( 2\pi \), as \( \sin \theta = \sin (\theta + 2\pi) \).

Thus \( \theta = \frac{13\pi}{6} \) and \( \frac{17\pi}{6} \) are also solutions.

\[ \begin{align*}
\text{Example 19} & \\
\text{Find two values of } x: & \\
\text{a} & \sin x = -0.3 \text{ in the range } 0 \leq x \leq 2\pi \\
\text{b} & \cos x^\circ = -0.7 \text{ in the range } 0 \leq x \leq 360
\end{align*} \]

\textbf{Solution}

\textbf{a} First we must solve the equation \( \sin x = 0.3 \).

Use your calculator to find the solution for \( x \in \left[0, \frac{\pi}{2}\right] \); \( x = 0.30469 \ldots \)

Now the value of \( \sin \) is negative for \( P(x) \) in the 3rd and 4th quadrants. From the symmetry relationships (or from the graph of \( y = \sin x \)):

3rd quadrant: \( x = \pi + 0.30469 \ldots = 3.446 \) (correct to 3 decimal places)

4th quadrant: \( x = 2\pi - 0.30469 \ldots = 5.978 \) (correct to 3 decimal places)

\( \therefore \) if \( \sin x = -0.3 \), \( x = 1.875 \), or \( x = 5.978 \)

\textbf{b} First we solve the equation \( \cos x^\circ = 0.7 \).

Use your calculator to find the solution for \( x \in [0^\circ, 90^\circ] \).

Now the value of \( \cos \) is negative for \( P(x) \) in the 2nd and 3rd quadrants.

2nd quadrant: \( x = 180 - 45.57 = 134.43 \)

3rd quadrant: \( x = 180 + 45.57 = 225.57 \)

\( \therefore \) if \( \cos x^\circ = -0.7 \), \( x = 134.43, 225.57 \)
Find all the values of $\theta$ between 0 and 360 for which:

\begin{align*}
\text{a} & \quad \cos \theta^\circ = \frac{\sqrt{3}}{2} \\
\text{b} & \quad \sin \theta^\circ = -\frac{1}{2} \\
\text{c} & \quad \cos \theta^\circ - \frac{1}{\sqrt{2}} = 0
\end{align*}

\textbf{Solution}

\text{a} \quad \cos \theta^\circ \text{ is positive, } \therefore P(\theta^\circ) \text{ lies in the 1st or 4th quadrants.}

\[
\cos \theta^\circ = \frac{\sqrt{3}}{2} \\
\theta = 30 \quad \text{or} \quad 360 - 30 \\
\theta = 30 \quad \text{or} \quad 330
\]

\text{b} \quad \sin \theta^\circ \text{ is negative, } \therefore P(\theta^\circ) \text{ is in the 3rd or 4th quadrants.}

\[
\sin \theta^\circ = -\frac{1}{2} \\
\theta = 180 + 30 \quad \text{or} \quad 360 - 30 \\
\theta = 210 \quad \text{or} \quad 330
\]

\text{c} \quad \cos \theta^\circ - \frac{1}{\sqrt{2}} = 0

\therefore \cos \theta^\circ = \frac{1}{\sqrt{2}}

\text{and since } \cos \theta^\circ \text{ is positive, } P(\theta^\circ) \text{ lies in the 1st or 4th quadrants.}

\therefore \cos \theta^\circ = \frac{1}{\sqrt{2}}

\theta = 45 \quad \text{or} \quad \theta = 360 - 45 \\
\theta = 45 \quad \text{or} \quad 315

\textbf{Using the TI-Nspire}

For Example 20a, make sure the calculator is in degree mode.

Complete as shown.
Using the Casio ClassPad

Ensure the mode is set to Degrees (Deg in the status bar at the bottom of the Main screen.)

Enter the equation

\[
\cos(x) = \frac{\sqrt{3}}{2} \mid 0 \leq x \leq 360.
\]

Use your stylus to highlight only \( \cos(x) = \frac{\sqrt{3}}{2} \), then tap Interactive—Equation/inequality—solve and ensure the variable is set to \( x \).

Example 21

Solve the equation \( \sin\left(\frac{2\theta}{3}\right) = -\frac{\sqrt{3}}{2} \) for \( \theta \in [-\pi, \pi] \).

Solution

It is clear that there are four solutions.

To solve the equation, let \( x = \frac{2\theta}{3} \).

Note:

If \( \theta \in [-\pi, \pi] \)
then \( 2\theta = x \in [-2\pi, 2\pi] \)

Consider the equation \( \sin x = -\frac{\sqrt{3}}{2} \) for \( x \in [-2\pi, 2\pi] \).

The first quadrant solution to the equation
\[
\sin x = \frac{\sqrt{3}}{2}
\]
is \( x = \frac{\pi}{3} \).

Symmetry gives the solutions to
\[
\sin x = -\frac{\sqrt{3}}{2}
\]
for \( x \in [0, 2\pi] \) as
\[
x = \pi + \frac{\pi}{3} \text{ and } x = 2\pi - \frac{\pi}{3}
\]
i.e. \( x = \frac{4\pi}{3} \) or \( x = \frac{5\pi}{3} \)
The other two solutions are obtained by subtracting $2\pi$: $\frac{4\pi}{3} - 2\pi$ and $\frac{5\pi}{3} - 2\pi$

\[ ∴ \text{the required solutions for } x \text{ are } \frac{-2\pi}{3} \text{ or } \frac{-\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } \frac{5\pi}{3} \]

\[ ∴ \text{the required solutions for } 0 \text{ are } \frac{-\pi}{3} \text{ or } \frac{-\pi}{6} \text{ or } \frac{2\pi}{3} \text{ or } \frac{5\pi}{6} \]

**Using the TI-Nspire**

Complete as shown.

**Using the Casio ClassPad**

Ensure the mode is set to Radians (Rad in the status bar at bottom of the Main screen).

Enter the equation

\[ \sin(2x) = \sqrt{3}/2 \text{ where } -\pi \leq x \leq \pi. \]

With your stylus highlight only $\sin(2x) = -\sqrt{3}/2$ then tap **Interactive**—**Equation/Inequality**—solve and ensure the variable is set to $x$.

**Exercise 16I**

1. Without using a calculator, find all the values of $x$ between 0 and $2\pi$ for each of the following:
   
   a. $\sqrt{2} \sin(x) + 1 = 0$
   
   b. $\sqrt{2} \cos(x) - 1 = 0$
Chapter 16 — Circular Functions

2 Find, correct to 2 decimal places, all the values of \( x \) between 0 and 2\( \pi \) for which:

\[ \begin{align*}
  a \quad & \sin x = 0.8 \\
  b \quad & \cos x = -0.4 \\
  c \quad & \sin x = -0.35 \\
  d \quad & \sin x = 0.4 \\
  e \quad & \cos x = -0.7 \\
  f \quad & \cos x = -0.2 \\
\end{align*} \]

3 Without using a calculator, find all the values of \( \theta \) between 0° and 360° for each of the following:

\[ \begin{align*}
  a \quad & \cos \theta = -\frac{\sqrt{3}}{2} \\
  b \quad & \sin \theta = \frac{1}{2} \\
  c \quad & \cos \theta = -\frac{1}{2} \\
  d \quad & 2 \cos (\theta) + 1 = 0 \\
  e \quad & 2 \sin 0^\circ = \sqrt{3} \\
  f \quad & \sqrt{2} \sin (\theta) - 1 = 0 \\
\end{align*} \]

4 Find all the values of \( x \) between 0 and 4\( \pi \) for which:

\[ \begin{align*}
  a \quad & \sin x = 0.6 \\
  b \quad & \sin x = -\frac{1}{\sqrt{2}} \\
  c \quad & \sin x = \frac{\sqrt{3}}{2} \\
\end{align*} \]

5 Find all the values of \( x \) between \(-\pi\) and \(\pi\) for which:

\[ \begin{align*}
  a \quad & \cos x = -\frac{1}{\sqrt{2}} \\
  b \quad & \sin x = \frac{\sqrt{3}}{2} \\
  c \quad & \cos x = -\frac{1}{2} \\
\end{align*} \]

6 Sketch the graph of \( f : [-2\pi, 2\pi] \to \mathbb{R}, f(x) = \cos x \).

b On the graph mark the points which have \( y \)-coordinate \( \frac{1}{2} \) and give the associated \( x \)-values.

c On the graph mark the points which have \( y \)-coordinate \( -\frac{1}{2} \) and give the associated \( x \)-values.

7 Solve the following equation for \( \theta \in [0, 2\pi] \):

\[ \begin{align*}
  a \quad & \sin (2\theta) = -\frac{1}{2} \\
  b \quad & \cos (2\theta) = \frac{\sqrt{3}}{2} \\
  c \quad & \sin (2\theta) = \frac{1}{2} \\
  d \quad & \sin (3\theta) = -\frac{1}{\sqrt{2}} \\
  e \quad & \cos (2\theta) = -\frac{\sqrt{3}}{2} \\
  f \quad & \sin (2\theta) = -\frac{1}{\sqrt{2}} \\
\end{align*} \]

8 Solve the following equations for \( \theta \in [0, 2\pi] \):

\[ \begin{align*}
  a \quad & \sin (2\theta) = -0.8 \\
  b \quad & \sin (2\theta) = -0.6 \\
  c \quad & \cos (2\theta) = 0.4 \\
  d \quad & \cos (3\theta) = 0.6 \\
\end{align*} \]

16.10 Sketch graphs of \( y = a \sin n(t \pm \varepsilon) \pm b \) and \( y = a \cos n(t \pm \varepsilon) \pm b \)

Translations parallel to the \( y \)-axis are now considered.

Example 22

Sketch each of the following graphs. Use a calculator to help establish the shape.

\[ \begin{align*}
  a \quad & y = 3 \sin 2 \left( t - \frac{\pi}{4} \right) + 2 \text{ for } \frac{\pi}{4} \leq t \leq \frac{5\pi}{4} \\
  b \quad & y = 2 \cos 3 \left( t + \frac{\pi}{3} \right) - 1 \text{ for } -\frac{\pi}{3} \leq t \leq \frac{\pi}{3} \\
\end{align*} \]
Observations

1. The graph of \( y = 3 \sin 2\left(t - \frac{\pi}{4}\right) + 2 \) is the same shape as the graph of \( y = 3 \sin 2\left(t - \frac{\pi}{4}\right) \) but it is translated 2 units in the positive direction of the \( y \)-axis.

2. Similarly, the graph of \( y = 2 \cos 3\left(t + \frac{\pi}{3}\right) - 1 \) is the same shape as the graph of \( y = 2 \cos 3\left(t + \frac{\pi}{3}\right) \) but it is translated 1 unit in the negative direction of the \( y \)-axis.

In general, the effect of \( \pm b \) is to translate the graph \( \pm b \) units parallel to the \( y \)-axis.

Finding axis intercepts

Example 23

Sketch the graphs of each of the following for \( x \in [0, 2\pi] \). Clearly indicate axis intercepts.

a. \( y = \sqrt{2} \sin(x) + 1 \)

b. \( y = 2 \cos(2x) - 1 \)

c. \( y = 2 \sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3} \)

Solution

a. To determine the axis intercepts, the equation \( \sqrt{2} \sin(x) + 1 = 0 \) must be solved.

\[
\sqrt{2} \sin(x) + 1 = 0
\]

\[
\therefore \sin x = -\frac{1}{\sqrt{2}}
\]

\[
\therefore x = \pi + \frac{\pi}{4} \quad \text{or} \quad 2\pi - \frac{\pi}{4}
\]

\[
\therefore x = \frac{5\pi}{4} \quad \text{or} \quad \frac{7\pi}{4}
\]

\[ \therefore \text{Intercepts: } \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right) \]
**b** \(2 \cos(2x) - 1 = 0\)

\[\therefore \cos(2x) = \frac{1}{2}\]

\[\therefore 2x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ or } \frac{7\pi}{3} \text{ or } \frac{11\pi}{3}\]

\[\therefore x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}\]

\[\therefore \text{Intercepts: } \left(\frac{\pi}{6}, 0\right), \left(\frac{5\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)\]

**c** \(\sin(2(x - \frac{\pi}{3}) = \frac{\sqrt{3}}{2}\)

\[\therefore 2(x - \frac{\pi}{3}) = \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ or } \frac{7\pi}{3} \text{ or } \frac{8\pi}{3}\]

\[\therefore x - \frac{\pi}{3} = \frac{\pi}{6} \text{ or } \frac{\pi}{3} \text{ or } \frac{\pi}{2} \text{ or } \frac{5\pi}{3}\]

\[\therefore x = \frac{2\pi}{3} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{3}\]

\[\therefore \text{Intercepts: } \left(\frac{\pi}{2}, 0\right), \left(\frac{2\pi}{3}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{5\pi}{3}, 0\right)\]
Exercise 16J

1 Sketch the graphs of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval:
   a $y = 2 \sin(x) + 1$
   b $y = 2 \sin(2x) - \sqrt{3}$
   c $y = \sqrt{2} \cos(x) + 1$
   d $y = 2 \sin(2x) - 2$
   e $y = \sqrt{2} \sin(x - \frac{\pi}{4}) + 1$

2 Sketch the graphs of each of the following for $x \in [-2\pi, 2\pi]$: 
   a $y = 2 \sin(3x) - 2$
   b $y = 2 \cos \left( x - \frac{\pi}{4} \right)$
   c $y = 2 \sin(2x) - 3$
   d $y = 2 \cos(2x) + 1$
   e $y = 2 \cos \left( x - \frac{\pi}{3} \right) - 1$
   f $y = 2 \sin \left( x + \frac{\pi}{6} \right) + 1$

3 Sketch the graphs of each of the following for $x \in [-\pi, \pi]$: 
   a $y = 2 \sin \left( x + \frac{\pi}{3} \right) + 1$
   b $y = -2 \sin \left( x + \frac{\pi}{6} \right) + 1$
   c $y = 2 \cos \left( x + \frac{\pi}{4} \right) + \sqrt{3}$

16.11 Further symmetry properties and the Pythagorean identity

Complementary relationships

\[
\sin \left( \frac{\pi}{2} - \theta \right) = a
\]
and since $a = \cos \theta$

\[
\sin \left( \frac{\pi}{2} - \theta \right) = \cos \theta
\]

Similarly

\[
\cos \left( \frac{\pi}{2} - \theta \right) = b
\]
and since $b = \sin \theta$

\[
\cos \left( \frac{\pi}{2} - \theta \right) = \sin \theta
\]

\[
\sin \left( \frac{\pi}{2} + \theta \right) = a = \cos \theta
\]

\[
\cos \left( \frac{\pi}{2} + \theta \right) = -b = -\sin \theta
\]
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Example 24

If \( \sin \theta = 0.3 \) and \( \cos \alpha = 0.8 \) find the values of:

a. \( \sin \left( \frac{\pi}{2} - \alpha \right) \)

b. \( \cos \left( \frac{\pi}{2} + \theta \right) \)

c. \( \sin(-\theta) \)

Solution

a. \( \sin \left( \frac{\pi}{2} - \alpha \right) = \cos \alpha = 0.8 \)

b. \( \cos \left( \frac{\pi}{2} + \theta \right) = -\sin \theta = -0.3 \)

c. \( \sin(-\theta) = -\sin \theta = -0.3 \)

The Pythagorean identity

Consider a point, \( P(\theta) \) on the unit circle.

By Pythagoras’ Theorem

\[ OP^2 = OM^2 + MP^2 \]

\[ \therefore 1 = (\cos \theta)^2 + (\sin \theta)^2 \]

Now \( (\cos \theta)^2 \) and \( (\sin \theta)^2 \) may be written as \( \cos^2 \theta \) and \( \sin^2 \theta \).

\[ \therefore 1 = \cos^2 \theta + \sin^2 \theta \]

As this is true for all values of \( \theta \) it is called an identity. In particular this is called the Pythagorean identity.

\[ \cos^2 \theta + \sin^2 \theta = 1 \]

Example 25

Given that \( \sin x = \frac{3}{5} \) and \( \frac{\pi}{2} < x < \pi \), find \( \cos x \) and \( \tan x \).

Solution

\[ 1 = \cos^2 x + \sin^2 x. \]

For \( \sin x = \frac{3}{5} \), \( 1 = \cos^2 x + \frac{9}{25} \)

Then \( \cos^2 x = 1 - \frac{9}{25} = \frac{16}{25} \)

Therefore \( \cos x = \pm \frac{4}{5} \)
But $x$ is in the second quadrant, hence $\cos x = -\frac{4}{5}$.

$$\tan x = \frac{\sin x}{\cos x} = \frac{3}{5} \div -\frac{4}{5} = \frac{3}{5} \times -\frac{5}{4} = -\frac{3}{4}$$

### Exercise 16K

1. If $\sin x = 0.3$, $\cos \alpha = 0.6$ and $\tan \theta = 0.7$, find the values of:
   - $\cos (-\alpha)$
   - $\sin (-x)$
   - $\tan \left(\frac{3\pi}{2} + \alpha\right)$
   - $\sin \left(\frac{\pi}{2} + \alpha\right)$
   - $\cos \left(\frac{\pi}{2} + x\right)$

2. Given that $0 < \theta < \frac{\pi}{2}$ and $\cos \theta = \sin \frac{\pi}{2}$, find the value of $\theta$.
   - $\tan (-\theta)$
   - $\tan \left(\frac{\pi}{2} - \theta\right)$
   - $\sin \left(\frac{\pi}{2} - \alpha\right)$

3. Given that $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

4. Given that $\sin x = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

5. Given that $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

### 16.12 The tangent function

A table of values for $y = \tan x$ is given below. Use a calculator to check these values, and plot the graph of $y = \tan x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\pi$</th>
<th>$-\frac{3\pi}{4}$</th>
<th>$-\frac{\pi}{2}$</th>
<th>$-\frac{\pi}{4}$</th>
<th>0</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{5\pi}{4}$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$\frac{7\pi}{4}$</th>
<th>$2\pi$</th>
<th>$\frac{9\pi}{4}$</th>
<th>$\frac{5\pi}{2}$</th>
<th>$\frac{11\pi}{4}$</th>
<th>$\frac{3\pi}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>1</td>
<td>ud</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>
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### Observations from the graph

- The graph repeats itself every $\pi$ units, i.e.
the period of tan is $\pi$.
- Range of tan is $R$.
- The equation of the asymptotes are of the form $y = \frac{(2k + 1)\pi}{2}$, where $k$ is an integer.
- The $x$-axis intercepts occur for $x = k\pi$, where $k$ is an integer.

### Transformations of $y = \tan x$

Consider a dilation of factor $\frac{1}{2}$ from the $y$-axis and a dilation of factor 3 from the $x$-axis being applied to the graph of $y = \tan x$.

$$(x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)$$. If the image of $(x, y)$ under the transformation is $(x', y')$ then $x' = \frac{1}{2}x$ and $y' = 3y$.

Hence $x = 2x'$ and $y = \frac{y'}{3}$. Thus the graph of $y = \tan x$ is transformed to the graph of $\frac{y'}{3} = \tan 2x'$; that is, it is transformed to the graph of $y = 3\tan 2x$.

The period of the graph will be $\frac{\pi}{2}$.

In general, for $a$ and $n$ positive numbers, the following are important properties of the function $f(t) = a \tan (nt)$:

- The period of the function is $\frac{\pi}{n}$.
- The graph of $y = a \tan (nt)$ is obtained from the graph of $y = \tan t$ by a dilation of factor $a$ from the $t$-axis and a factor of $\frac{1}{n}$ from the $y$-axis.
- The range of the function is $R$.
- The asymptotes have equations $x = \frac{(2k + 1)\pi}{2n}$, where $k$ is an integer.
- The $t$-axis intercepts are $x = \frac{k\pi}{n}$, where $k$ is an integer.

Note: $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ and $\frac{5\pi}{2}$ are asymptotes.

The $x$-axes intercepts where $\sin x = 0$, are $x = 0$ or $\pi$ or $2\pi$ etc.

In general, $x = k\pi$, where $k$ is an integer.

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Example 26

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3 \tan (2x)$  

b $y = -2 \tan (3x)$

Solution

a $y = 3 \tan (2x)$

- Period $= \frac{\pi}{n} = \frac{\pi}{2}$
- Asymptotes: $x = \frac{(2k + 1)\pi}{4}, k \in \mathbb{Z}$
- Axes intercepts: $x = \frac{k\pi}{2}, k \in \mathbb{Z}$

b $y = -2 \tan (3x)$

- Period $= \frac{\pi}{n} = \frac{\pi}{3}$
- Asymptotes: $x = \frac{(2k + 1)\pi}{6}, k \in \mathbb{Z}$
- Axes intercepts: $x = \frac{k\pi}{3}, k \in \mathbb{Z}$

Example 27

Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\tan x = -1$  

b $\tan (2x) = \sqrt{3}$  

c $2 \tan (3x) = 0$

Solution

a $\tan x = -1$

- $x = \frac{3\pi}{4}$ or $-\frac{\pi}{4}$

b $\tan (2x) = \sqrt{3}$

- $2x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$ or $-\frac{2\pi}{3}$ or $-\frac{5\pi}{3}$
- $x = \frac{\pi}{6}$ or $\frac{4\pi}{6}$ or $-\frac{2\pi}{6}$ or $-\frac{5\pi}{6}$
- $= \frac{\pi}{6}$ or $\frac{2\pi}{3}$ or $-\frac{\pi}{3}$ or $-\frac{5\pi}{6}$

C $2 \tan (3x) = 0$

- $3x = -3\pi$ or $-2\pi$ or $-\pi$ or $0$ or $\pi$ or $2\pi$ or $3\pi$

Hence $x = -\pi$ or $-\frac{2\pi}{3}$ or $-\frac{\pi}{3}$ or $0$ or $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ or $\pi$
Example 28

Sketch the graph of \( y = \tan (2x) + 1 \) for \( x \in [-\pi, \pi] \).

Solution

The graph of \( y = \tan (2x) + 1 \) is formed from the graph of \( y = \tan (2x) \) by a translation of 1 unit in the positive direction of the y-axis.

The y-axis intercept occurs when \( x = 0 \). When \( x = 0 \) then \( y = 0 \).

For the x-axis intercepts consider \( \tan (2x) + 1 = 0 \)

This implies \( \tan (2x) = -1 \)

Hence \( 2x = \frac{3\pi}{4} \) or \( -\frac{\pi}{4} \) or \( \frac{7\pi}{4} \) or \( -\frac{5\pi}{4} \)

and \( x = \frac{3\pi}{8} \) or \( -\frac{\pi}{8} \) or \( \frac{7\pi}{8} \) or \( -\frac{5\pi}{8} \)

The asymptotes are the same as those for \( y = \tan (2x) \). That is, \( x = \frac{(2k + 1)\pi}{4} \), \( k \in \mathbb{Z} \).

Exercise 16L

1. For each of the following state the period:
   - a. \( y = \tan (4x) \)
   - b. \( y = \tan \left( \frac{2x}{3} \right) \)
   - c. \( y = -3 \tan (2x) \)

2. Sketch the graph of each of the following for \( x \in [-\pi, \pi] \):
   - a. \( y = \tan (2x) \)
   - b. \( y = 2 \tan (3x) \)
   - c. \( y = -2 \tan (3x) \)
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Example 27
3 Solve each of the following equations for $x \in [-\pi, \pi]$:
   a $2 \tan(2x) = 2$
   b $3 \tan(3x) = \sqrt{3}$
   c $2 \tan(2x) = 2\sqrt{3}$
   d $3 \tan(3x) = -\sqrt{3}$

Example 28
4 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:
   a $y = 3 \tan(x) + \sqrt{3}$
   b $y = 2 \tan(x) + 2$
   c $y = 3 \tan(x) - 3$

16.13 Numerical methods with a CAS calculator

Using the TI-Nspire

Example 29
Solve the equation $\frac{x}{2} = \sin x$, giving your answer correct to 2 decimal places.

Use $\text{Solve( )}$ (Menu 3 1) as shown.

Press $\text{on} \rightarrow \text{to obtain the answer as a decimal number.}$

Fitting data
Consider the points (1, 2.08), (2, 2.3), (3, 0.49), (4, −1.77) and (6, −0.96).

Enter the data in a Lists & Spreadsheet application as shown.
Choose Sinusoidal Regression from the list of available regressions and complete as shown.

This now gives the values of $a$, $b$, $c$ and $d$, and the equation has been entered in $f_1(x)$.

The curve can be shown in a Graphs & Geometry application together with the Scatter Plot using an appropriate Window.
Using the Casio ClassPad

Example 29

Solving the equation \( \frac{x}{2} = \sin(x) \) is done numerically by drawing the graph of the each side of the equation, then finding the intersection on the calculator.

Note: The window has been set for \( 0 \leq x \leq 2\pi \) and \( -2 \leq y \leq 2 \) in order to see where all the solutions lie.

Use Zoom—Box to zoom on the non-zero solution, then with the graph window selected (bold box), select G-solve—Intersect to obtain the numerical solution.

Fitting data

Consider the points (1, 2.08), (2, 2.3), (3, 0.49), (4, -1.77) and (6, -0.96).

From the Menu select and enter the data in lists 1 and 2 as shown.

Select Calc—Sinusoidal Reg and check the entries are correct.

Note: Set Copy Formula to y1 as this will put the formula for the graph drawn in the section automatically for later use if required.
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Note the formula, then click OK again to produce the graph.

Exercise 16M

1. Solve each of the following equations for \( x \) correct to 2 decimal places:
   \[ \begin{align*}
   \text{a} & \quad \cos x = x \\
   \text{b} & \quad \sin x = 1 - x \\
   \text{c} & \quad \cos x = x^2 \\
   \text{d} & \quad \sin x = x^2
   \end{align*} \]

2. For each of the following sets of data find a suitable trigonometric rule (model).

   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   & \theta & 0 & \frac{\pi}{4} & \frac{\pi}{2} & \frac{3\pi}{4} & \pi \\
   \hline
   \text{a} & \theta & 0 & 2.4 & 0 & 2.4 & 1 \\
   \hline
   \text{b} & \theta & 0 & 0.2 & 0.4 & 0.6 & 0.8 \\
   \hline
   \text{c} & \theta & 0 & 0.2 & 0.4 & 0.6 & 0.8 \\
   \hline
   \end{array}
   \]

16.14 General solution of circular function equations

Solution of equations of circular functions has been discussed in Section 16.9 for functions over a restricted domain. In this section, we consider the general solutions of such equations over the maximal domain for each function.

If the equation of a circular function has one or more solutions in one ‘cycle’, then it will have corresponding solutions in each cycle of its domain, i.e. there will be an infinite number of solutions.

For example, if \( \cos x = a \), then the solution in the interval \([0, \pi]\) is given by:

\[ x = \cos^{-1}(a) \]

By the symmetry properties of the cosine function, other solutions are given by:

\[ -\cos^{-1}(a), \pm2\pi + \cos^{-1}(a), \pm2\pi - \cos^{-1}(a), \pm4\pi + \cos^{-1}(a), \pm4\pi - \cos^{-1}(a), \ldots \]
In general, if \( \cos x = a \):

\[
x = 2n \pi \pm \cos^{-1}(a), \text{ where } n \in \mathbb{Z} \text{ and } a \in [-1, 1].
\]

Similarly, if \( \tan x = a \):

\[
x = n \pi + \tan^{-1}(a), \text{ where } n \in \mathbb{Z} \text{ and } a \in \mathbb{R}.
\]

If \( \sin x = a \):

\[
x = 2n \pi + \sin^{-1}(a) \text{ or } x = (2n + 1) \pi - \sin^{-1}(a), \text{ where } n \in \mathbb{Z} \text{ and } a \in [-1, 1].
\]

**Note:** An alternative and more concise way to express the general solution of \( \sin x = a \) is:

\[
x = n \pi + (-1)^n \sin^{-1}(a), \text{ where } n \in \mathbb{Z} \text{ and } a \in [-1, 1].
\]

---

**Example 30**

Find the general solution to each of the following equations.

- **a** \( \cos x = 0.5 \)
- **b** \( \sqrt{3} \tan (3x) = 1 \)
- **c** \( 2 \sin x = \sqrt{2} \)

**Solution**

**a** \( x = 2n \pi \pm \cos^{-1}(0.5) \)

\[
x = 2n \pi \pm \frac{\pi}{3}, \quad n \in \mathbb{Z}
\]

**b** \( \tan (3x) = \frac{1}{\sqrt{3}} \)

\[
3x = n \pi + \tan^{-1}\left( \frac{1}{\sqrt{3}} \right)
\]

\[
x = \frac{n \pi}{3} + \frac{\pi}{6} = \frac{(6n + 1) \pi}{18}, \quad n \in \mathbb{Z}
\]

**c** \( \sin x = \frac{1}{\sqrt{2}} \)

\[
x = 2n \pi + \sin^{-1}\left( \frac{1}{\sqrt{2}} \right) \text{ or } x = (2n + 1) \pi - \sin^{-1}\left( \frac{1}{\sqrt{2}} \right)
\]

\[
x = 2n \pi + \frac{\pi}{4} = \frac{(8n + 1) \pi}{4}, \quad n \in \mathbb{Z}
\]

\[
x = (2n + 1) \pi - \frac{\pi}{4} = \frac{(8n + 3) \pi}{4}, \quad n \in \mathbb{Z}
\]
Chapter 16 — Circular Functions

Using the TI-Nspire

Check that the calculator is in Radian mode.

a Use \( \text{Solve(} \) from the \( \text{Algebra} \) menu (menu \( \text{3} \) \( \text{1} \)) and complete as shown.

Note the use of \( \frac{1}{2} \) rather than 0.5 to ensure that the answer is exact.

b Complete as shown.

c Complete as shown.

Using the Casio ClassPad

Check the calculator is in Radian mode.

a In enter and highlight the equation \( \cos(x) = 0.5 \). Select \( \text{Interactive—Equation/inequality—solve} \), then click \( \text{Exe} \) to solve the equation.

The solutions are

\[ x = 2m\pi - \pi/3, 2n\pi + \pi/3, m, n \in \mathbb{Z}. \]
b Enter and highlight the equation
\[ \sqrt{3} \tan(3x) = 1 \] and follow the steps as in a.

The solutions are
\[ x = \frac{(6m + 1)\pi}{18}, m \in \mathbb{Z} \]

c Enter and highlight the equation
\[ \sqrt{2} \sin(x) = 1 \] and follow the steps as in a.

The solutions are
\[ x = 2m\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}, m, n \in \mathbb{Z}. \]

**Example 31**

Find the first three positive solutions to each of the following equations.

- **a** \( \cos x = 0.5 \)
- **b** \( \sqrt{3} \tan (3x) = 1 \)
- **c** \( 2 \sin x = \sqrt{2} \)

**Solution**

**a** The general solution (from Example 30) is given by \( x = \frac{(6n + 1)\pi}{3}, n \in \mathbb{Z} \).

When \( n = 0 \), \( x = \pm \frac{\pi}{3} \) and, when \( n = 1 \), \( x = \frac{5\pi}{3} \) or \( x = \frac{7\pi}{3} \).

The first three positive solutions of \( \cos x = 0.5 \) are \( x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \).

**b** The general solution (from Example 30) is given by \( x = \frac{\pi (6n + 1)}{18}, n \in \mathbb{Z} \).

When \( n = 0 \), \( x = \frac{\pi}{18} \) and, when \( n = 1 \), \( x = \frac{7\pi}{18} \) and when \( n = 2 \), \( x = \frac{13\pi}{18} \).

The first three positive solutions of \( \sqrt{3} \tan (3x) = 1 \) are \( x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18} \).

**c** The general solution (from Example 30) is given by \( x = \frac{\pi (8n + 3)}{4}, n \in \mathbb{Z} \).

When \( n = 0 \), \( x = \frac{\pi}{4} \) or \( \frac{3\pi}{4} \) and, when \( n = 1 \), \( x = \frac{9\pi}{4} \) or \( x = \frac{11\pi}{4} \).

The first three positive solutions of \( 2 \sin x = \sqrt{2} \) are \( x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4} \).

**Exercise 16N**

1 Find the general solution to each of the following equations.

- **a** \( \sin x = 0.5 \)
- **b** \( 2 \cos (3x) = \sqrt{3} \)
- **c** \( \sqrt{3} \tan x = -3 \)

2 Find the first two positive solutions to each of the following equations.

- **a** \( \sin x = 0.5 \)
- **b** \( 2 \cos (3x) = \sqrt{3} \)
- **c** \( \sqrt{3} \tan x = -3 \)

3 Find the general solution to \( 2 \cos \left( 2x + \frac{\pi}{4} \right) = \sqrt{2} \), and hence find all the solutions for \( x \) in the interval \( (-2\pi, 2\pi) \).
4 Find the general solution to \( \sqrt{3} \tan \left( \frac{\pi}{6} - 3x \right) - 1 = 0 \), and hence find all the solutions for \( x \) in the interval \([-\pi, 0]\).

5 Find the general solution to \( 2 \sin (4\pi x) + \sqrt{3} = 0 \), and hence find all the solutions for \( x \) in the interval \([-1, 1]\).

### 16.15 Applications of trigonometric functions

**Example 32**

It is suggested that the height \( h(t) \) metres of the tide above mean sea level on 1 January at Warnung is given approximately by the rule \( h(t) = 4 \sin \left( \frac{\pi}{6} t \right) \), where \( t \) is the number of hours after midnight.

- **a** Draw the graph of \( y = h(t) \) for \( 0 \leq t \leq 24 \).
- **b** When was high tide?
- **c** What was the height of the high tide?
- **d** What was the height of the tide at 8 am?
- **e** A boat can only cross the harbour bar when the tide is at least 1 metre above mean sea level. When could the boat cross the harbour bar on 1 January?

**Solution**

- **a** We note: period = \( 2\pi \div \frac{\pi}{6} = 12 \) hours, i.e. high tide occurs at 03.00 and 15.00 (3 pm).
- **b** High tide occurs when \( h(t) = 4 \sin \left( \frac{\pi}{6} t \right) = 4 \)

  \[
  \text{implies } \sin \left( \frac{\pi}{6} t \right) = 1
  \]

  \[
  \therefore \quad \frac{\pi}{6} t = \frac{\pi}{2}, \frac{5\pi}{2}
  \]

  \[
  \therefore \quad t = 3, 15
  \]

- **c** The high tide has height 4 metres above the mean height.
- **d** \( h(8) = 4 \sin \left( \frac{8\pi}{6} \right) = 4 \sin \left( \frac{4\pi}{3} \right) = 4 \times \frac{-\sqrt{3}}{2} = -2\sqrt{3} \).

  The water is \( 2\sqrt{3} \) metres below the mean height at 8 am.

- **e** We first consider \( 4 \sin \frac{\pi}{6} t = 1 \).

  \[
  \text{Thus } \sin \frac{\pi}{6} t = \frac{1}{4}
  \]

  \[
  \therefore \quad \frac{\pi}{6} t = 0.2526, 2.889, 6.5358, 9.172
  \]

  \[
  \therefore \quad t = 0.4824, 5.176, 12.4824, 17.5173
  \]

  i.e. the water is at height 1 metre at 00:29, 05:31, 12:29, 17:31.

  Thus the boat can pass across the harbour bar between 00:29 and 05:31 and between 12:29 and 17:31.
Exercise 160

1. The depth, \( D(t) \) metres, of water at the entrance to a harbour at \( t \) hours after midnight on a particular day is given by \( D(t) = 10 + 3 \sin \left( \frac{\pi t}{6} \right) \), \( 0 \leq t \leq 24 \).
   a. Sketch the graph of \( D(t) \) for \( 0 \leq t \leq 24 \).
   b. Find the value of \( t \) for which \( D(t) \geq 8.5 \).
   c. Boats which need a depth of \( w \) metres are permitted to enter the harbour only if the depth of the water at the entrance is at least \( w \) metres for a continuous period of 1 hour. Find, correct to 1 decimal place, the largest value of \( w \) which satisfies this condition.

2. The depth of water at the entrance to a harbour \( t \) hours after high tide is \( D \) metres, where \( D = p + q \cos (rt) \) for suitable constants \( p, q, r \). At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.
   a. Show that \( r = 30 \) and find the values of \( p \) and \( q \).
   b. Sketch the graph of \( D \) against \( t \) for \( 0 \leq t \leq 12 \).
   c. Find how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

3. A particle moves on a straight line, \( OX \), and its distance \( x \) metres from \( O \) at time \( t \) (s) is given by \( x = 3 + 2 \sin (3t) \).
   a. Find its greatest distance from \( O \).
   b. Find its least distance from \( O \).
   c. Find the times at which it is 5 metres from \( O \) for \( 0 \leq t \leq 5 \).
   d. Find the times at which it is 3 metres from \( O \) for \( 0 \leq t \leq 3 \).
   e. Describe the motion of the particle.
Chapter summary

- Definition of a radian

One radian (written $1^c$) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

$$1^c = \frac{180^\circ}{\pi} \quad 1^\circ = \frac{\pi^c}{180}$$

- Sine and cosine

$x$-coordinate of $P(\theta)$ in unit circle,

$$x = \cos \theta, \theta \in R$$

$y$-coordinate of $P(\theta)$ in unit circle,

$$y = \sin \theta, \theta \in R$$

Abbreviated to

$$x = \cos \theta \quad y = \sin \theta$$

- Tangent

If the tangent to the unit circle at $A$ is drawn then the $y$-coordinate of $B$ is called tangent $\theta$ (abbreviated to $\tan \theta$).

Also by using similar triangles:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

- Circular functions and trigonometric ratios

$$\sin \theta = \frac{O}{H} = \frac{y}{1} = y$$

$$\cos \theta = \frac{A}{H} = \frac{x}{1} = x$$

$$\tan \theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$
Symmetry properties of circular functions

**Quadrant 2** (sin is positive)
\[
\sin(\pi - \theta) = b = \sin \theta
\]

**Quadrant 1** (all functions are positive)
\[
e.g. \sin \theta = b
\]

**Quadrant 3** (tan is positive)
\[
\sin(2\pi - \theta) = -b = -\sin \theta
\]

**Quadrant 4** (cos is positive)
\[
\sin(2\pi - \theta) = -b = -\sin \theta
\]

Solutions of trigonometric equations of the type \(\sin x^\circ = a\) and \(\cos x^\circ = a\)
e.g. If \(\cos x^\circ = -0.7\), find the two values of \(x\) in the range \(0 \leq x \leq 360\).

If \(\cos x^\circ = 0.7\) then \(x = 45.6\)

\(\cos\) is negative in the 2nd and 3rd quadrants

\(\therefore \ x = 180 - 45.6 = 134.4\)

and \(\ x = 180 + 45.6 = 225.6\)

Further symmetry properties

Negative angles:
\[
\cos (-\theta) = \cos \theta
\]
\[
\sin (-\theta) = -\sin \theta
\]
\[
\tan (-\theta) = -\frac{\sin \theta}{\cos \theta} = -\tan \theta
\]

Complementary angles:
\[
\sin \left(\frac{\pi}{2} - \theta\right) = \cos \theta, \quad \sin \left(\frac{\pi}{2} + \theta\right) = \cos \theta
\]
\[
\cos \left(\frac{\pi}{2} - \theta\right) = \sin \theta, \quad \cos \left(\frac{\pi}{2} + \theta\right) = -\sin \theta
\]
### Exact values of circular functions

<table>
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<th>cos θ</th>
<th>tan θ</th>
</tr>
</thead>
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<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
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<td>1/2</td>
<td>√3/2</td>
<td>1/√3</td>
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<tr>
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<td>1</td>
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<tr>
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<td>√3/2</td>
<td>1/2</td>
<td>√3</td>
</tr>
<tr>
<td>π/2</td>
<td>1</td>
<td>0</td>
<td>undefined</td>
</tr>
</tbody>
</table>

### Graphs of circular functions

- **y = sin θ**
  - Amplitude = 1
  - Period = 2π

- **y = cos θ**
  - Amplitude = 1
  - Period = 2π

- **y = tan θ**
  - Amplitude is undefined
  - Period = π
Essential Mathematical Methods 1 & 2 CAS

Review

Graphs of circular functions of the type \( y = a \sin n(t \pm \epsilon) \pm b \) and \( y = a \cos n(t \pm \epsilon) \pm b \)
e.g. \( y = 2 \cos 3 \left(t + \frac{\pi}{3}\right) - 1 \)
Amplitude, \( a = 2 \)
Period \( \frac{2\pi}{n} = \frac{2\pi}{3} \)
The graph is the same shape as \( y = 2 \cos (3t) \)
but is translated:

i. \( \frac{\pi}{3} \) units in the negative direction of the \( t \)-axis, and

ii. 1 unit in the negative direction of the \( y \)-axis.

Pythagorean identity
\[ \cos^2 \theta + \sin^2 \theta = 1 \]

Multiple-choice questions

1. In a right-angled triangle, the two shorter side lengths are 3 cm and 4 cm. To the nearest degree, the value of the smallest angle is
   A. 1°  B. 23°  C. 37°  D. 53°  E. 92°

2. The minimum value of \( 3 - 10 \cos (2x) \) is
   A. −13  B. −17  C. −23  D. −7  E. −10

3. The range of the function \( f : [0, 2\pi] \to \mathbb{R}, f(x) = 4 \sin \left(2x - \frac{\pi}{2}\right) \) is
   A. \( \mathbb{R} \)  B. \([0, 4]\)  C. \([-4, 0]\)  D. \([0, 8]\)  E. \([-4, 4]\)

4. The period of the graph of \( y = 3 \sin \left(\frac{1}{2}x - \pi\right) + 4 \) is
   A. \( \pi \)  B. 3  C. 4π  D. \( \pi + 4 \)  E. \( 2\pi \)

5. The graph of \( y = \sin x \) is dilated by factor \( \frac{1}{2} \) from the \( y \)-axis and translated \( \frac{\pi}{4} \) units in the positive direction of the \( x \)-axis. The equation of the image is
   A. \( y = \sin \left(\frac{1}{2}x + \frac{\pi}{4}\right) \)  B. \( y = \sin \left(\frac{1}{2}x - \frac{\pi}{4}\right) \)  C. \( y = 2 \sin \left(x - \frac{\pi}{4}\right) \)
   D. \( y = \sin \left(2x - \frac{\pi}{4}\right) \)  E. \( y = \sin \left(2 \left(x - \frac{\pi}{4}\right)\right) \)

6. The period of the function \( f : \mathbb{R} \to \mathbb{R}, \text{where } f(x) = a \sin (bx) + c \) and \( a, b \) and \( c \) are positive constants, is
   A. \( a \)  B. \( b \)  C. \( \frac{2\pi}{a} \)  D. \( \frac{2\pi}{b} \)  E. \( \frac{b}{2\pi} \)

7. One cycle of the graph of function with equation \( y = \tan ax \) has vertical asymptotes at \( x = -\frac{\pi}{6} \) and \( x = \frac{\pi}{6} \). A possible value of \( a \) is
   A. 6  B. \( \pi \)  C. \( \frac{\pi}{6} \)  D. \( \frac{1}{3} \)  E. 3
Chapter 16 — Circular Functions

8 The equation $3 \sin(x) + 1 = b$, where $b$ is a positive real number, has one solution in the interval $[0, 2\pi]$. The value of $b$ is

A 1  B  1.5  C  2  D  3  E  4

9 The number of solutions of the equation $b = a \sin x$, where $x \in [-2\pi, 2\pi]$ and $a$ and $b$ are positive real numbers with $a > b$, is

A 2  B  3  C  4  D  5  E  6

10 The depth of water, in metres, in a harbour at a certain point at time $t$ hours is given by $D(t) = 8 + 2 \sin \left( \frac{\pi t}{6} \right)$, $0 \leq t \leq 24$. The depth of the water is first $9$ m at

A $t = 0$  B $t = 1$  C $t = 2$  D $t = 3$  E $t = 4$

Short-answer questions (technology-free)

1 Change each of the following to radian measure in terms of $\pi$:

a $330^\circ$  b $810^\circ$  c $1080^\circ$  d $1035^\circ$  e $135^\circ$

f $405^\circ$  g $390^\circ$  h $420^\circ$  i $80^\circ$

2 Change each of the following to degree measure:

a $\frac{5\pi^c}{6}$  b $\frac{7\pi^c}{4}$  c $\frac{11\pi^c}{4}$  d $\frac{3\pi^c}{12}$  e $\frac{15\pi^c}{2}$

f $\frac{-3\pi^c}{4}$  g $\frac{-\pi^c}{4}$  h $\frac{11\pi^c}{4}$  i $\frac{-23\pi^c}{4}$

3 Give exact values of each of the following:

a $\sin \left( \frac{11\pi}{4} \right)$  b $\cos \left( \frac{-7\pi}{4} \right)$  c $\sin \left( \frac{11\pi}{6} \right)$  d $\cos \left( \frac{-7\pi}{6} \right)$

e $\cos \left( \frac{13\pi}{6} \right)$  f $\sin \left( \frac{23\pi}{3} \right)$  g $\cos \left( \frac{23\pi}{3} \right)$  h $\sin \left( \frac{-17\pi}{4} \right)$

4 State the amplitude and period of each of the following:

a $2 \sin \left( \frac{\theta}{2} \right)$  b $-3 \sin (4\theta)$  c $\frac{1}{2} \sin (3\theta)$

d $-3 \cos (2x)$  e $-4 \sin \left( \frac{x}{3} \right)$  f $\frac{2}{3} \sin \left( \frac{2x}{3} \right)$

5 Sketch the graphs of each of the following (showing one cycle):

a $y = 2 \sin (2x)$  b $y = -3 \cos \left( \frac{x}{3} \right)$  c $y = -2 \sin (3x)$

d $y = 2 \sin \left( \frac{x}{3} \right)$  e $y = \sin \left( x - \frac{\pi}{4} \right)$  f $y = \sin \left( x + \frac{2\pi}{3} \right)$

g $y = 2 \cos \left( x - \frac{5\pi}{6} \right)$  h $y = -3 \cos \left( x + \frac{\pi}{6} \right)$
6 Solve each of the following equations for \( R \):

\[
\begin{align*}
\text{a} & \quad \sin \theta = -\frac{\sqrt{3}}{2}, \quad \theta \in [-\pi, \pi] \\
\text{b} & \quad \sin(2\theta) = -\frac{\sqrt{3}}{2}, \quad \theta \in [-\pi, \pi] \\
\text{c} & \quad \sin \left( \theta - \frac{\pi}{3} \right) = -\frac{1}{2}, \quad \theta \in [0, 2\pi] \\
\text{d} & \quad \sin \left( \theta + \frac{\pi}{3} \right) = -1, \quad \theta \in [0, 2\pi] \\
\text{e} & \quad \sin \left( \frac{\pi}{3} - \theta \right) = -\frac{1}{2}, \quad \theta \in [0, 2\pi]
\end{align*}
\]

Extended-response questions

1 The number of hours of daylight at a point on the Antarctic Circle is given approximately by

\[ d = 12 + 12 \cos \frac{\pi}{6} \left( t + \frac{1}{3} \right) \]

where \( t \) is the number of months which have elapsed since 1 January.

\text{a} Find \( d \):

\text{i} on 21 June (\( t \approx 5.7 \))

\text{ii} on 21 March (\( t \approx 2.7 \))

\text{b} When will there be 5 hours of daylight?

2 The temperature \( A \)°C inside a house at \( t \) hours after 4 am is given by \( A = 21 - 3 \cos \left( \frac{\pi t}{12} \right) \)

for \( 0 \leq t \leq 24 \), and the temperature \( B \)°C outside the house at the same time is given by

\[ B = 22 - 5 \cos \left( \frac{\pi t}{12} \right) \]

for \( 0 \leq t \leq 24 \).

\text{a} Find the temperature inside the house at 8 am.

\text{b} Write down an expression for \( D = A - B \), the difference between the inside and outside temperatures.

\text{c} Sketch the graph of \( D \) for \( 0 \leq t \leq 24 \).

\text{d} Determine when the inside temperature is less than the outside temperature.

3 At a certain time of the year the depth of water \( d \) m in the harbour at Bunk Island is given by the rule

\[ d = 3 + 1.8 \cos \left( \frac{\pi t}{6} \right) \]

where \( t \) is the time in hours after 3 am.

\text{a} Sketch the graph of the function \( d = 3 + 1.8 \cos \left( \frac{\pi t}{6} \right) \) over a 24-hour period from 3 am to 3 am.

\text{b} At what time(s) does high tide occur for \( t \in [0, 24] \)?

\text{c} At what time(s) does low tide occur for \( t \in [0, 24] \)?

A passenger ferry operates between Main Beach and Bunk Island. It takes 50 minutes to go from Main Beach to Bunk Island. The ferry only runs between the hours of 8 am and 8 pm and is only able to enter the harbour at Bunk Island if the depth of water is at least 2 metres.

\text{d} What is the earliest time the ferry should leave Main Beach so that it arrives at Bunk Island and can immediately enter the harbour?
The time to go from Bunk Island to Main Beach is also 50 minutes. The minimum time the ferry takes at Bunk Island harbour is 5 minutes. The minimum time at Main Beach is also 5 minutes.

i What is the latest time the ferry can leave Main Beach to complete a round trip in 105 minutes?

ii How many complete round trips can the ferry make in a day?

The depth of water $D$ at the end of Brighton pier $t$ hours after low tide is given by the rule $D = p - 2 \cos (rt)$, where $p$ and $r$ are suitable constants.

At low tide ($t = 0$) the depth is 2 metres; at high tide, which occurs 8 hours later, the depth is 6 metres.

a Show that $r = \frac{\pi}{8}$ and $p = 4$.

b Sketch the graph of $D = 4 - 2 \cos \left( \frac{\pi}{8} t \right)$ for $0 \leq t \leq 16$.

c If the first low tide occurs at 4 am, when will the next low tide occur?

d At what times will the depth be equal to 4 metres?

The poles that support the Brighton pier stand 7.5 metres above the sea bed.

e How much of a particular pole is exposed at:

i high tide? 

ii 2 pm?

Over the years mussels have attached themselves to the pole. A particular mussel is attached 4 metres from the top of the pole so that some of the time it is exposed and some of the time it is covered by water.

f For how long will the mussel be covered by water during the time from one low tide to the next?