

# Directed graphs

- How do we define a directed graph and apply the definition to problems?
- What are reachability and dominance and how are they determined?
- How do we determine critical activities for a project and hence a critical path for the project?
- How do we find maximum flows for a network using the 'minimum cut = maximum flow' theorem?
- How do we define a bipartite graph?
- How do we find the optimal solution for an allocation problem using the Hungarian algorithm?

## 24.1 Introduction, reachability and dominance

A graph where direction is indicated for every edge is called a **directed graph**. This is often abbreviated to **digraph**. Figure 24.1 is a directed graph. A **network** is a connected directed graph with **no** loops.

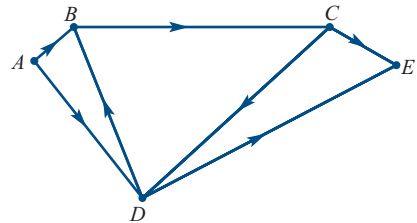


Figure 24.1

For example, the results of a competition between teams *A*, *B*, *C* and *D* can be shown in a graph, as in Figure 24.2. The figure represents the following information:

- D* defeats *A*
- D* defeats *B*
- D* defeats *C*
- A* defeats *B*
- A* defeats *C*
- B* defeats *C*

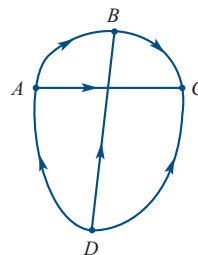


Figure 24.2

This information can be represented with an **adjacency matrix**.

The way to read this is:

across the top row,  $A$  defeats  $B$  and  $A$  defeats  $C$

across the second row,  $B$  defeats  $C$

across the third row,  $C$  does not defeat any other team

across the fourth row,  $D$  defeats  $A$ ,  $B$  and  $C$

A '0' is also used to indicate no match.

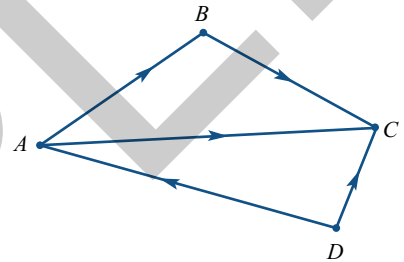
$$\begin{array}{c} \begin{array}{cccc} & A & B & C & D \\ A & 0 & 1 & 1 & 0 \\ B & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 0 \\ D & 1 & 1 & 1 & 0 \end{array} \end{array}$$

## Reachability

One point in a network is said to be reachable from another different point in a directed graph if a path exists between the two points. In this situation, each edge and vertex can only be traversed once.

For example, in the directed graph shown:

- $B$  is reachable from  $A$  in one step along the path  $AB$ .
- $C$  is reachable from  $A$  in one step along the path  $AC$ .
- $C$  is also reachable from  $A$  in two steps along the path  $A-B-C$ .
- $D$  is not reachable from  $A$ .



To fully describe the reachability properties of a directed graph we need to determine whether there are paths between each of the vertices in a network, their number and their length. A compact way of recording the reachability properties of a directed graph is to construct a series of matrices as shown below.

The first matrix  $R^1$  records the number of one-step paths between each of the vertices.

For example, reading across row  $D$ , we can see that there are two one-step paths from  $D$ , one to  $A$  and one to  $C$ .

$$R^1 = \begin{array}{c} \begin{array}{cccc} & A & B & C & D \\ A & 0 & 1 & 1 & 0 \\ B & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 0 \\ D & 1 & 0 & 1 & 0 \end{array} \end{array}$$

The second matrix  $R^2$  records the number of two-step paths between each of the vertices.

For example, reading across row  $A$ , we can see that there is one two-step path from  $A$  to  $C$ .

$$R^2 = \begin{array}{c} \begin{array}{cccc} & A & B & C & D \\ A & 0 & 0 & 1 & 0 \\ B & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 \\ D & 0 & 1 & 1 & 0 \end{array} \end{array}$$

The third matrix  $R^3$  records the number of three-step paths between each of the vertices.

There is only one three-step path, from  $D$  to  $C$ .

$$R^3 = \begin{array}{c} \begin{array}{cccc} & A & B & C & D \\ A & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 \\ D & 0 & 0 & 1 & 0 \end{array} \end{array}$$

The matrices  $R^2$  and  $R^3$  above have been formed by counting the number of two-step and three-step paths in the graph.

Finally, we can combine this information into a single reachability matrix  $R$ , which will show all possible paths in the directed graph. This can be obtained by either counting all the possible paths in the network or by simply combining the information in the three matrices by adding corresponding elements.

$$R = R^1 + R^2 + R^3 = \begin{matrix} & A & B & C & D \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} \end{matrix}$$

Using this matrix, we can then calculate the total reachability of each of the vertices by adding each of the columns and recording the sums in a table.

Vertex	Reachability
A	1
B	2
C	6
D	0

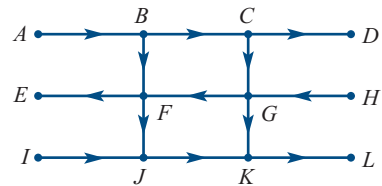
From this table, we can see:

- C has a total reachability of 6 (2 + 1 + 0 + 3). That is, there are six different ways you can get to C from other vertices.
- A has a total reachability of 1. That is, there is only one way you can get to vertex A.
- B has a total reachability of 2. That is, there are two different ways that you can get to B from other vertices.
- D with a total reachability of 0 is not reachable from any other vertex in the matrix.

**Example 1**      **Determining reachability in a directed graph**

The graph opposite represents a one-way road system.

- a Which locations in the system are reachable from A?
- b Which locations in the system are not reachable from C?
- c Starting at G, how many ways can you get from G to K? Identify these paths.



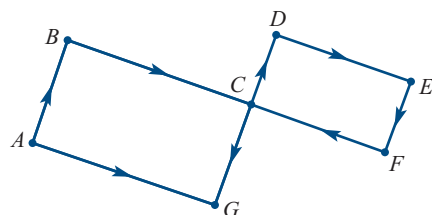
**Solution**

- a B, C, D, E, F, G, J, K and L
- b A, B, H, I
- c 2; G-K and G-F-J-K

**Example 2**      **Constructing a reachability matrix and table**

The graph opposite represents a one-way road system.

- a Construct a reachability matrix  $R^1$  showing all one step-paths.
- b Construct a reachability matrix showing all possible paths?



**Solution**

- a** 1 Construct a matrix with seven rows and seven columns, label  $A$  to  $G$ .
- 2 Complete row  $A$  which records all one step paths in the graph from  $A$  to another vertex. If a one-step path exists between the vertices, enter 1, otherwise, enter 0. There are two one-step paths from  $A$ ,  $A$  to  $B$  and  $A$  to  $G$ .
- 3 Repeat the process for each row in the matrix.

$$R^1 = \begin{matrix} & A & B & C & D & E & F & G \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

- b** 1 Construct a matrix with seven rows and seven columns, label  $A$  to  $G$ .
- 2 Complete row  $A$  which records all possible paths in the graph from  $A$  to another vertex. These can be multi-stepped. There is
- a one-step path from  $A$  to  $B$ ; enter 1
  - a two step path from  $A$  to  $C$ ; enter 1
  - a three-step path from  $A$  to  $D$ ; enter 1
  - a four-step path from  $A$  to  $E$ ; enter 1
  - a five step path from  $A$  to  $F$ ; enter 1
  - and both a one-step and three-step path from  $A$  to  $G$ ; enter 2
- 3 Repeat the process for each row in the matrix.

$$R = \begin{matrix} & A & B & C & D & E & F & G \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

**Dominance**

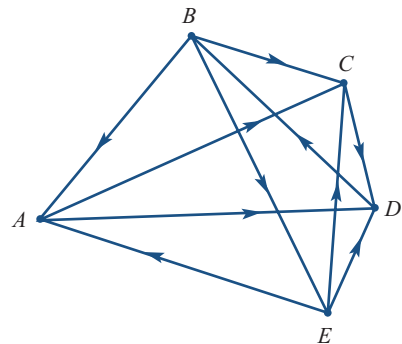
A group of five tennis players  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$ , play each other in a round-robin competition to see who is the best player.

The results are as follows:

- $A$  defeated  $C$  and  $D$
- $B$  defeated  $A$ ,  $C$  and  $E$
- $C$  defeated  $D$
- $D$  defeated  $B$
- $E$  defeated  $A$ ,  $C$  and  $D$

Both  $B$  and  $E$  had three wins so there is a tie.

How can we resolve the situation and see who is the best player?



We can represent the results graphically as shown above. We call this a **dominance graph**. The directed edge from  $B$  to  $A$  tells us that  $B$  defeated (or ‘dominates’)  $A$ .

We can then use the graph to form a series of dominance matrices.

The first dominance matrix  $D^1$  records the number of one-step dominances between the players. For example  $A$  dominates  $C$  because  $A$  played and beat  $C$ .

The matrix can be then used to calculate a dominance score for each player, by summing each of the rows of the matrix. According to this analysis,  $B$  and  $C$  are equally dominant with a dominance score of 3.

Now let us take into account two-step dominances between players. For example,  $B$  has a two-step dominance over  $D$  because  $B$  defeated  $C$  who defeated  $D$ . These two step dominances are shown in matrix  $D^2$ .

$$D^1 = \begin{array}{c|ccccc|c} & A & B & C & D & E & \text{Dominance} \\ \hline A & 0 & 0 & 1 & 1 & 0 & 2 \\ B & 1 & 0 & 1 & 0 & 1 & 3 \\ C & 0 & 0 & 0 & 1 & 0 & 1 \\ D & 0 & 1 & 0 & 0 & 0 & 1 \\ E & 1 & 0 & 1 & 1 & 0 & 3 \end{array}$$

$$D^2 = \begin{array}{c|ccccc|c} & A & B & C & D & E & \text{Dominance} \\ \hline A & 0 & 1 & 0 & 1 & 0 & 2 \\ B & 1 & 0 & 2 & 3 & 0 & 6 \\ C & 0 & 1 & 0 & 0 & 0 & 1 \\ D & 1 & 0 & 1 & 0 & 1 & 3 \\ E & 0 & 1 & 1 & 2 & 0 & 4 \end{array}$$

Combining the information in the two matrices by adding the elements, we now have a new dominance matrix  $D$  which takes into account both one-step and two-step dominances.

We can now use this matrix to calculate a new set of dominance scores that can be used to rank the players.

$$D = D^1 + D^2$$

$$D = \begin{array}{c|ccccc|c} & A & B & C & D & E & \text{Dominance} \\ \hline A & 0 & 1 & 1 & 2 & 0 & 4 \\ B & 2 & 0 & 3 & 3 & 1 & 9 \\ C & 0 & 1 & 0 & 1 & 0 & 2 \\ D & 1 & 1 & 1 & 0 & 1 & 4 \\ E & 1 & 1 & 2 & 3 & 0 & 7 \end{array}$$

Using these dominance scores:

- $B$  is the top ranked player with a dominance score of 9
- $E$  is second with a score of 7
- $A$  and  $D$  are equal third with a score of 4
- $C$  is the bottom ranked player with a score of 2

### Example 3

#### Determining dominance

Four people  $A$ ,  $B$ ,  $C$  and  $D$  have been asked to form a committee to decide on the location of a new toxic waste dump.

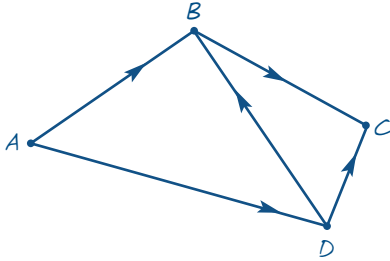
From previous experience, it is known that

- $A$  influences the decisions of  $B$  and  $D$
- $B$  influences the decisions of  $C$
- $C$  influences the decisions of no-one
- $D$  influences the decisions of  $C$  and  $B$

- a** Construct a graph to represent this situation.
- b** Use the graph to construct a dominance matrix which takes into account both one-step and two-step dominances. From this matrix, determine the most influential person on the committee.

**Solution**

a



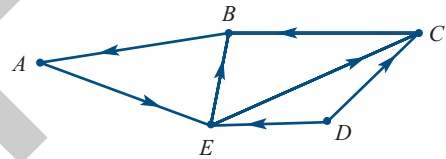
b

$$D = \begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{matrix}$$

A is the most influential person on the committee with a dominance score of 5.

**Exercise 24A**

- 1 a** Give the adjacency matrix for the directed graph shown.
- b**
- Is vertex  $C$  reachable from vertex  $A$
  - Is vertex  $A$  reachable from vertex  $D$
  - Is vertex  $D$  reachable from vertex  $A$
- c** Construct a reachability matrix that takes into account all possible paths between vertices.



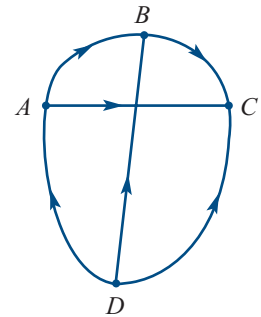
- 2** The adjacency matrix for a directed graph with six vertices  $A, B, C, D, E, F$  is shown.

- a** Draw the directed graph corresponding to this adjacency matrix.
- b** Can you reach:
- $E$  from  $B$ ?
  - $B$  from  $A$ ?
  - $C$  from  $A$ ?
- c** Construct a matrix from the graph showing all two-step paths in the network.

$$\begin{matrix} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

- 3** The results of a competition between teams  $A, B, C$  and  $D$  are displayed opposite. An arrow from  $D$  to  $C$  indicates that team  $D$  defeated team  $C$ .

- a** Construct a dominance matrix showing one-step dominance between the teams. Rank the teams according to one-step dominance.
- b** Construct a dominance matrix showing two-step dominances between the teams. Rank the teams, taking into account both one-step and two-step dominances.



- 4 Five students play each other at chess. The matrix shows the winner of each game with a '1' and the loser or no match with a '0'. For example, row 2 indicates that  $B$  loses to  $A$ ,  $D$  and  $E$  but beats  $C$ .

	$A$	$B$	$C$	$D$	$E$
$A$	0	1	1	1	1
$B$	0	0	1	0	0
$C$	0	0	0	0	0
$D$	0	1	1	0	0
$E$	0	1	0	1	0

- Construct a graph to represent the outcomes of these chess games.
- Calculate a one-step dominance score for each student and use these scores to rank the students.

## 24.2 Network flows

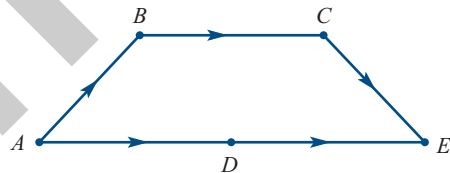
One application of weighted digraphs involves the concept of flows. A **flow** is the quantity of material that can move along a given channel; for example, traffic flow along a highway or water flow through a pipe.

### Example 4

### Determining the maximum flow by inspection

In the figure shown,  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  are towns. The edges of the graph represent roads between the towns. The maximum traffic flows per hour are:

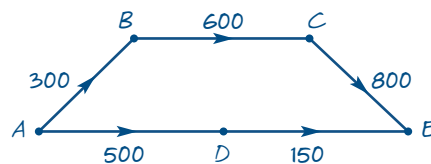
- $A$  to  $B$ : 300 cars
- $B$  to  $C$ : 600 cars
- $C$  to  $E$ : 800 cars
- $A$  to  $D$ : 500 cars
- $D$  to  $E$ : 150 cars



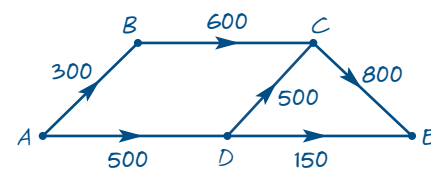
- Find the maximum traffic flow from  $A$  to  $E$  through  $D$ .
- A new road is built connecting  $D$  to  $C$ . The new road can carry 500 cars per hour. What is the maximum flow from  $A$  to  $E$  (not necessarily through  $D$ ) now?

### Solution

- The information can be represented as a weighted digraph.
  - The maximum flow from  $A$  to  $E$  through  $D$  is 150 cars, as this is the maximum flow from  $D$  to  $E$ .
- The information can be represented as a weighted digraph. The new road avoids having to use the road  $DE$ .
  - The maximum flow is now 800 cars 300 along  $A$  to  $B$  to  $C$  and 500 along  $A$  to  $D$  to  $C$ . The road  $CA$  can then take 800 cars.



Maximum flow = 150



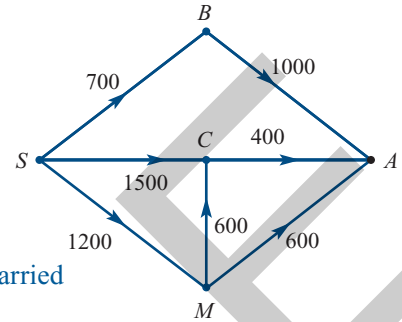
Maximum flow = 300 + 500 = 800

**Example 5****Determining the maximum flow by inspection**

Flights connect the airports at Sydney, Melbourne, Brisbane, Adelaide and Canberra. The numbers in this graph represent the maximum number of passengers, that can be carried in a fixed time.

**Note:** There are no direct flights between Melbourne and Brisbane or between Sydney and Adelaide in the time period considered.

- Find the maximum number of passengers that can be carried between Sydney and Adelaide (in the fixed time).
- Indicate how this can be achieved.

**Solution**

- The maximum number between Sydney and Adelaide is  $700 + 400 + 600 = 1700$ .
- 700 passengers through Brisbane to Adelaide. 1000 passengers to Melbourne and of these 600 go directly to Adelaide and an additional 400 go via Canberra.

As the weighted graphs become more complicated the problem of finding the maximum flow becomes more difficult.

**Maximum flow**

One method of determining the maximum flow for a network is to use the minimum capacity of cuts. A **cut** is defined as a collection of edges that, if removed from the directed graph, produces a zero flow between the start ( $S$ ) and the terminal ( $T$ ). For example, in figure 24.3 the cut (shown as a broken line) is defined by the edges  $BT$ ,  $BA$  and  $SA$ .

The **capacity** of a cut is the sum of the capacities (weights) of the edges directed from  $S$  to  $T$  that the cut passes through. The capacity of the cut in figure 24.3 is  $3 + 2 + 2 = 7$ .

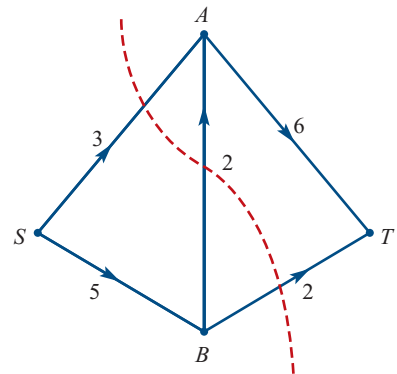


Figure 24.3

Four possible cuts are shown in Figure 24.4. Cut 1 passes through edges  $BD$ ,  $BA$  and  $CA$ . Cut 2 passes through edges  $CB$ ,  $BA$  and  $AD$ . Cut 3 passes through edges  $CB$  and  $CA$ . Cut 4 passes through edges  $BD$  and  $AD$ . The capacity of each of the four cuts is given in the table below.

Cut	Capacity
$C_1$	7
$C_2$	11
$C_3$	8
$C_4$	8

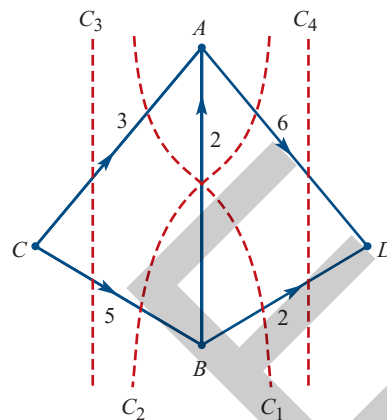


Figure 24.4

The cut  $C_2$  defined by edges  $CB$ ,  $BA$  and  $AD$  has capacity 11 ( $6 + 0 + 5$ ). This is because for edge  $BA$ , the flow is from the right side to the left side of the cut. Thus, this edge can contribute nothing to the flow from  $C$  to  $D$ . It counts as zero. This leads to the following convention.

Edges that cannot contribute to the flow of a network in the required direction are counted as zero.

From the weighted digraph it can be seen that:

$$\text{maximum flow} \leq \text{capacity of any cut}$$

and, in particular:

$$\text{maximum flow} \leq \text{capacity of 'minimum' cut}$$

This can be strengthened to

$$\text{maximum flow} = \text{capacity of minimum cut}$$

For the example discussed here the maximum flow = 7.

### Example 6

### Using the minimum cut to find maximum flow

Determine the capacities of each of the cuts in the graph shown, and also the value of the minimum cut and hence the maximum flow.

#### Solution

The capacity of  $C_1 = 20 + 15 = 35$ .

The capacity of  $C_2 = 15 + 6 + 15 + 20 = 56$ .

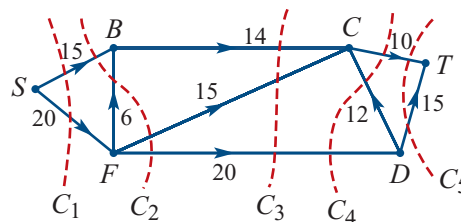
The capacity of  $C_3 = 14 + 15 + 20 = 49$ .

The capacity of  $C_4 = 20 + 0 + 10 = 30$ .

The capacity of  $C_5 = 10 + 15 = 25$ .

$\therefore$  the maximum flow = 25.

**Note:** For cut  $C_4$  the flow along edge  $DC$  is from the right side to the left side of the cut and therefore the capacity of this cut is 30.



**Example 7**

**Using the minimum cut to find maximum flow**

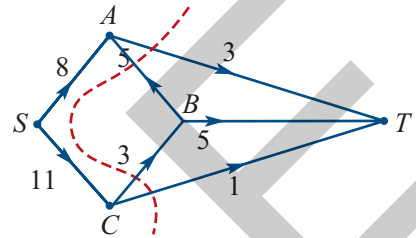
Determine the maximum flow for the directed graph.

**Solution**

The cut defined by the edges  $CT$ ,  $CB$  and  $AT$  yields the maximum flow.

The capacity of the cut =  $1 + 3 + 0 + 3 = 7$ .

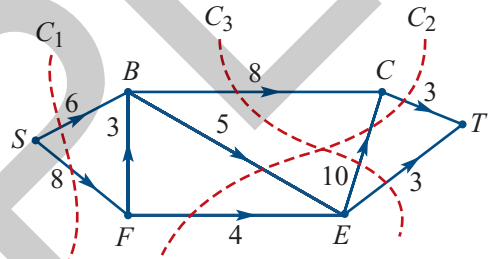
**Note:** For edge  $BA$ , the flow is from the right to the left of the cut illustrated. Therefore it contributes nothing to the flow from left to right.



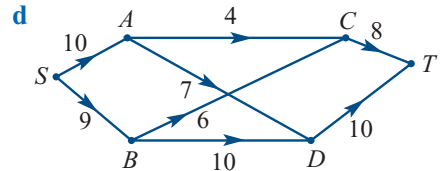
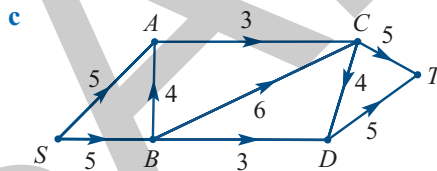
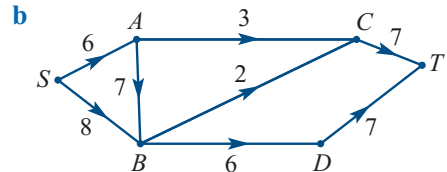
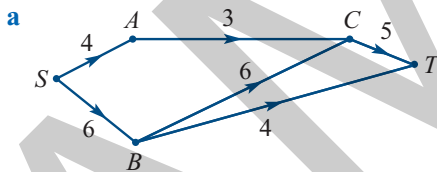
**Exercise 24B**



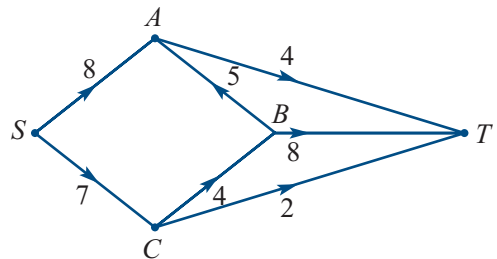
1 Determine the capacity of each of the cuts of this digraph.



2 Find the maximum flow for each of the following.



3 Consider the network shown opposite. Draw up a table listing all the cuts, and the capacity of each cut.



## 24.3 The critical path problem

Developing and manufacturing a product frequently involves many interrelated activities. It is often the case that some of these activities cannot be started until other activities are completed. Digraphs can be used to represent such situations.

Figure 24.5 represents a procedure where activities  $A$  and  $B$  must be completed before activities  $C$  and  $D$ .

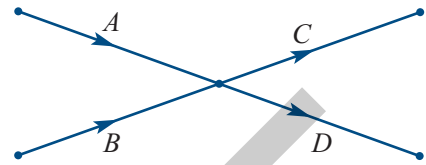


Figure 24.5

A company is to produce a type of bicycle light. In order to produce the light two machines,  $M_1$  and  $M_2$ , are required. Machine  $M_1$  produces component ‘front’ and machines  $M_1$  and  $M_2$  are needed to produce component ‘back’. The ‘back’ and ‘front’ are assembled and shipped.

The situation can be represented in a table.

Activity	Description	Estimated duration (days)	Immediate predecessors
$A$	Purchase and install $M_1$	8	None
$B$	Purchase and install $M_2$	6	None
$C$	Test $M_1$	1	$A$
$D$	Test $M_2$	2	$B$
$E$	Produce ‘front’	3	$C$
$F$	Produce ‘back’	1	$C, D$
$G$	Assemble ‘front’ and ‘back’	2	$E, F$
$H$	Ship product	1	$G$

A weighted digraph can then be drawn to represent this situation.

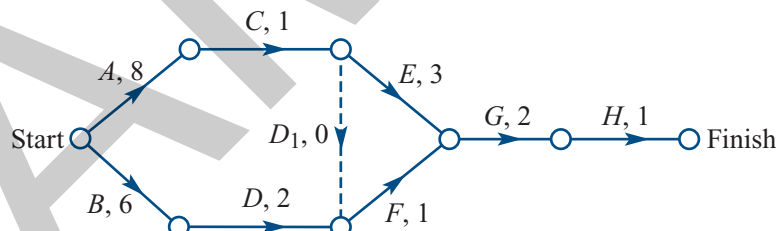


Figure 24.6

**Note:** A dummy activity ‘ $D_1$ ’ of 0 hours’ duration was introduced as activity  $F$  needs both  $C$  and  $D$ .

There are some activities in this process which determine the minimum time for the production and which are not flexible. These activities are determined by finding a path of maximal weight from the initial vertex ‘start’ to the terminal vertex ‘customer’. Such a path is called a **critical path** and its edges represent the project’s **critical activities**.

A **critical activity** is any task that, if delayed, will also delay the entire project.

In Figure 24.6 the critical path is:  $A C E G H$ . In some cases there will be more than one critical path.

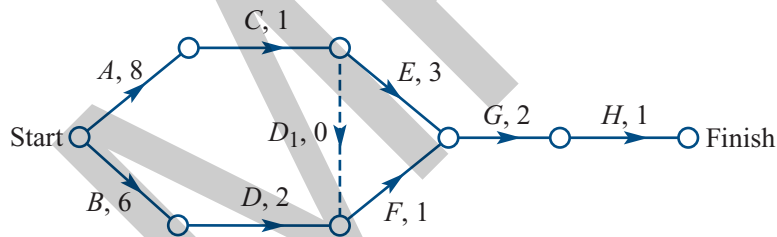
Two important facts about critical paths are:

- The weight of the critical path is the minimal length of time required to complete the project.
- Increasing the time required for any critical activity will also increase the time necessary to complete the project.

When drawing weighted digraphs for problems involving critical paths the following conventions are used:

- The edges (or arcs) represent the activities.
- The vertices (or nodes) represent **events**. The start/finish of one or more activities is called an **event**.
- An edge should lead from a start vertex to represent each activity that has no predecessors.
- A vertex (called the **finish node**) representing the completion of the project should be included in the network.
- An activity should not be represented by more than one edge in the network.
- Two nodes can be connected directly by, at most, one edge.

In order to satisfy the final two conventions, it is sometimes necessary to introduce a **dummy activity** that takes zero time. Following these conventions the weighted digraph can be redrawn as shown.



The need for the dummy activity ( $D_1$ ) is now apparent.

## Earliest starting times (EST)

The **earliest starting time** is the earliest time an activity can be commenced.

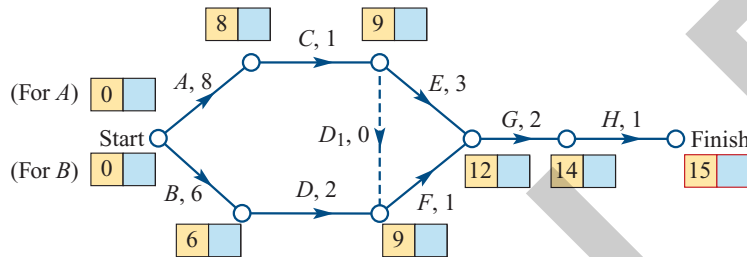
The activities without predecessors have earliest start time 0. You work from left to right through the digraph to determine earliest starting time for an activity. At the start of each activity a double box is drawn. In the left-hand box the earliest starting time is entered. For example:

- to find the earliest starting time for activity  $C$ : The activity  $A$  is the only immediate predecessor. Therefore the earliest starting time for activity  $C$  is 8.
- to find the earliest starting time for activity  $E$ : The activity  $C$  is the only immediate predecessor. Therefore the earliest starting time for activity  $E$  is equal to

$$\text{earliest start time for } C + \text{duration of } C = 8 + 1 = 9$$

If there are two edges (activities) meeting at a vertex (event) then the ‘maximum’ value is chosen as shown in the following.

- Activity  $G$  has two immediate predecessors  $E$  and  $F$ .  
The earliest starting time for  $E + \text{duration of } E = 12$ .  
The earliest starting time for  $F + \text{duration of } F = 9$ .  
Therefore earliest starting time for  $G$  is 12.



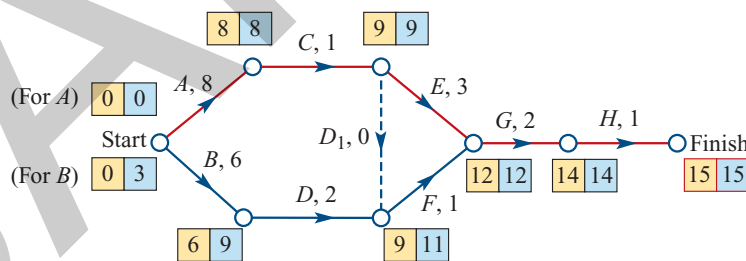
The last box to the right gives the time for completion.

## Latest starting times (LST)

The **latest starting time** is the latest time an activity can be left if the whole project is to be completed on time.

Latest starting times are established by working backwards through the network. They are entered in the right-hand box.

Start in the boxes to the far right with both entries equal and then subtract duration times to move back through the diagram. To obtain the other final event times, subtract the activity time (working backwards). If two or more edges ‘lead back’ to the vertex, then two boxes are used. This happens at the start. The graph with this extra information can now be redrawn.

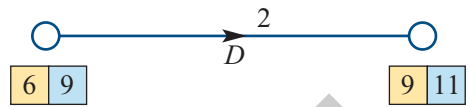


If the **difference** between the earliest starting time for an activity and the latest starting time for an activity is 0, then the **activity is critical**.

Therefore, as stated earlier, the critical activities are  $A$ ,  $C$ ,  $E$ ,  $G$ , and  $H$  and the path defined by these activities is the **critical path** (in red).

## Float

For Activity  $D$ , a section of the graph displays the start and finish vertices. Activity  $D$  is not a critical activity.



The **float** or **slack** of a non-critical activity is the amount by which the **latest starting time** is greater than its **earliest starting time**:

$$\text{float} = \text{latest starting time} - \text{earliest starting time}$$

$\therefore$  The float for Activity  $D = 9 - 6 = 3$ .

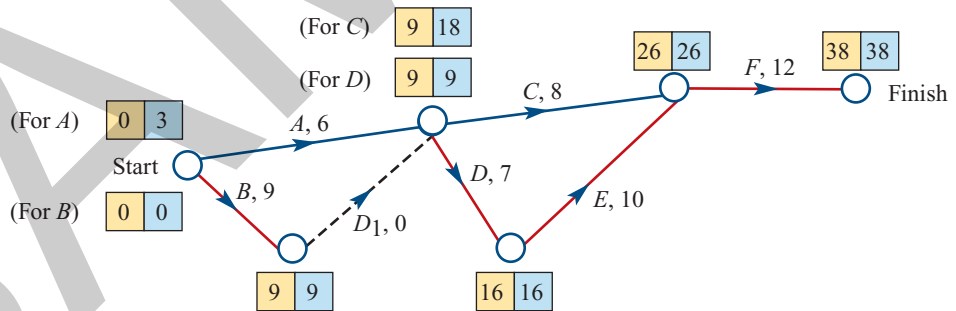
The existence of a float means that an activity can start later than its earliest possible start time, or the duration of the activity can be increased.

### Example 8

### A critical path analysis

A project has these activities, order of activities and duration of activities. A project diagram with earliest and latest starting times is shown. Find the critical path and give the floats of the non-critical activities. Form a table showing EST and LST for each activity.

Activity	Predecessors	Duration (days)
A Train workers	—	6
B Purchase raw material	—	9
C Produce product 1	A, B	8
D Produce product 2	A, B	7
E Test product 2	D	10
F Assemble products 1 and 2	C, E	12



### Solution

1 Identify the critical activities: those with no slack time.

2 Float time =  $LST - EST$ .

Determine this for each activity.

**Note:** A dummy activity ( $D_1$ ) is necessary as edges  $A$  and  $B$  would otherwise join the same vertices

Critical path:  $B \rightarrow D \rightarrow E \rightarrow F$

The floats of the activities are:

$$\text{Activity A: } 3 - 0 = 3 \quad \text{Activity B: } 0 - 0 = 0$$

$$\text{Activity } D_1: 9 - 9 = 0 \quad (\text{dummy activity})$$

$$\text{Activity C: } 18 - 9 = 9 \quad \text{Activity D: } 9 - 9 = 0$$

$$\text{Activity E: } 16 - 16 = 0 \quad \text{Activity F: } 26 - 26 = 0$$

The float for Activity A is 3.

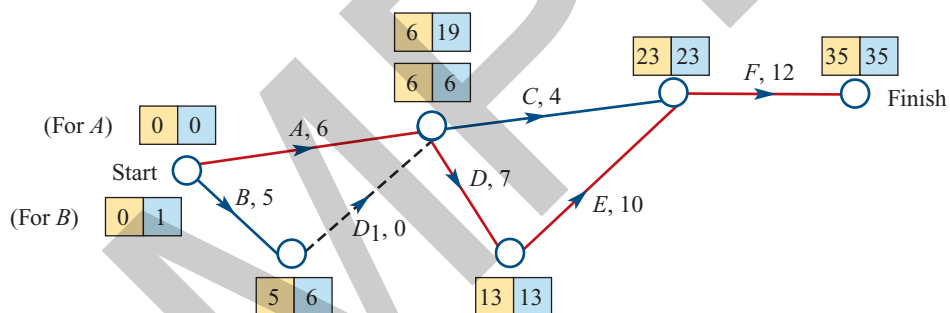
The float for Activity C is 9.

- 3 Use the project diagram to form a table of earliest and latest start times for activities.

Activity	Earliest start time (Day)	Latest start time (Day)
A	0	3
B	0	0
C	9	18
D	9	9
E	16	16
F	26	26

## Project crashing

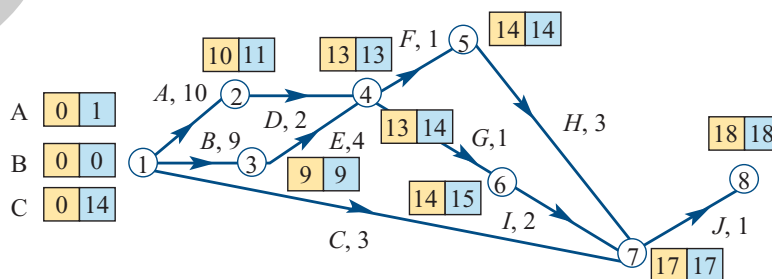
It may be possible to speed up a project by completing some activities more quickly. For example, in Example 8 it may be possible to shorten the time for activities *B* and *C*. This is called **crashing**. If the duration of *B* is reduced to 5 days and the duration of *C* is reduced to 4 days, then a new critical path is formed. The project diagram is now as shown.



There is a new critical path:  $A \rightarrow D \rightarrow E \rightarrow F$   
 The minimum time for finishing is now 35 days.

## Exercise 24C

- 1 a The project diagram is given below for the table on page 600, with earliest and latest starting times for the activities.
- Find the critical path.
  - Give the floats of the non-critical activities.

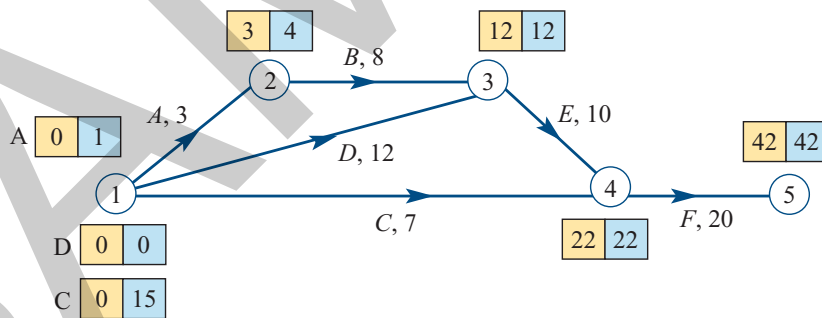


Activity	Duration (weeks)	Preceding activities
A	10	—
B	9	—
C	3	—
D	2	A
E	4	B
F	1	D, E
G	1	D, E
H	3	F
I	2	G
J	1	C, H, I

b The project diagram is given for the following, with earliest and latest starting times for the activities.

- i Find the critical path.      ii Give the floats of the non-critical activities.

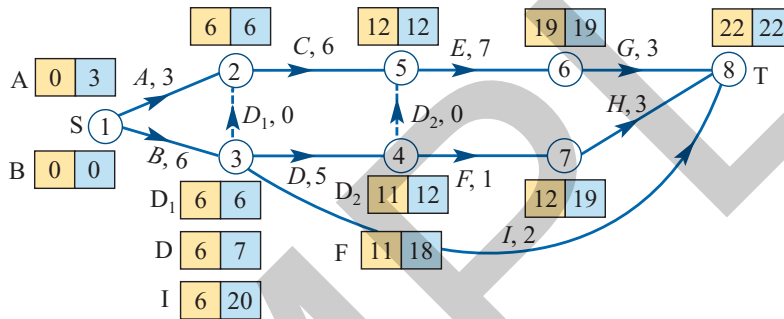
Activity	Duration (minutes)	Preceding activities
A Shower	3	—
B Dress	8	A
C Fetch car	7	—
D Make breakfast	12	—
E Eat breakfast	10	B, D
F Drive to work	20	C, E



c The project diagram is given for the following with earliest and latest starting times for the activities.

- i Complete the missing information in the table.  
 ii Find the critical path.  
 iii Give the floats of the non-critical activities.

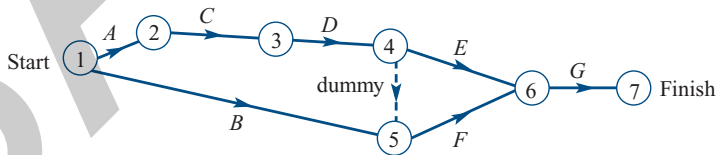
Activity	Duration (weeks)	Preceding activities
A	3	—
B	6	—
C	6	
D		B
E	7	
F	1	D
G		E
H	3	
I	2	B



2 a Draw a digraph (project diagram) to represent this project: Repair a component of an engine.

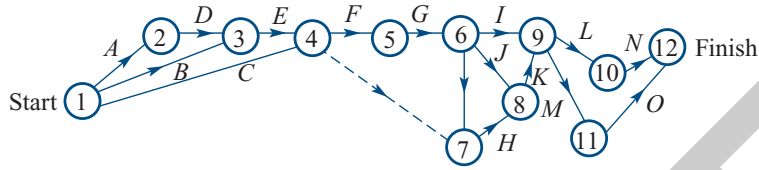
Activity	Description	Immediate predecessors
A	Remove engine	—
B	Remove component	A
C	Order necessary parts	B
D	Repair component	C
E	Replace component	D
F	Reinstall engine	E

b The following is the project diagram for the repair of a car.



Activity	Description
A	Remove panel
B	Order component
C	Remove broken component
D	Pound out dent
E	Repaint
F	Install new component
G	Replace panel

- i Which events are the immediate predecessors of event ‘remove broken component’?
- ii Which events are the immediate predecessors of the event ‘install new component’?
- c The following is the project diagram for building a house.



Activity	Description	Activity	Description
A	Prepare foundations	I	Glazing
B	Order bricks	J	Wiring
C	Order tiles	K	Plastering
D	Lay drains	L	Fittings
E	Erect shell	M	Clear site
F	Roofing	N	Paint and clean
G	Flooring	O	Lay paths
H	Plumbing		

- i Which events are the immediate predecessors of event ‘fittings’?
- ii Which events are the immediate predecessors of event ‘erect shell’?
- d Project: Produce and assemble two new products. Draw the associated project diagram for the project.

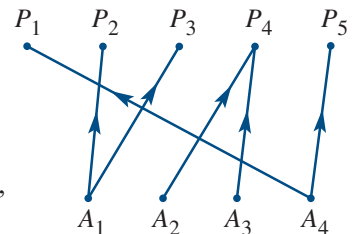
Activity	Description	Immediate predecessors
A	Order machine A	—
B	Order machine B	—
C	Install machine A	A
D	Install machine B	B
E	Order machine J	—
F	Install machine J	E
G	Produce part a	C
H	Produce part b	D
I	Use machine J to assemble parts a and b	F, G, H

## 24.4 Allocation problems

A management company for a project needs to fill several job positions. The company has various types of vacant positions, and various applicants who are qualified for a certain number of jobs.

Let  $A_1, A_2, A_3$  and  $A_4$  be the set of applicants, and  $P_1, P_2, P_3, P_4$  and  $P_5$  the different positions available.

Figure 24.7 is an example of a **bipartite** graph. There are two separate sets of vertices (‘Positions’ and ‘Applicants’), joined by edges.



The edges indicate the positions for which an applicant is qualified.

Applicant	Position
$A_1$	$P_2, P_3$
$A_2$	$P_4$
$A_3$	$P_4$
$A_4$	$P_1, P_5$

It is clear from this example that not all jobs can be filled. Figure 24.8 represents the options that would be available if there were a fifth applicant,  $A_5$ , who can do jobs  $P_3$  and  $P_5$ .

It is still impossible to fill all the jobs.

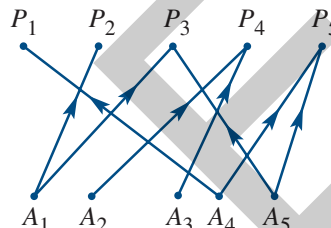


Figure 24.8

In the following section, allocations are considered where the edges of the corresponding bipartite graphs are weighted.

### The Hungarian algorithm

Four supermarkets ( $A, B, C$  and  $D$ ) are supplied from four distribution outlets ( $W, X, Y$  and  $Z$ ). The cost in dollars of supplying one vanload of goods is given in the table opposite. This table is called a **cost matrix**.

	$A$	$B$	$C$	$D$
$W$	30	40	50	60
$X$	70	30	40	70
$Y$	60	50	60	30
$Z$	20	80	50	70

The aim is to supply each of the supermarkets at the lowest cost. This can be done by trial and error but that would be time consuming. The **Hungarian algorithm** gives a method for determining this minimum cost.

#### Step 1

Simplify the cost matrix by subtracting the minimum entry in each row from each of the elements in that row.

- i.e. 30 is subtracted from all entries in Row 1
- 30 is subtracted from all entries in Row 2
- 30 is subtracted from all entries in Row 3
- 20 is subtracted from all entries in Row 4

	$A$	$B$	$C$	$D$
$W$	0	10	20	30
$X$	40	0	10	40
$Y$	30	20	30	0
$Z$	0	60	30	50

The process is repeated for columns where there is no '0' entry.

	$A$	$B$	$C$	$D$
$W$	0	10	10	30
$X$	40	0	0	40
$Y$	30	20	20	0
$Z$	0	60	20	50

**Step 2**

Cover the zero elements with the minimum number of lines. If this minimum number equals the number of rows, then it is possible to obtain a maximum matching using all vertices immediately. Otherwise, continue to Step 3.

	A	B	C	D
W	0	10	10	30
X	40	0	0	40
Y	30	20	20	0
Z	0	60	20	50

**Step 3**

The minimum uncovered element is '10'. Add this number to the rows and columns that are covered, i.e. Columns 1 and 4 and Row 2.

	A	B	C	D
W	10	10	10	40
X	60	10	10	60
Y	40	20	20	10
Z	10	60	20	60

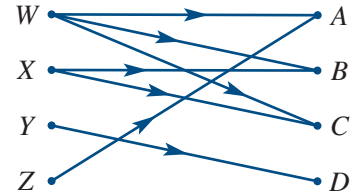
**Note**

Where lines intersect, this number is added twice.

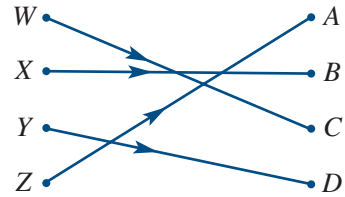
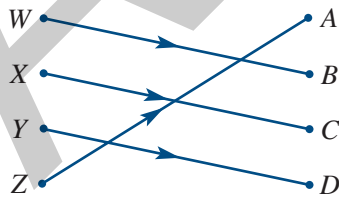
The minimum uncovered element (10) is now subtracted from all entries and **Step 2 is repeated**. In this example, the minimum number of lines is equal to the number of rows.

	A	B	C	D
W	0	0	0	30
X	50	0	0	50
Y	30	10	10	0
Z	0	50	10	50

A bipartite graph can be used to show the possibilities. The edges are chosen through the zero entries in the table. The graph shows all possible allocations.



The possibilities with four edges (one task per person) are as follows, i.e. there are two possible choices.



Cost (\$)	Cost (\$)
W to B = 40	W to C = 50
X to C = 40	X to B = 30
Y to D = 30	Y to D = 30
Z to A = 20	Z to A = 20
<b>Total = 130</b>	<b>Total = 130</b>

**Example 9****Allocating tasks using the Hungarian algorithm**

Find the minimum cost of the allocation problem using a cost matrix.

	W	X	Y	Z
A	3	7	6	2
B	4	6	5	3
C	5	7	6	3
D	6	4	3	7

**Solution**

- 1 Subtract the minimum element of each row from each element of that row.

	W	X	Y	Z
A	1	5	4	0
B	1	3	2	0
C	2	4	3	0
D	3	1	0	4

- 2 Subtract the minimum element of each column from each element of that column, then cover the zero elements with the minimum number of lines.

	W	X	Y	Z
A	0	4	4	0
B	0	2	2	0
C	1	3	3	0
D	<del>2</del>	0	0	<del>4</del>

- 3 Add the minimum uncovered number (2) to each of the rows and columns that are covered.

	W	X	Y	Z
A	2	4	4	2
B	2	2	2	2
C	3	3	3	2
D	6	2	2	8

**Note**

Where lines intersect, this number is added twice.

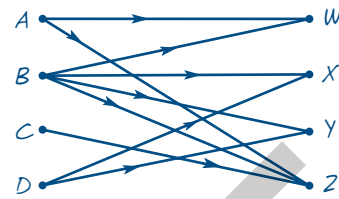
Now subtract the minimum uncovered number (2) from all entries.

	W	X	Y	Z
A	0	2	2	0
B	0	0	0	0
C	1	1	1	0
D	4	0	0	6

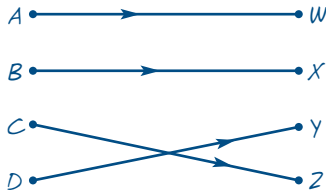
- 4 Repeat step 2 by covering the zero elements with the minimum number of lines.

	W	X	Y	Z
A	0	2	2	0
B	0	0	0	0
C	1	1	1	0
D	<del>4</del>	0	0	<del>6</del>

- 5 Construct a bipartite graph to show all possible allocations. The ‘O’ entries indicate the required allocations.

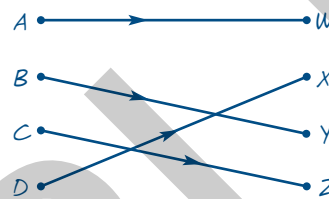


- 6 Choose only those edges that allow allocation of only one task to each person. Determine the minimum cost for each.



$$\text{Total cost} = 3 + 6 + 3 + 3 = 15$$

or



$$\text{Total cost} = 3 + 5 + 3 + 4 = 15$$

## Exercise 24D

- 1 a A cost matrix is shown. Find the allocation(s) by the Hungarian algorithm that will give the minimum cost.

	A	B	C	D
W	110	95	140	80
X	105	82	145	80
Y	125	78	140	75
Z	115	90	135	85

- b Find the minimum cost for the given cost matrix and give the possible allocation.

	A	B	C	D
W	2	4	3	5
X	3	5	3	4
Y	2	3	4	2
Z	2	4	2	3

- 2 A school is to enter four students in four track events: 100 m, 400 m, 800 m and 1500 m. The four students' times (in seconds) are given in the table. The rules permit each student to enter only one event.

	100 m	400 m	800 m	1500 m
Dimitri	11	62	144	379
John	13	60	146	359
Carol	12	61	149	369
Elizabeth	13	63	142	349

The aim is to obtain the minimum total time.

Use the Hungarian algorithm to select the ‘best’ team.

- 3 Three volunteer workers, Joe, Meg and Ali, are available to help with three jobs. The time (in minutes) that each worker can complete each task is given in the table opposite.

	<i>Job</i>		
	A	B	C
<i>Joe</i>	20	20	36
<i>Meg</i>	16	20	44
<i>Ali</i>	26	26	44

Which allocation of workers to jobs will enable the jobs to be completed in the minimum time?

- 4 A company has four machine operators and four different machines that they can operate. The table shows the hourly cost in dollars of running each machine for each operator. How should the machinists be allocated to the machines to maximise the hourly output from each of the machines with the staff available?

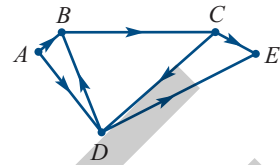
	<i>Machine</i>			
<i>Operator</i>	<i>W</i>	<i>X</i>	<i>Y</i>	<i>Z</i>
<i>A</i>	38	35	26	54
<i>B</i>	32	29	32	26
<i>C</i>	44	26	23	35
<i>D</i>	20	26	32	29



## Key ideas and chapter summary

**Directed graph**

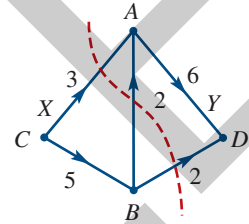
A **directed graph** is a graph where direction is indicated for every edge. This is often abbreviated to **digraph**.

**Cut**

A **cut** is a line dividing a directed graph into two parts (shown as a broken line dividing the graph below into two sections, labelled  $X$  and  $Y$ ).

**Capacity of a cut**

The sum of the capacities (weights) of the edges directed from  $X$  to  $Y$  that the cut passes through. For the weighted digraph shown the capacity of the cut is 7.

**Minimum cut**

The **minimum cut** is the cut with the minimum capacity.

**Maximum flow**

Maximum flow = capacity of 'minimum' cut.

**Critical path analysis convention**

When drawing weighted digraphs for problems involving critical paths the following conventions are used:

- The edges (or arcs) represent the activities.
- The vertices (or nodes) represent events. The start/finish of one or more activities is called an event.
- The start. The 'start vertices' have no predecessors.
- A vertex (called the finish node) representing the completion of the project should be included in the network.
- An activity should not be represented by more than one edge in the network.
- Two vertices (nodes) can be connected directly by, at most, one edge.

**Dummy activity**

In order to satisfy the final two conventions it is sometimes necessary to introduce a **dummy activity** that takes zero time.

**Earliest start time (EST)**

The **earliest start time** is the earliest time the activity can be commenced. The earliest start time for activities without predecessors is chosen to be 0.

**Latest start time (LST)**

The **latest start time** is the latest time an activity can be left if the whole project is to be completed on time. Latest event times are established by working backwards through the network.

**Critical activity**

A **critical activity** is any task that, if delayed, will also delay the entire project. If the difference between the earliest start time at the beginning of an activity and the latest start time at the end of the activity is equal to the duration of that activity, then the activity is critical.

**Critical path**

The **critical path** in a project diagram is the path that has the longest completion time

**Crashing**

**Crashing** is the process of shortening the length of time in which a project can be completed by completing some activities more quickly. This can usually only be done by increasing the cost of the project.

**Bipartite graph**

A **bipartite graph** is a graph where the vertices can be partitioned into two disjoint sets and each edge has a vertex in each set.

Weighted bipartite graphs are used to represent allocation problems.

**The Hungarian algorithm**

Provides a method of finding the optimal solution for an allocation problem.

**Skills check**

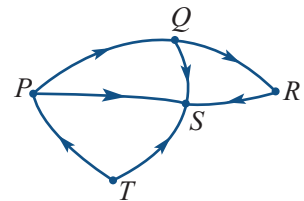
Having completed this chapter you should be able to:

- determine the capacity of a cut
- determine the maximum flow for a network
- determine the critical path for a project
- determine the earliest start time and latest start time for an event
- Use the Hungarian algorithm to solve allocation problems

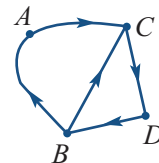
**Multiple-choice questions**

- 1 For the digraph opposite, which point is **not** reachable from  $P$ ?

A  $P$       B  $Q$       C  $R$       D  $S$       E  $T$



- 2 For the digraph opposite, which of the following matrices could be used to show all possible paths between the vertices  $A$ ,  $B$ ,  $C$  and  $D$ ?



**A**

	$A$	$B$	$C$	$D$
$A$	0	1	1	1
$B$	0	0	0	1
$C$	0	1	0	1
$D$	0	0	0	0

**B**

	$A$	$B$	$C$	$D$
$A$	0	2	1	2
$B$	0	0	0	1
$C$	0	1	0	1
$D$	0	0	0	0

**C**

	$A$	$B$	$C$	$D$
$A$	0	0	0	0
$B$	2	0	1	0
$C$	1	0	0	0
$D$	2	1	1	0

**D**

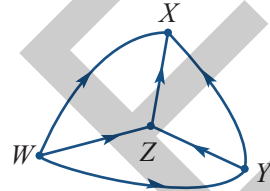
	A	B	C	D
A	0	1	1	1
B	1	0	2	2
C	1	1	0	1
D	1	1	2	0

**E**

	A	B	C	D
A	0	1	1	1
B	1	0	1	1
C	1	1	0	1
D	1	1	1	0

The following information relates to questions 3 and 4

Four teams  $W$ ,  $X$ ,  $Y$  and  $Z$  play each other in a round robin competition. The results of the competition are represented in the digraph opposite. In the graph, an arrow from  $W$  to  $X$  indicates that team  $W$  defeated team  $X$ .



3 In this competition, the one-step dominance score of team  $W$  is:

- A** 0      **B** 1      **C** 2      **D** 3      **E** 4

4 A matrix showing both one- and two-step dominances of each of the team is:

**A**

	W	X	Y	Z
W	0	1	1	1
X	0	0	0	0
Y	0	1	0	1
Z	0	1	0	0

**B**

	W	X	Y	Z
W	0	3	1	2
X	0	0	0	0
Y	0	2	0	1
Z	0	1	0	0

**C**

	W	X	Y	Z
W	0	0	0	0
X	1	0	1	1
Y	1	0	0	0
Z	1	0	1	0

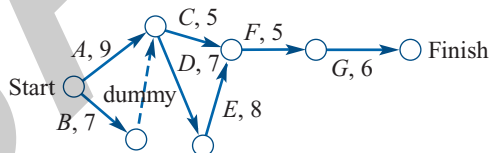
**D**

	W	X	Y	Z
W	0	0	0	1
X	3	0	2	0
Y	1	0	0	1
Z	2	0	1	0

**E**

	W	X	Y	Z
W	1	0	0	0
X	0	1	0	0
Y	0	0	1	0
Z	0	0	0	1

5 The directed graph represents a project development with activities and their durations (in days) listed on the arcs of the graph. Note that the dummy activity takes zero time.

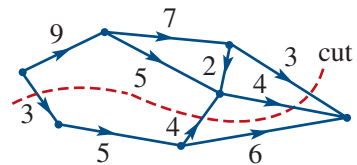


The earliest time (in days) that activity  $F$  can begin is:

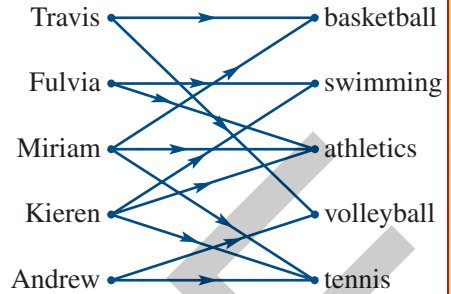
- A** 0      **B** 12      **C** 14      **D** 22      **E** 24

6 A cut of network is as shown. The capacity of the cut is:

- A** 3      **B** 6      **C** 9  
**D** 10      **E** 14



7 A group of five students represent their school in five different sports. The information is displayed in a bipartite graph. From this graph we can conclude that:

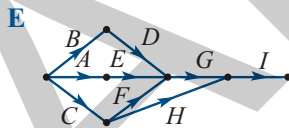
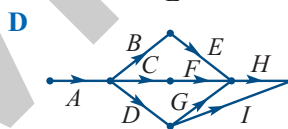
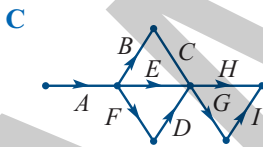
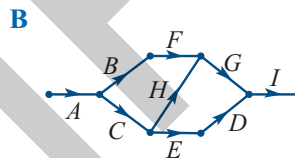
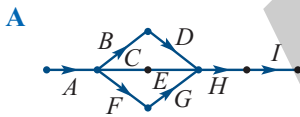


- A Travis and Miriam played all the sports between them.
- B In total Miriam and Fulvia played fewer sports than Andrew and Travis.
- C Kieren and Miriam each played the same number of sports.
- D In total Kieren and Travis played fewer different sports than Miriam and Fulvia.
- E Andrew played fewer sports than any of the others.

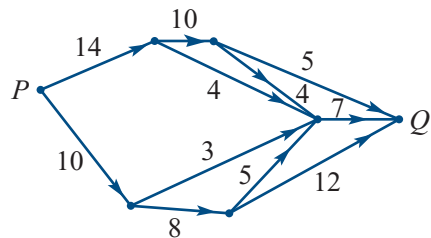
8 A project involves nine basic tasks:  $A, B, C, D, E, F, G, H, I$ . These tasks must be performed by obeying the following sequence rules:

- Task  $A$  must be done before tasks  $B, C$  and  $E$ .
- Task  $C$  must be done before tasks  $D$  and  $H$ .
- Task  $F$  must be done after task  $B$ .

Which one of the following networks could represent this schedule?



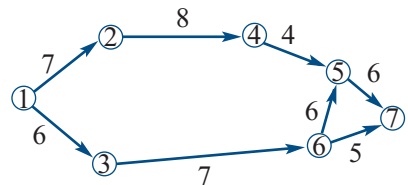
9 In the communications network shown, the numbers represent transmission capacities for information (data) in scaled units. What is the maximum flow of information from station  $P$  to station  $Q$ ?



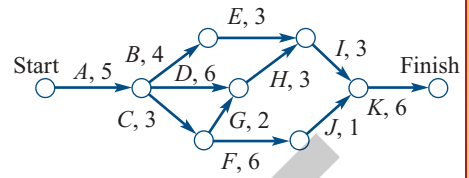
- A 20      B 23      C 22
- D 24      E 30

10 The maximum flow in the network opposite linking node 1 and node 7 is:

- A 10      B 11      C 12
- D 13      E 14

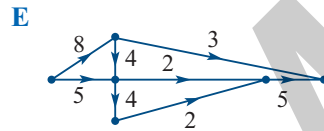
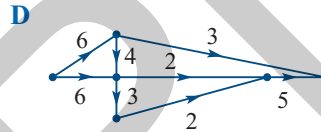
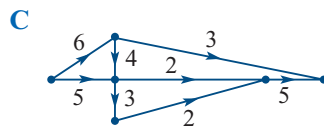
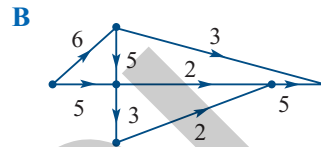
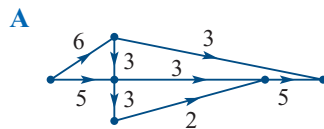


- 11** This graph represents the activity schedule for a project where the component times in days are shown. The critical path for the network of this project is given by:

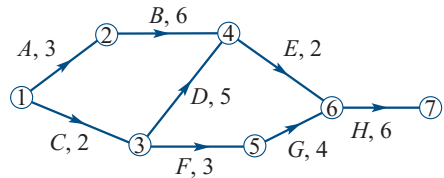


- A** A-B-E-I-K    **B** A-D-H-I-K    **C** A-C-G-H-I-K  
**D** A-C-F-J-K    **E** A-D-G-F-J-K

- 12** In the networks shown below the capacity of each edge is indicated. Which network has the greatest possible flow value?

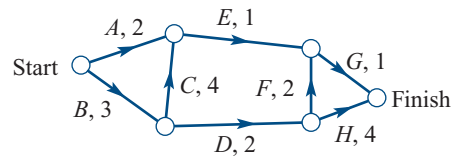


- 13** The edges in this directed graph correspond to the tasks involved in the preparation of an examination. The numbers indicate the time, in weeks, needed for each task. The total number of weeks needed for the preparation of the examination is:



- A** 14    **B** 15    **C** 16    **D** 17    **E** 27

- 14** The directed graph represents a manufacturing process with activities and their duration (in hours) listed on the arcs of the graph. The earliest time (in hours) after the start that activity *G* can begin is:

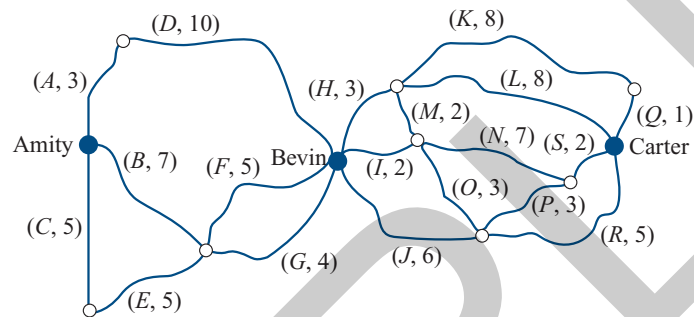


- A** 3    **B** 5    **C** 6    **D** 7    **E** 8

## Extended-response questions

- 1 The map below shows the roads that connect the towns of Amity, Bevin and Carter. The towns and the major intersections (indicated by open circles) form the nodes of this network of roads.

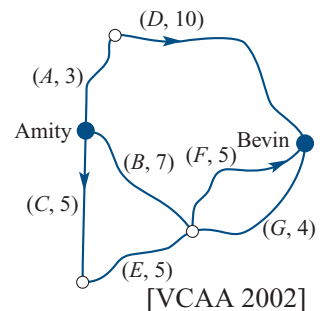
Labels on roads indicate their names and lengths in kilometres. For instance,  $(E, 5)$  indicates Road  $E$  is 5 km long.



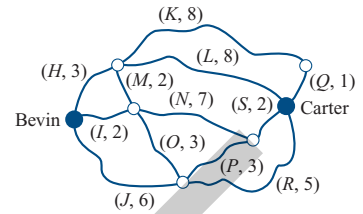
- a** What is the length of the shortest path from Amity to Bevin?
- b** The Amity Cycling Club conducts a race beginning at Amity with checkpoints at every node in this network. The race covers the full length of every road on the network in any order or direction chosen by the riders. A rider may pass through each checkpoint more than once, but must travel along each road exactly once.
- i** One competitor claims this cannot be done. By referring to the degrees of the nodes in this network of roads, explain why it is possible to travel on every road once only when cycling according to the club's rules.
- ii** Under the club's rules for the race, where is the finish line?
- iii** One competitor begins his race along roads  $A-D-I-M-H$  in that order. To continue the race, which road should this competitor avoid at the end of road  $H$ ? Justify your answer.
- c** For 2007, the club wants to start the race at Amity and finish at Carter over the shortest route that still requires riders to ride the full length of every road in this network. The rules will be modified to allow riders to travel twice along one of the roads.

Which road must be travelled twice in 2007?

- d** A suggestion for the proposed race in 2007 is to permit riders to travel along roads only in a specified direction between Amity and Bevin. For this section of the race, the suggested directions for roads  $C$ ,  $D$  and  $F$  are as shown by arrows on the map section opposite. Clearly indicate the correct direction for each of the other roads.



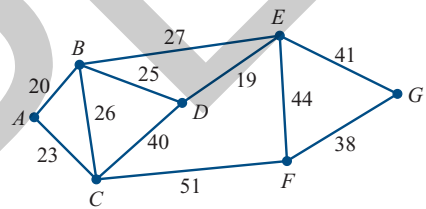
- 2 The Water Authority wants to lay water mains along the roads in order to put a fire hydrant at every node on the network shown on this map section. It decides that a minimal spanning tree for this network is suitable.



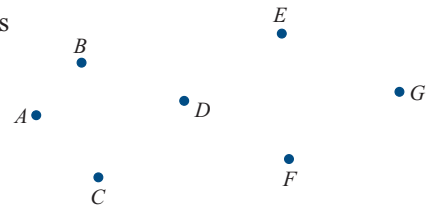
- Draw in a minimal spanning tree for this network.
- Each week, Andrew, who lives in Bevin, must travel through this network to inspect each of the fire hydrants and then return to Bevin.
  - Write down, in order, the road sections that Andrew must travel to complete a circuit of shortest length, beginning at Bevin. He does this by travelling along a circuit that prevents him from travelling along any road more than once. List the shortest circuit for this.
  - What is the total length of this circuit?

[VCAA 2002]

- 3 A car rally is being held on a network of roads linking seven outback towns. The network is shown opposite. The distance in kilometres along each road is shown in the diagram.



- In the network, what is the length of the shortest route from  $A$  to  $G$ ?
- At present it is not possible to travel along each road in this network once and only once starting at  $A$  and returning to  $A$ . Which towns would need to be linked by an additional road to make this possible?
- The towns are linked by a telecommunications grid to a remote education centre at  $G$ . The cable of this grid runs along the road network. Draw the minimal-length spanning tree for the cable network using the diagram opposite as a basis.
- What is the minimum length of cable required for the telecommunication grid?



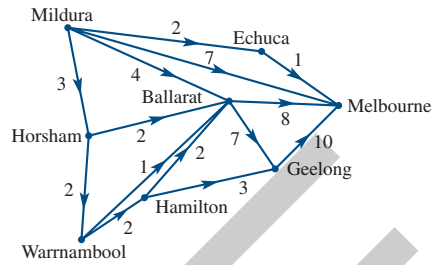
- Four of the towns,  $B$ ,  $C$ ,  $D$  and  $E$ , need race marshalls assigned to them. The four marshalls are Ann, Bianca, Con and David. The table shows the times (in minutes) that the marshalls would take to travel from their homes to each town.

	$B$	$C$	$D$	$E$
<i>Ann</i>	25	30	15	35
<i>Bianca</i>	22	34	20	45
<i>Con</i>	32	20	33	35
<i>David</i>	40	30	28	26

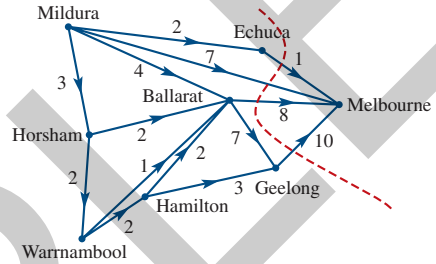
Using the Hungarian algorithm or otherwise, assign the marshalls to the towns in order to minimise the travel time overall.

[VCAA 2000]

- 4 WestAir Company flies routes in western Victoria. The network shows the layout of connecting flight paths for WestAir, which originate in Mildura and terminate either in Melbourne or on the way to Melbourne. On this network the available spaces for passengers flying out of various locations on one morning are listed.



- a The network is cut as shown. What does this cut tell us about the maximum number of passengers who could depart Mildura and arrive in Melbourne on this morning using WestAir?
- b What is the maximum number of passengers who could travel from Mildura to Melbourne for the morning via WestAir services?

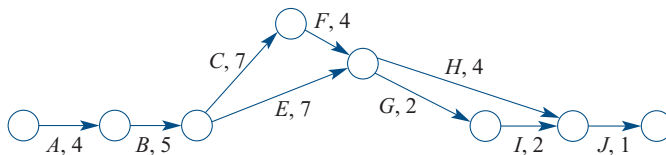


[VCAA 2001]

- 5 LiteAero Company designs and makes light aircraft for the civil aviation industry. They identify 10 activities required for production of their new model, the MarchFly. These, and the associated activity durations, are given in the table opposite.

Activity	Duration (weeks)	Immediate predecessor(s)
A	4	—
B	5	A
C	7	B
D	6	B
E	7	B
F	4	C
G	2	E, F
H	4	F
I	2	D, G
J	1	H, I

- a A network for this project is shown. The network is incomplete as activity D must be included. Complete the network by drawing and labelling activity D.



(cont'd.)

- b** Use the information from the table to complete the earliest and latest start times.
- c** State the critical path(s) for this network.

<i>Activity</i>	<i>Earliest start time</i>	<i>Latest start time</i>
<i>A</i>	0	0
<i>B</i>	4	4
<i>C</i>	9	9
<i>D</i>	9	
<i>E</i>	9	13
<i>F</i>	16	16
<i>G</i>		20
<i>H</i>	20	20
<i>I</i>	22	22
<i>J</i>	24	24

[VCAA 2001]

- 6** A school swimming team wants to select a  $4 \times 200$  metre relay team. The fastest times of its four best swimmers in each of the strokes are shown in the table below. Which swimmer should swim which stroke to give the team the best chance of winning, and what would be their time to swim the relay?

	<i>Stroke</i>			
	<i>Backstroke</i>	<i>Breaststroke</i>	<i>Butterfly</i>	<i>Freestyle</i>
<i>Rob</i>	76	78	70	62
<i>Joel</i>	74	80	66	62
<i>Henk</i>	72	76	68	58
<i>Sav</i>	78	80	66	60