

Transformations

Objectives

- To define **translations**
- To define **reflections** in the axes and in the line $y = x$
- To define **dilations** from the x axis and the y axis
- To apply these transformations to points and figures
- To find **algebraic rules** for these transformations
- To find the **composition** of two transformations and give the rule for this transformation
- To apply **transformations** to graphs
- To **determine the rule** which transforms one graph to another (within a suitable family of graphs)
- To sketch the graph of the **absolute value function**, the **integer part function** and transformations of these graphs
- To describe transformations with function notation

Introduction

In this chapter three different types of transformations of the cartesian plane are discussed. These are

- Translations
- Reflections
- Dilations.

These transformations are very useful in the graphing of functions.

A transformation is a rule which ‘associates’ each point in the cartesian plane to another point in the plane. These points uniquely define each other through the rule.

For example, one transformation can be defined by:

‘Add 5 to each x coordinate.’

This can be expressed algebraically

$$(x, y) \rightarrow (x + 5, y)$$

$$\text{e.g., } (1, 6) \rightarrow (6, 6)$$

This is read as:

‘The point with coordinates $(1, 6)$ is mapped to the point with coordinates $(6, 6)$ ’.

A formal definition is the following:

A transformation T is a mapping from R^2 to R^2 such that if $T(a, b) = T(c, d)$ then $a = c$ and $b = d$.

The transformation defined above can be written

$$T : R^2 \rightarrow R^2, T(x, y) = (x + 5, y)$$

This formal notation is avoided in this book and the transformation will be defined by the rule given in the form

$$(x, y) \rightarrow (x + 5, y)$$

8.1 Translations

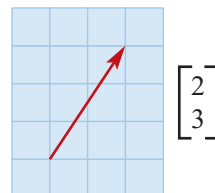
The transformation defined above, i.e. $(x, y) \rightarrow (x + 5, y)$, is an example of a **translation**. A **translation** is a transformation for which each point in the plane is moved the same distance in the same direction.

In this section, a **vector** will mean a column of two numbers. The first number indicates a ‘move’ in the positive or negative direction of the x axis and the second indicates a ‘move’ in the positive or negative direction of the y axis. A directed line segment is used to illustrate a vector.

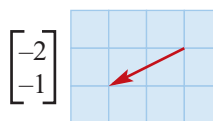
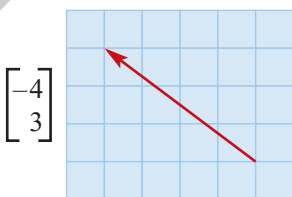
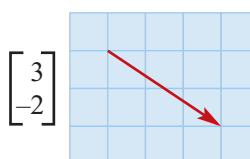
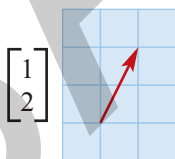
For example, the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the vector ‘2 to the right and 3 up’.

The image of the point $(1, 2)$ would be $(3, 5)$ under the translation

determined by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.



The top number gives the displacement in the positive or negative direction of the x axis and the lower number gives the displacement in the positive or negative direction of the y axis.



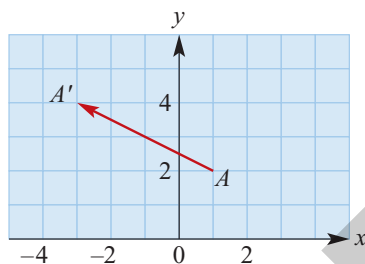
If the top number is negative, the displacement is to the left and if the lower number is negative, the displacement is downwards.

Vectors can be used to describe translations. They will be studied in a more general context in Chapter 15.

Example 1

The point A has coordinates $(1, 2)$. Find the image of A under the translation defined by the vector $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

Solution



The image of A is $(-3, 4)$.

Example 2

Find the vector which defines the translation which takes $A(3, 4)$ to $A'(2, 6)$.

Solution

Let $(3, 4) \rightarrow (2, 6)$

The vector $\begin{bmatrix} a \\ b \end{bmatrix}$ defines this translation if $(3 + a, 4 + b) = (2, 6)$

This implies

$$\begin{aligned} 3 + a &= 2 & \text{and} & & 4 + b &= 6 \\ \text{i.e., } a &= -1 & \text{and} & & b &= 2 \end{aligned}$$

The vector is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Example 3

A translation is defined by the rule $(x, y) \rightarrow (x - 3, y + 2)$ and the point A with coordinates (a, b) is mapped to $A'(7, -1)$. Find the values of a and b .

Solution

$$\begin{aligned} (a, b) &\rightarrow (a - 3, b + 2) = (7, -1) \\ \therefore a - 3 &= 7 & \text{and} & & b + 2 &= -1 \\ a &= 10 & \text{and} & & b &= -3 \end{aligned}$$

Exercise 8A

Example 1 1 Find the images of the points in each of the following under the translations described by the given vectors.

a $(3, 1), \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

b $(4, 5), \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

c $(-2, 4), \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

d $(3, 2), \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

e $(4, 5), \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

Example 2 2 For each of the following, find the vectors describing the translations that map A to A' .

a $A(1, 2), A'(5, 3)$

b $A(3, 8), A'(2, 9)$

c $A(1, 2), A'(5, 4)$

d $A(-3, 0), A'(4, 6)$

e $A(-4, -3), A'(0, 0)$

3 In each of the following the given point A' is the image of an object point A under the translation described by the given vector. Find the coordinates of A .

a $A'(7, 9), \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b $A'(3, 6), \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

c $A'(0, 6), \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

4 Give the vectors describing the translations which map

a $\triangle ABC$ to $\triangle PQR$

b $\triangle ABC$ to $\triangle LMN$

c $\triangle XYZ$ to $\triangle ABC$

d $\triangle ABC$ to $\triangle ABC$

5 Draw axes for x and y from -4 to 5 . Draw the following triangles.

$\triangle ABC$ with $A(2, 2), B(4, 2), C(2, 5)$

$\triangle PQR$ with $P(1, -2), Q(3, -2), R(1, 1)$

$\triangle XYZ$ with $X(-3, 1), Y(-1, 1), Z(-3, 4)$.

Give the vectors describing the translations which map

a $\triangle ABC$ to $\triangle PQR$

b $\triangle PQR$ to $\triangle ABC$

c $\triangle PQR$ to $\triangle XYZ$

d $\triangle ABC$ to $\triangle ABC$

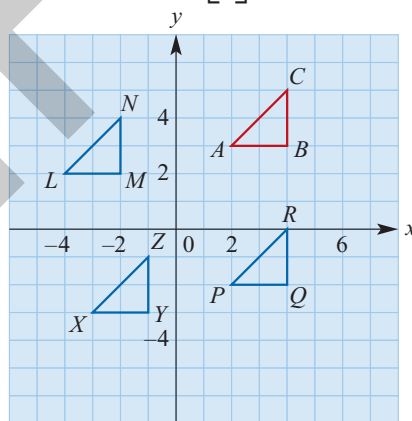
6 **a** Find the image of the point $(2, 3)$ under the translation determined by the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

followed by the translation determined by the vector $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

b Find the image of the point $(-5, 6)$ under the translation determined by the vector

$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ followed by the translation determined by the vector $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

c Describe how one translation may be used to obtain the final image in **a** and **b**.

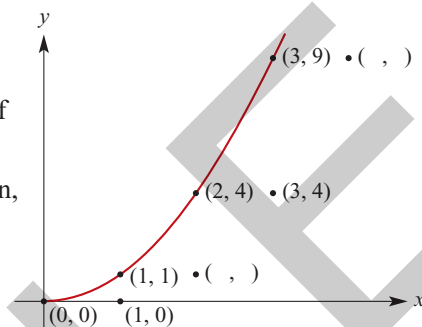


Example 3 7 A translation has a rule $(x, y) \rightarrow (x - 5, y + 3)$. Find

- a the image of the point $(1, 3)$ under this translation
- b a and b if $(a, b) \rightarrow (6, 7)$ under this translation.

8 A translation has a rule $(x, y) \rightarrow (x + 1, y)$.

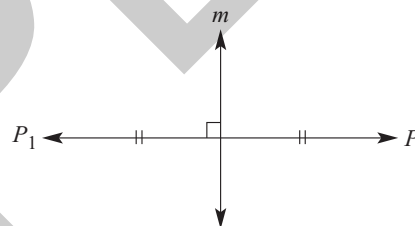
- a The points $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 9)$ all lie on the graph of $y = x^2$. Find the image of each of these points under the translation.
- b Sketch the graph of $y = x^2$ for $x \geq 0$ as shown, and complete.
- c Describe the image of all the points on the graph of $y = x^2$ under this transformation.



8.2 Reflections

P_1 is the image of P under a reflection in the line m .

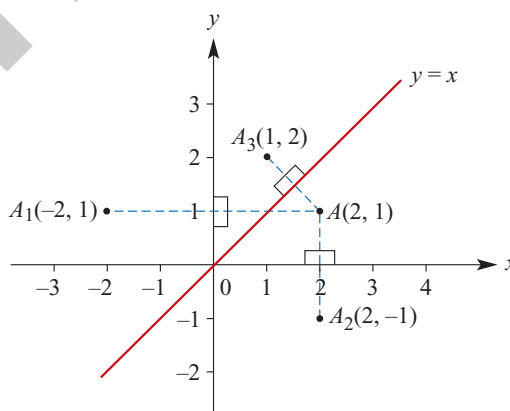
The line m is the perpendicular bisector of line PP_1 .



A_1 is the image of A under the transformation 'reflection in the y axis'.

A_2 is the image of A under the transformation 'reflection in the x axis'.

A_3 is the image of A under the transformation 'reflection in the line $y = x$ '.



For reflection in the x axis the rule is $(x, y) \rightarrow (x, -y)$

For reflection in the y axis the rule is $(x, y) \rightarrow (-x, y)$

For reflection in the line $y = x$ the rule is $(x, y) \rightarrow (y, x)$

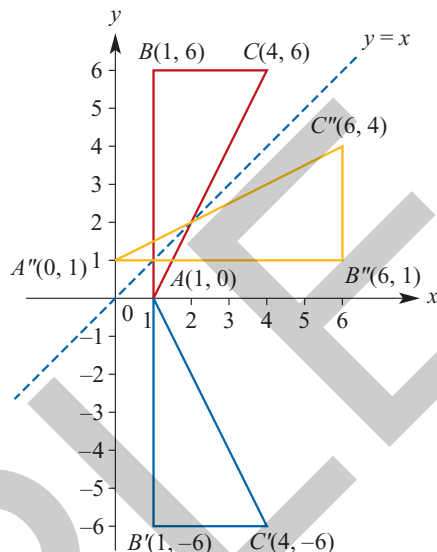
Example 4

The triangle ABC has coordinates $A(1, 0)$, $B(1, 6)$ and $C(4, 6)$. Find the image of triangle ABC under a reflection in

- a the x axis
- b the line $y = x$.

Solution

- a** $A(1, 0) \rightarrow A'(1, 0)$
 $B(1, 6) \rightarrow B'(1, -6)$
 $C(4, 6) \rightarrow C'(4, -6)$
- b** $A(1, 0) \rightarrow A''(0, 1)$
 $B(1, 6) \rightarrow B''(6, 1)$
 $C(4, 6) \rightarrow C''(6, 4)$

**Exercise 8B****Example 4**

- 1 Draw axes for x from -5 to 5 and for y from 0 to 5 . Draw triangle ABC by plotting $A(1, 2)$, $B(3, 2)$ and $C(3, 5)$. Draw the image $A'B'C'$ when $\triangle ABC$ is reflected in the y axis.
- 2 Draw axes for x from 0 to 5 and for y from -2 to 2 . Draw triangle PQR where P is $(1, -1)$, Q is $(5, -1)$ and R is $(4, 0)$. Draw the image $P'Q'R'$ when $\triangle PQR$ is reflected in the x axis.
- 3 Draw axes for x and y from -5 to 0 . Draw rectangle $WXYZ$ where W is $(-3, -1)$, X is $(-3, -2)$, Y is $(-5, -2)$ and Z is $(-5, -1)$. Draw the mirror line $y = x$. Draw the image $W'X'Y'Z'$ when $WXYZ$ is reflected in the mirror line.
- 4 Draw axes for x and y from -1 to 8 . Plot the points $A(2, 1)$, $B(5, 1)$, $C(7, 3)$ and $D(4, 3)$. Draw the parallelogram $ABCD$ and its image by reflection in the line $y = x$.
- 5 Draw axes for x and y from -6 to 7 . Draw triangle ABC where A is $(-6, -2)$, B is $(-3, -4)$ and C is $(-2, -1)$. Draw the following images of triangle ABC
 - a** triangle $A_1B_1C_1$ by reflection in the y axis
 - b** triangle $A_2B_2C_2$ by reflection in the line $y = -x$ (this is the straight line through the points $(2, -2)$, $(-4, 4)$)
 - c** triangle $A_3B_3C_3$ by reflection in the x axis.
- 6 Find the image of $(6, -2)$ under each of the following
 - a** reflection in the line $y = x$
 - b** reflection in the line $x = 0$
 - c** reflection in the line $y = 0$

7 Find the image of $(0, -1)$ under each of the following

- a** reflection in the line $y = x$ **b** reflection in the line $x = 0$
c reflection in the line $y = 0$ **d** reflection in the line $y = -x$

8.3 Dilations from the axes



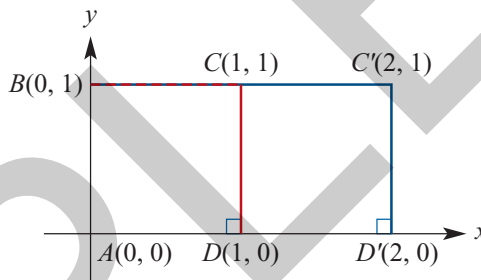
For reflections and translations, lengths and angles are preserved. In this section dilations from the axes are introduced. These transformations do not preserve distances or angles.

The transformation ‘**dilation** from the y axis of factor k ’, is defined by the rule

$$(x, y) \rightarrow (kx, y); k \in \mathbb{R}^+$$

For example, for $k = 2$, the unit square

$A(0, 0), B(0, 1), C(1, 1), D(1, 0)$
 is transformed to the rectangle
 $A(0, 0), B(0, 1), C'(2, 1), D'(2, 0)$



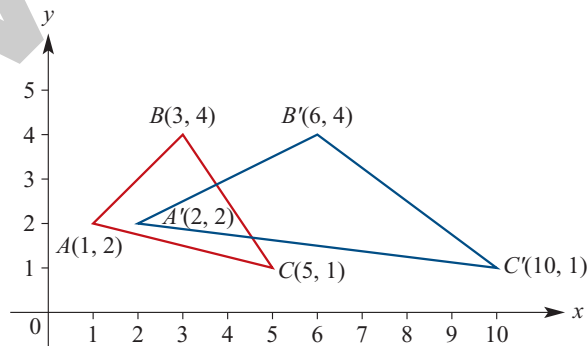
Example 5

Triangle ABC has vertices $A(1, 2), B(3, 4), C(5, 1)$.

Find the image of the triangle under a dilation of factor 2 from the y axis.

Solution

$$\begin{aligned}
 (3, 4) &\rightarrow (6, 4) \\
 (1, 2) &\rightarrow (2, 2) \\
 (5, 1) &\rightarrow (10, 1)
 \end{aligned}$$



The transformation ‘**dilation** from the x axis of factor k ’ is defined by the rule:

$$(x, y) \rightarrow (x, ky); k \in \mathbb{R}^+$$

For example, a triangle with vertices $A(1, 2), B(3, 4), C(5, 1)$ is mapped to the triangle with vertices $A'(1, 4), B'(3, 8), C'(5, 2)$ under a dilation of factor 2 from the x axis.

The unit square with vertices

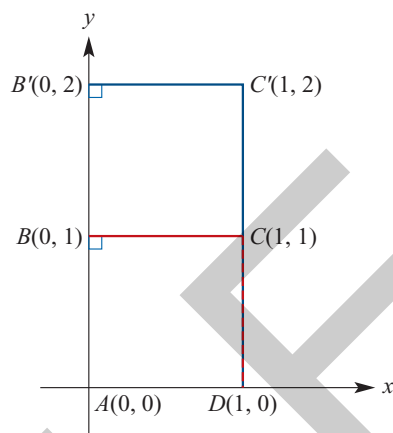
$$A(0, 0), B(0, 1),$$

$$C(1, 1), D(1, 0)$$

is mapped to the rectangle with vertices

$$A(0, 0), B'(0, 2),$$

$$C'(1, 2), D'(1, 0)$$



Exercise 8C

Example 5

- 1 Find the image of the point $(1, 3)$ under each of the following
 - a a dilation of factor 3 from the x axis
 - b a dilation of factor 2 from the y axis
 - c a dilation of factor 4 from the y axis.
- 2 Give the rule for the dilation from the x axis which maps $(1, 2) \rightarrow (1, 8)$.
- 3 Give the rule for the dilation from the y axis which maps $(3, -2) \rightarrow (9, -2)$.
- 4 a Find the image of the unit square $A(0, 0), B(0, 1), C(1, 1), D(1, 0)$ under a dilation from the x axis of factor 3.
 b Find the image of the unit square $A(0, 0), B(0, 1), C(1, 1), D(1, 0)$ under a dilation from the y axis of factor 3.
- 5 Triangle ABC has vertices $A(0, 0), B(3, 0), C(3, 4)$.
 - a Find the image of triangle ABC under
 - i a dilation of factor $\frac{1}{2}$ from the x axis
 - ii a dilation of factor $\frac{1}{2}$ from the y axis.
 - b Illustrate triangle ABC and its images.

8.4 Rules for transformations

Transformations can be described through a rule given in terms of ordered pairs. For example, it has been seen that $(x, y) \rightarrow (x + 2, y + 3)$ describes the transformation of 2 units in the positive direction of the x axis and 3 units in the positive direction of the y axis.

Example 6

Find the rules for each of the following transformations.

- a** a translation determined by the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$
b a reflection in the line $y = -x$ **c** a dilation of factor $\frac{1}{4}$ from the x axis.

Solution

- a** $(x, y) \rightarrow (x + 3, y - 2)$ **b** $(x, y) \rightarrow (-y, -x)$ **c** $(x, y) \rightarrow \left(x, \frac{1}{4}y\right)$

A point (a, b) is said to be **invariant** under a transformation if $(a, b) \rightarrow (a, b)$ under that transformation.

The transformation which maps $(x, y) \rightarrow (x, y)$ for all $(x, y) \in R^2$ is called the **identity transformation**.

Example 7

A transformation has rule $(x, y) \rightarrow (2x + 3, y + 4)$.

- a** Find the image of $(2, 3)$ under this transformation.
b Find the coordinates of the point which maps to $(11, 12)$.

Solution

- a** If $x = 2$ and $y = 3$
 $(x, y) \rightarrow (2 \times 2 + 3, 3 + 4) = (7, 7)$
 i.e., $(2, 3) \rightarrow (7, 7)$
- b** If $(2x + 3, y + 4) = (11, 12)$
 Then $2x + 3 = 11$ and $y + 4 = 12$
 $\therefore x = 4$ and $y = 8$

Exercise 8D**Example 6**

- 1** For each of the following transformations find
i the rule **ii** the invariant points (if they exist).

- a** a translation determined by the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ **c** a dilation from the y axis of factor $\frac{1}{3}$
b a dilation from the x axis of factor 4
d a translation determined by the vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$
e a reflection in the line $y = x$ **f** a reflection in the y axis

Example 7

- 2** A transformation has rule $(x, y) \rightarrow (3 - x, 2y + 1)$
a Find the image of $(2, 3)$ under this transformation.

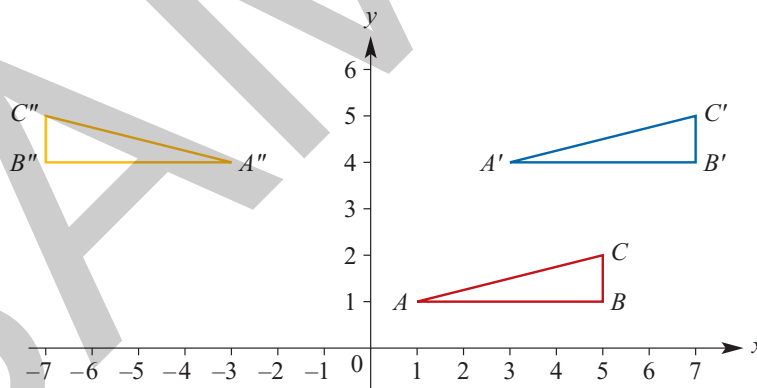
- b** Find the coordinates of the point which maps to $(-6, 12)$.
 - c** Find the coordinates of the point which is invariant under this transformation.
- 3** A transformation has rule $(x, y) \rightarrow (3 - y, 4 - x)$. Find the coordinates of the invariant point.
- 4** A transformation has rule $(x, y) \rightarrow (3 - 2x, -2y + 1)$.
- a** Find the image of $(4, -1)$ under this transformation.
 - b** Find the coordinates of the point which maps to $(7, 12)$.
 - c** Find the coordinates of the point which is invariant under this transformation.
- 5** A transformation has rule $(x, y) \rightarrow (-x, -2y)$.
- a** Find the image of $(-1, 3)$ under this transformation.
 - b** Find the coordinates of the point which maps to $(0, 0)$.
 - c** Find the coordinates of the point which is invariant under this transformation.

8.5 Composition of transformations

Consider a transformation determined by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ followed by a reflection in the line $x = 0$.

The rules for these two transformations are $(x, y) \rightarrow (x + 2, y + 3)$ and $(x, y) \rightarrow (-x, y)$ respectively.

The triangle $A(1, 1)$, $B(5, 1)$, $C(5, 2)$ is mapped to $A''(-3, 4)$, $B''(-7, 4)$, $C''(-7, 5)$



One rule can be used to describe the transformation of triangle ABC to triangle $A''B''C''$. Consider

$$(x, y) \rightarrow (x + 2, y + 3) \rightarrow (-x - 2, y + 3)$$

translation reflection

The rule is $(x, y) \rightarrow (-x - 2, y + 3)$

Note: $(1, 1) \rightarrow (-3, 4)$ $(5, 1) \rightarrow (-7, 4)$ $(5, 2) \rightarrow (-7, 5)$

This new rule is called the **composition** of the two transformations.

Example 8

Find the rule for the transformation defined by a dilation of factor 2 from the x axis followed by a translation determined by the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Solution

The dilation is given by the rule $(x, y) \rightarrow (x, 2y)$ and the translation by the rule $(x, y) \rightarrow (x - 1, y + 3)$.

The composition is determined by

$$(x, y) \xrightarrow{\text{dilation}} (x, 2y) \xrightarrow{\text{translation}} (x - 1, 2y + 3)$$

i.e., the rule is $(x, y) \rightarrow (x - 1, 2y + 3)$

Exercise 8E**Example 8**

1 For each of the following find the rule for

- i transformation of column A followed by transformation of column B
- ii transformation of column B followed by transformation of column A .

Column A

a a translation determined by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b a dilation from the x axis of factor 2

c a translation determined by the vector $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

d a translation determined by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

e a reflection in the line $y = x$

f a reflection in the line $x = 2$

g a dilation from the y axis of factor $\frac{1}{2}$

h a dilation from the y axis of factor 2

Column B

a translation determined by the vector $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$

a dilation from the y axis of factor 2

a dilation from the x axis of factor 3

a reflection in the y axis

a reflection in the line $y = -x$

a reflection in the x axis

a translation determined by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

a translation determined by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

8.6 Applying transformations to graphs of functions



As mentioned in the introduction, transformations have an important role in graphing functions. In this section techniques are developed for graphing functions using transformations.

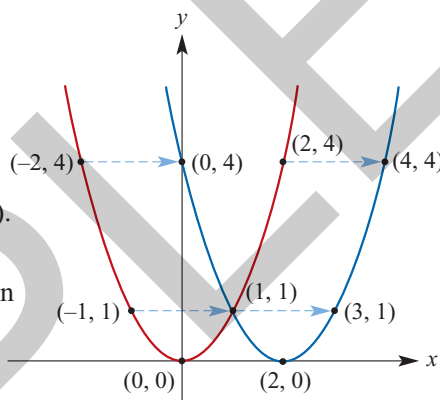
Consider $\{(x, y) : y = x^2\}$. This is the set of points on the graph of $y = x^2$. Transformations can be applied to this set of points.

For example, consider the translation determined by the vector $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ applied to this set of points.

It can be seen that all the images of the points of $y = x^2$ lie on the graph of $y = (x - 2)^2$

The rule for this transformation is $(x, y) \rightarrow (x + 2, y)$.

Each coordinate (x', y') of the image must be of this form, i.e. $x' = x + 2$, $y' = y$. The relationship between x and y is known to be $y = x^2$. Therefore if (x', y') is the image of a point on $y = x^2$, and $x = x' - 2$, then $y' = (x' - 2)^2$.



Example 9

Find the equation of the image of the line $y = x + 1$ under the translation determined by the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Sketch the graph of $y = x + 1$ and its image.

Solution

The rule for the translation is $(x, y) \rightarrow (x + 2, y + 1)$.

Let (x', y') be the coordinates of the image of (x, y)

$$x' = x + 2, y' = y + 1 \quad \text{and} \quad y = x + 1$$

$$\text{Thus} \quad x = x' - 2 \quad \text{and} \quad y = y' - 1$$

and the relation $y = x + 1$

$$\text{becomes} \quad y' - 1 = x' - 2 + 1$$

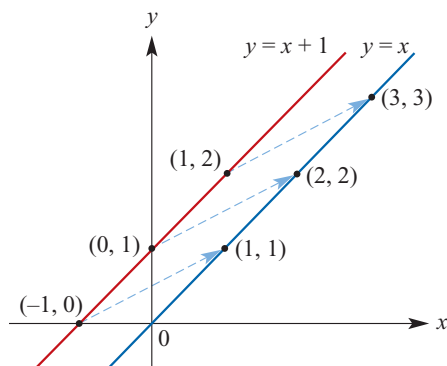
$$\therefore \quad y' = x'$$

The points $(-1, 0)$, $(0, 1)$, $(1, 2)$ and their images are shown,

$$\text{i.e.,} \quad (-1, 0) \rightarrow (1, 1)$$

$$(0, 1) \rightarrow (2, 2)$$

$$(1, 2) \rightarrow (3, 3)$$



Example 10

Find the equation of the image of the parabola $y = x^2$ under a dilation from the y axis of factor $\frac{1}{2}$. Sketch the graph of $y = x^2$ and its image.

Solution

The rule for the dilation is $(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$

Therefore, if (x', y') is a point of the image of $y = x^2$

Then $x' = \frac{x}{2}$ and $y' = y$

Also $y = x^2$

Therefore the relationship between x' and y' is

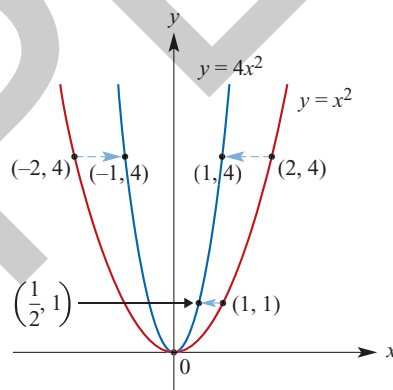
$$\begin{aligned} y' &= (2x')^2 \\ &= 4(x')^2 \end{aligned}$$

Note:

$$(1, 1) \rightarrow \left(\frac{1}{2}, 1\right)$$

$$(2, 4) \rightarrow (1, 4)$$

$$(-2, 4) \rightarrow (-1, 4)$$

**Example 11**

For $y = x^2$

- find the equation for the image of $y = x^2$ under a dilation of factor $\frac{1}{2}$ from the x axis
- sketch the graph of $y = x^2$ and its image under this transformation.
- Find the equation for the image of $y = x^2$ under the dilation of **a** followed by a translation determined by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
 - Sketch the graph of this second image on the same set of axes as **b**.

Solution

a The rule for dilation is $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$

If (x', y') is a point on the image

$$x' = x \text{ and } y' = \frac{1}{2}y$$

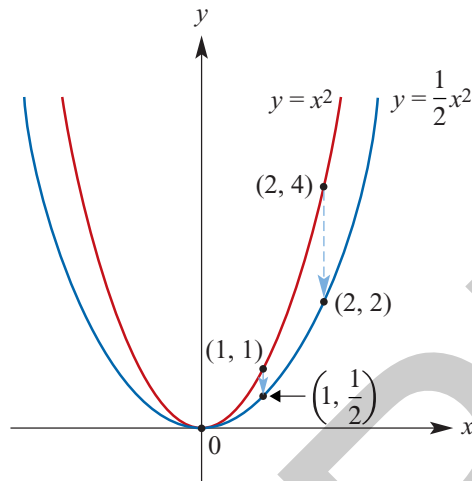
$$\text{i.e., } x' = x \text{ and } y = 2y'$$

$$\text{Also } y = x^2$$

$\therefore$ The image will have equation $2y' = (x')^2$

i.e., $y' = \frac{1}{2}(x')^2$

b



c i The rule becomes

$$(x, y) \rightarrow \left(x, \frac{1}{2}y\right) \rightarrow \left(x+1, \frac{1}{2}y+2\right)$$

dilation translation

i.e., let (x', y') be a point on the image.

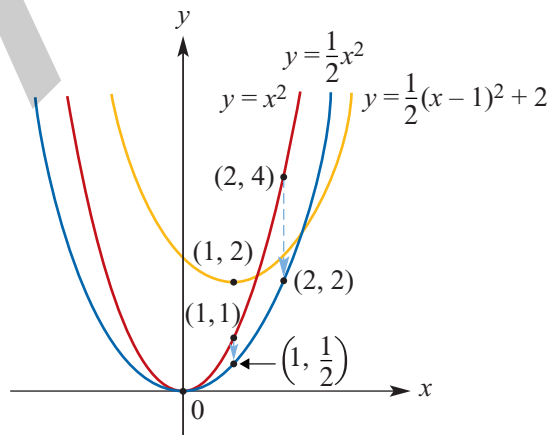
$$x' = x + 1, y' = \frac{1}{2}y + 2$$

$$\therefore x = x' - 1 \text{ and } y = 2(y' - 2)$$

$$\therefore y = x^2 \text{ is mapped to } 2(y' - 2) = (x' - 1)^2$$

$$\therefore y' = \frac{1}{2}(x' - 1)^2 + 2$$

ii



The figure displays six graphs of functions on Cartesian coordinate systems:

- $y = x$: A straight line passing through the origin (0,0).
- $y = x^2$: A parabola opening upwards with its vertex at the origin (0,0).
- $y = \frac{1}{x}$: A hyperbola with two branches, one in the first quadrant and one in the third quadrant, approaching the x-axis and y-axis as asymptotes.
- $x^2 + y^2 = 1$: A circle centered at the origin (0,0) with a radius of 1.
- $y = 2^x$: An exponential growth curve passing through the point (0, 1).

Find the image of each of the following curves under a dilation of factor 3 from the y axis followed by a translation determined by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

b $y = \frac{1}{x}$

c $x^2 + y^2 = 1$

The rule is given by $(x, y) \rightarrow (3x, y) \rightarrow (3x - 1, y + 2)$
dilation translation

$$\therefore x' = 3x - 1 \text{ and } y' = y + 2$$

i.e., $x = \frac{x' + 1}{3}$ and $y = y' - 2$

a $y = x^2$ is mapped to $y' - 2 = \left(\frac{x' + 1}{3}\right)^2$

The image has equation $y = \left(\frac{x+1}{3}\right)^2 + 2$

b $y = \frac{1}{x}$ is mapped to $y' - 2 = \frac{1}{\frac{x'+1}{3}}$

$$\therefore y' = \frac{3}{x' + 1} + 2$$

The image has equation $y = \frac{3}{x+1} + 2$

c $x^2 + y^2 = 1$ is mapped to $\left(\frac{x' + 1}{3}\right)^2 + (y' - 2)^2 = 1$

The image has equation $\frac{(x + 1)^2}{9} + (y - 2)^2 = 1$

The image is an ellipse.

Exercise 8F

Example 9

- 1** Find the image of each of the following curves under the translation determined by the vector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$. State the equation of the image and sketch the graph of both the original relation and its image on the one set of axes.

a $y = x$

b $y = x^2$

c $y = \frac{1}{x}$

d $y = \frac{1}{x^2}$

e $x^2 + y^2 = 1$

Example 10

- 2** Find the image of each of the following curves under the dilation from the x axis given by the rule $(x, y) \rightarrow (x, 2y)$, i.e., a dilation of factor 2 from the x axis. State the equation of the image and sketch the graph of both the original relation and its image on the one set of axes.

a $y = x$

b $y = x^2$

c $y = \frac{1}{x}$

d $y = \frac{1}{x^2}$

e $x^2 + y^2 = 1$

Examples 11, 12

- 3** For each of the curves considered in **1**, find the equation of its image under the transformation defined by a reflection in the line $x = 0$ followed by a translation determined by the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.
- 4** For each of the curves considered in **1**, find the equation of its image under the transformation defined by a dilation of factor 2 from the y axis followed by a translation determined by the vector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.
- 5** Find the image of the graph of $y = 2x + 3$ under the transformation defined by first reflecting in the line $y = x$ and then translating as determined by the vector $\begin{bmatrix} -4 \\ 6 \end{bmatrix}$.
- 6** Find the image of the line $\{(x, y): y = x + 2\}$ under each of the following transformations. Sketch a graph showing $\{(x, y): y = x + 2\}$ and its image in each case.
- a** a translation determined by the vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- b** a reflection in the x axis
- c** a dilation of factor 4 from the x axis
- d** a reflection in the line $y = x$
- e** a reflection in the y axis

- 7 Repeat **1** for the circle $\{(x, y): x^2 + y^2 = 4\}$.
- 8 Consider $\{(x, y): y = x^2\}$. Find the image after
- a reflection in the x axis then a dilation of factor 2 from the x axis
 - b translation where $(0, 0) \rightarrow (5, 2)$ then a dilation of factor $\frac{1}{2}$ from the y axis
 - c dilation of factor $\frac{1}{2}$ from the x axis followed by translation where $(0, 0) \rightarrow (5, 2)$
 - d translation where $(0, 0) \rightarrow (2, 1)$ followed by a reflection in the y axis
 - e reflection in the line $y = x$ followed by a translation where $(0, 0) \rightarrow (0, 2)$.
- 9 Sketch the graph of the image of $\{(x, y): y = x^2\}$ in **8 a–e** above.
- 10 Consider $\{(x, y): y = 2^x\}$. Find the image after
- a translation where $(0, 0) \rightarrow (1, 0)$
 - b translation where $(0, 0) \rightarrow (0, 1)$
 - c dilation of factor 2 from the y axis followed by dilation of factor $\frac{1}{3}$ from the x axis
 - d dilation of factor $\frac{1}{3}$ from the x axis followed by dilation of factor 2 from the y axis
 - e translation where $(0, 0) \rightarrow (2, 4)$ followed by reflection in the y axis
 - f translation where $(0, 0) \rightarrow (0, 2)$ then dilation of factor $\frac{1}{2}$ from the y axis then reflection in the x axis.
- 11 Sketch the graph of the image of $\{(x, y): y = 2^x\}$ in **10 a–f** above.

8.7 Determining transformations

In the previous section a method was presented for determining the equation of the image of a graph under a given transformation. In this section a procedure for determining the transformations which have produced a particular image is discussed.

Example 13

Find a sequence of transformations which maps $y = x^2$ to $y = 2(x + 3)^2 - 4$

Solution

The composition of transformations maps (x, y) to (x', y')

$$\therefore y' = 2(x' + 3)^2 - 4$$

Rearrange to make the transformation from $y = x^2$ more obvious

$$\frac{y' + 4}{2} = (x' + 3)^2$$

It can be seen that to obtain this, take

$$y = \frac{y' + 4}{2} \quad \text{and} \quad x = (x' + 3)$$

i.e., $y' = 2y - 4 \quad \text{and} \quad x' = x - 3$

A dilation of factor 2 from the x axis followed by a translation determined by the vector $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ gives this image.

Example 14

Find a sequence of transformations which maps $y = 2^x$ to $y = 3(2^{x-2}) - 4$.

Solution

Assume the composition of transformations maps (x, y) to (x', y')

Write $y' = 3(2^{x'-2}) - 4$

Rearrange to $\frac{y' + 4}{3} = 2^{x'-2}$

It can be seen to obtain this, take

$$y = \frac{y' + 4}{3} \quad \text{and} \quad x = x' - 2$$

$\therefore y' = 3y - 4 \quad \text{and} \quad x' = x + 2$

A dilation of factor 3 from the x axis followed by a translation determined by the vector $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ gives this image.



Exercise 8G

Example 13 1 Find the single transformation which maps

- a** $\{(x, y): y = x^2\}$ to $\{(x, y): y = 2x^2\}$
- b** $\{(x, y): y = x^2\}$ to $\{(x, y): y = (x + 2)^2\}$
- c** $\{(x, y): y = x^2\}$ to $\{(x, y): x = y^2\}$
- d** $\{(x, y): y = 2^x\}$ to $\{(x, y): y = 2^{2x}\}$
- e** $\{(x, y): y = 2^x\}$ to $\{(x, y): y = 3 \times 2^x\}$
- f** $\{(x, y): y = 2^x\}$ to $\{(x, y): y = 2^{x-3}\}$
- g** $\{(x, y): y = 2^x\}$ to $\{(x, y): y = 2^{-x}\}$
- h** $\{(x, y): y = 2^x\}$ to $\{(x, y): y = -2^x\}$

Example 14 2 Find a sequence of transformations which maps

- a** $\{(x, y): y = x^2\}$ to $\{(x, y): y = 2(x - 3)^2\}$
- b** $\{(x, y): y = x^2\}$ to $\{(x, y): y = 2x^2 - 3\}$
- c** $\{(x, y): y = x^2\}$ to $\{(x, y): y = 2(x - 3)^2 + 1\}$

$$\text{d } \left\{ (x, y) : y = \frac{1}{x} \right\} \text{ to } \left\{ (x, y) : y = \frac{2}{x-3} \right\}$$

$$\text{e } \left\{ (x, y) : y = \frac{1}{x} \right\} \text{ to } \left\{ (x, y) : y = \frac{2}{x} - 3 \right\}$$

$$\text{f } \left\{ (x, y) : y = \frac{1}{x} \right\} \text{ to } \left\{ (x, y) : y = \frac{1}{3-x} \right\}$$

$$\text{g } y = 2^x \text{ to } y = 2^{\frac{x-1}{3}} + 4$$

$$\text{h } y = \frac{1}{x} \text{ to } y = \frac{3}{2x-4}$$

8.8 Absolute value function and integer value function



Absolute value function

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

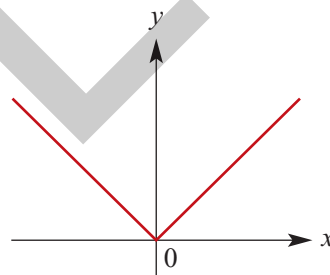
$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This function is written as $f(x) = |x|$

The graph of this function is as shown.

It is known as the **modulus function** or **absolute value function**.

The images of this graph under several transformations are considered.



Example 15

Find the image of $\{(x, y) : y = |x|\}$ and sketch the graph of the image for each of the following transformations.

a a translation defined by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b a dilation of factor 4 from the y axis

c a reflection in the line $y = x$

d a reflection in the x axis

Solution

$$\text{a } (x, y) \rightarrow (x + 2, y + 3)$$

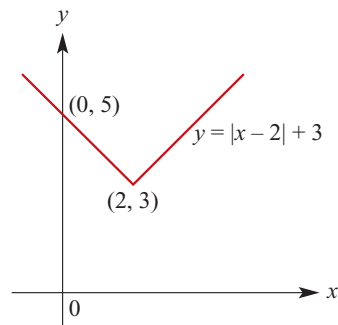
$$(x, y) \rightarrow (x', y')$$

$$\therefore x' = x + 2, y' = y + 3$$

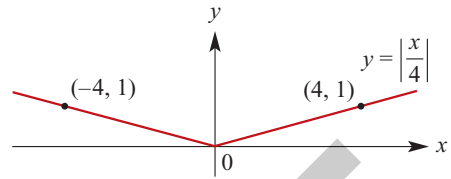
$$\therefore x = x' - 2 \text{ and } y = y' - 3$$

$$\{(x, y) : y = |x|\} \text{ is mapped to}$$

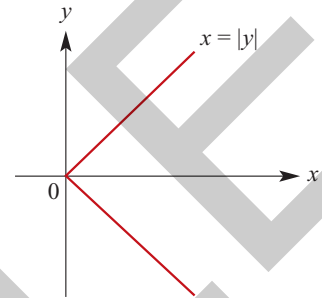
$$\{(x', y') : y' - 3 = |x' - 2|\}$$



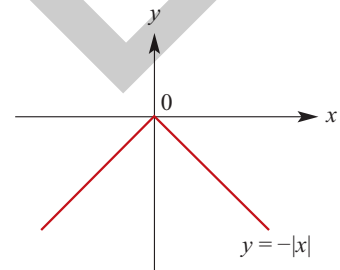
b $(x, y) \rightarrow (4x, y)$
 $(x, y) \rightarrow (x', y')$
 $\therefore x' = 4x \text{ and } y' = y$
 $\{(x, y): y = |x|\}$ is mapped to
 $\{(x', y'): y' = \left|\frac{x'}{4}\right|\}$



c $(x, y) \rightarrow (y, x)$
 $(x, y) \rightarrow (x', y')$
 $\therefore x' = y \text{ and } y' = x$
 $\{(x, y): y = |x|\}$ is mapped to
 $\{(x', y'): x' = |y'|\}$



d $(x, y) \rightarrow (x, -y)$
 $(x, y) \rightarrow (x', y')$
 $\therefore x' = x, y' = -y$
 $\{(x, y): y = |x|\}$ is mapped to
 $\{(x', y'): -y' = |x'|\}$

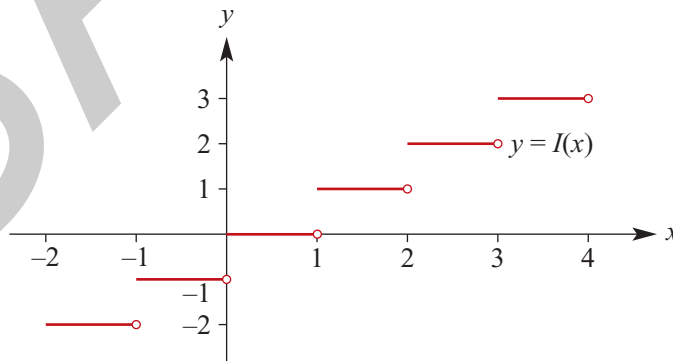


Integer value function

The integer value function $I: \mathbb{R} \rightarrow \mathbb{Z}$ is defined by $I(x) = [x]$ where $[x]$ is the greatest integer not exceeding x . For example,

$$[3.9] = 3, [5] = 5, [-4.1] = -5, [\pi] = 3$$

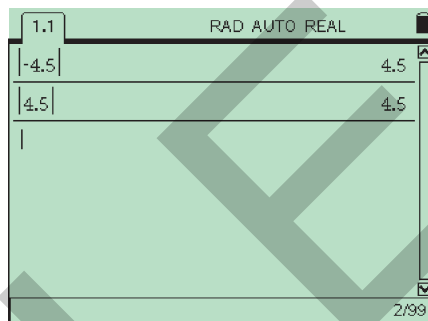
For all real numbers x , we have $x - 1 \leq [x] \leq x$



Using the TI-Nspire

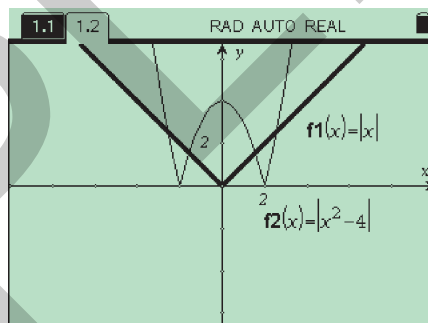
Absolute Value function

The **abs()** function can be found in the catalog (⌕ 1 A) or typed directly. It works as shown.



The graph of the absolute value function can be obtained in the usual way in a **Graphs & Geometry** application (⌕ 2).

The graph of $f1(x) = \text{abs}(x)$ is shown with medium line width. The graph of the composite function $f2(x) = \text{abs}(x^2 - 4)$ is also shown.



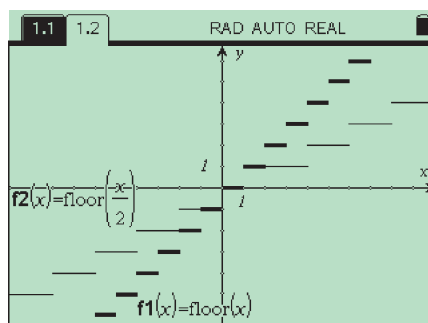
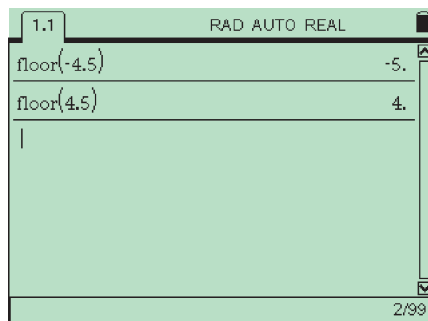
Integer Value function

The integer value function rounds down to the nearest integer, so on the TI-Nspire it is called the **floor()** function and can be found in the **Number Tools** submenu of the **Number** menu (⌕ 2 7 6). It works as shown.

(Note the difference from the Integer Part Function (⌕ 2 7 2) that returns the integer part of a number.)

The graph of the integer value function can be obtained in the usual way in a **Graphs & Geometry** application (⌕ 2).

The graph of $f1(x) = \text{floor}(x)$ is shown with medium line width. The graph of the composite function $f2(x) = \text{floor}\left(\frac{x}{2}\right)$ is also shown.



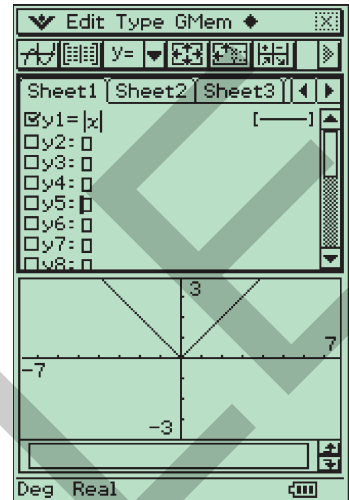
Using the Casio ClassPad

To enter the absolute value function, switch on the screen keyboard and in **[mth]** tap **|x|**.

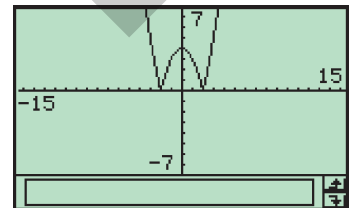
This is entered in the  menu.

Tick the selection box and then tap  to produce the graph.

Note: The Zoom is set to Quick Initialize for the graph shown.



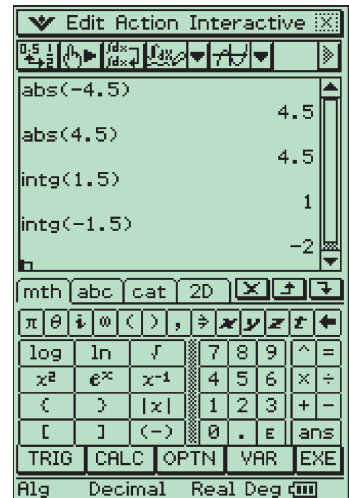
The absolute value function can be composed with other functions. The small graph shows $y = |x^2 - 4|$.



The CAS calculator has the absolute value and integer part functions as built-in features.

(absolute value function)

Turn on the screen keyboard and tap **[mth]** **|x|** then enter the number.



(integer part function)

This function is found in the catalogue. Tap **cat** and use the alphabetic shortcut at the bottom of the screen to find a function beginning with 'i'. The required function is **intg**(. This function returns the next whole number which is less than the given number.

Using the Casio ClassPad

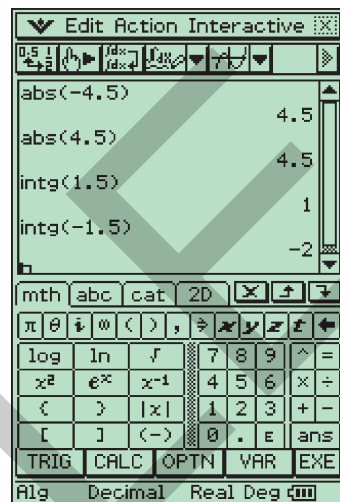
The CAS calculator has the absolute value and integer part functions as built in features.

(absolute value function)

Turn on the screen keyboard and tap **|mth|** **|x|** then enter the number.

(integer part function)

This function is found in the catalog. Tap **cat** and use the alphabetic shortcut at the bottom of the screen to find a function beginning with 'i'. The required function is **intg**(. This function returns the next whole number which is less than the given number.



Exercise 8H

Example 15

- Find the image of $\{(x, y): y = |x|\}$ and sketch the graph of the image for the following.
 - a translation corresponding to the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
 - a reflection in the line $y = -x$
 - a dilation from the y axis of factor 4
 - a reflection in the x axis
 - a translation corresponding to the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ followed by a reflection in the line $y = x$
 - a reflection in the line $y = x$ then a translation corresponding to the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
 - a dilation from the x axis of factor 2 followed by a reflection in the x axis.
- State the transformation(s) which map(s)
 - $\{(x, y): y = |x|\}$ to $\{(x, y): y = |x| + 3\}$
 - $\{(x, y): y = |x|\}$ to $\{(x, y): y = |x - 3| + 3\}$
 - $\{(x, y): y = |x|\}$ to $\{(x, y): y = |2x|\}$
 - $\{(x, y): y = |x|\}$ to $\{(x, y): y = -2|x|\}$
- Find the image of $\{(x, y): y = [x]\}$ and sketch the graph of the image of the following.
 - a dilation from the y axis of factor 2

- b** a translation corresponding to the vector $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$
- c** a translation corresponding to the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$
- d** a reflection in the line $y = x$
- e** a reflection in the line $y = -x$
- f** a translation corresponding to the vector $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$
- g** a dilation from the y axis of factor 2 followed by a reflection in the x axis
- h** a reflection in the x axis followed by a translation corresponding to the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

8.9 Function notation with transformations

In this section the image of the graph $y = f(x)$ (where f is an arbitrary function) under a transformation is considered. This is particularly useful when using a graphics or CAS calculator.

Example 16

Let $y = f(x)$ be the equation of a curve.

Find the image of $y = f(x)$ under each of the following transformations.

- a** a translation determined by the vector $\begin{bmatrix} h \\ k \end{bmatrix}$
- b** a reflection in the x axis
- c** a dilation of factor k from the y axis
- d** a dilation of factor k from the x axis

Solution

- a** The rule is $(x, y) \rightarrow (x + h, y + k)$

$$\therefore \text{Consider } x' = x + h \text{ and } y' = y + k$$

$$\therefore x = x' - h \text{ and } y = y' - k$$

$$\therefore y = f(x) \text{ is mapped to } y' - k = f(x' - h)$$

$$\text{i.e., the image is } y = f(x - h) + k$$

- b** The rule is $(x, y) \rightarrow (x, -y)$

$$\therefore \text{Consider } x' = x \text{ and } y' = -y$$

$$\text{i.e., the image is } y = -f(x)$$

- c** The rule is $(x, y) \rightarrow (kx, y)$

$$\therefore \text{Consider } x' = kx \text{ and } y' = y$$

$$\therefore x = \frac{x'}{k} \text{ and } y = y'$$

$$\therefore y = f(x) \text{ is mapped to } y' = f\left(\frac{x'}{k}\right)$$

$$\text{i.e., the image of } y = f(x) \text{ is } y = f\left(\frac{x}{k}\right)$$

d The rule is $(x, y) \rightarrow (x, ky)$

$\therefore$ Consider $x' = x$ and $y' = ky$

$\therefore x = x'$ and $y = \frac{y'}{k}$

$\therefore y = f(x)$ is mapped to $\frac{y'}{k} = f(x')$

i.e., the image of $y = f(x)$ is $y = kf(x)$

Example 17

For $f(x) = x^2$ find $4f(x + 3)$ and state the transformations which take $y = f(x)$ to $y = 4f(x + 3)$

Solution

$$4f(x + 3) = 4(x + 3)^2$$

$\therefore y = x^2$ is mapped to $y = 4(x + 3)^2$

If (x, y) is mapped to (x', y')

Then $\frac{y'}{4} = (x' + 3)^2$

$$x = x' + 3 \text{ and } y = \frac{y'}{4}$$

i.e., $x' = x - 3$ and $y' = 4y$

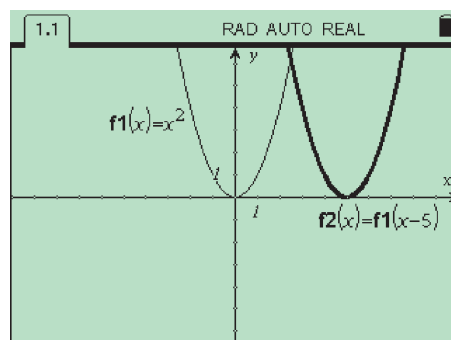
$\therefore$ a dilation of factor 4 from the x axis followed by a translation $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ takes

$$y = f(x) \text{ to } y = 4f(x + 3)$$

Using the TI-Nspire

The notation for transformations can be used with a calculator.

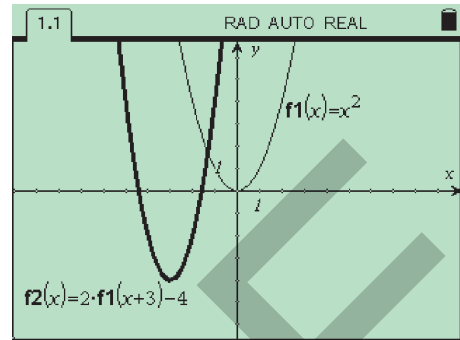
In the screen shown, the transformation which takes the curve of $y = f(x)$ to the curve of $y = f(x - 5)$ is applied to the curve with equation $f(x) = x^2$ by defining $f1(x) = x^2$ and $f2(x) = f1(x - 5)$.



Consider the following sequence of transformations:

- dilation of factor 2 from the x -axis
- translation of '3 to the left'
- translation '4 down'

These transformations will take the curve of $y = f(x)$ to the curve of $y = 2f(x+3) - 4$. In the screen shown, this is applied to the curve with equation $f(x) = x^2$.



Using the Casio ClassPad

The notation for transformations can be used with the CAS calculator.

Consider the transformation which takes the curve of $y = f(x)$ to the curve of $y = f(x - 5)$. The function for $y1$ can be entered and then $y2$ can be defined as shown.

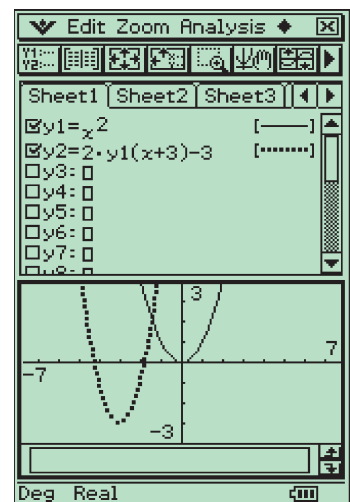
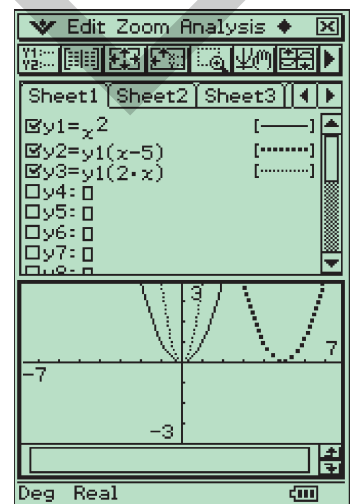
Note: The y must be selected from the **abc** tab on the screen keyboard, not the y on the physical keyboard.

A dilation factor $\frac{1}{2}$ from the y -axis is applied to the curve with rule $f(x) = x^2$. The rule has been entered as $y3 = y1(2x)$.

Consider the following sequence of transformations:

- dilation of factor 2 from the x -axis
- translation of '3 to the left'
- translation '4 down'

These transformations will take the curve of $y = f(x)$ to the curve of $y = 2f(x+3) - 4$. In the screen shown, this is applied to the curve with equation $y = x^2$.



Exercise 8

Example 16 1 Let $y = f(x)$ be the equation of a curve. Find the image of $y = f(x)$ under each of the following transformations.

- a** a translation determined by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **b** a reflection in the line $y = x$
c a dilation of factor $\frac{1}{2}$ from the y axis **d** a dilation of factor 2 from the x axis

Example 17 2 For $f(x) = 2^x$, find

- a** $f(x + 3)$ **b** $f(2x)$ **c** $f\left(\frac{x}{2}\right)$ **d** $4f(x - 3)$

3 For $f(x) = x^2$, find $4f\left(\frac{x}{2}\right)$ and state the transformations which take $y = f(x)$ to $y = 4f\left(\frac{x}{2}\right)$

4 For $f(x) = \frac{1}{x}$ find $f(2x - 3) + 4$ and state the transformations which take $y = f(x)$ to $y = f(2x - 3) + 4$

5 For $f(x) = x^2$ find $-3f(2 - x)$ and state the transformations which take $y = f(x)$ to $y = -3f(2 - x)$

8.10 Summary of transformations

A summary of some of the transformations and their rules is presented here. Suppose (x', y') is the image of (x, y) under the mapping in the first column of the table below.

Mapping	Rule
Reflection in the x axis	$x' = x \quad = x + 0y$ $y' = -y \quad = 0x + -y$
Dilation by factor k from the y axis	$x' = kx \quad = kx + 0y$ $y' = y \quad = 0x + y$
Rotation of $\frac{\pi^c}{2}$ about O in an anticlockwise direction	$x' = -y \quad = 0x + -y$ $y' = x \quad = x + 0y$
Expansion of factor k from the origin	$x' = kx \quad = kx + 0y$ $y' = ky \quad = 0x + ky$
Reflection in the line $y = x$	$x' = y \quad = 0x + y$ $y' = x \quad = x + 0y$
Translation defined by a vector $\begin{bmatrix} a \\ b \end{bmatrix}$	$x' = x + a$ $y' = y + b$

The first five mappings are special cases of a general kind of mapping defined by

$$\begin{aligned}x' &= ax + by \\ y' &= cx + dy\end{aligned}$$

where a, b, c, d are real numbers.

These equations can be rewritten as

$$\begin{aligned}x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y\end{aligned}$$

which yields the equivalent matrix equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A transformation of the form

$$(x, y) \rightarrow (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$$

is called a linear transformation.

Example 18

Consider a linear transformation such that $(1, 0) \rightarrow (3, -1)$ and $(0, 1) \rightarrow (2, -4)$. Find the image of $(-3, 5)$.

Solution

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\therefore a_{11} = 3, a_{21} = -1 \text{ and } a_{12} = 2, a_{22} = -4$$

i.e., the transformation can be defined by the 2×2 matrix $\begin{bmatrix} 3 & 2 \\ -1 & -4 \end{bmatrix}$

Let $(-3, 5) \rightarrow (x', y')$

$$\begin{aligned}\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 3 & 2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 \times -3 + 2 \times 5 \\ -1 \times -3 - 4 \times 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -17 \end{bmatrix}\end{aligned}$$

$$\therefore (-3, 5) \rightarrow (1, -17)$$

The image of $(-3, 5)$ is $(1, -17)$

Note: Non-linear mappings cannot be represented by a matrix in the way indicated above.

Thus for the translation defined by $(0, 0) \rightarrow (a, b)$

$$x' = x + a$$

$$y' = y + b$$

While this cannot be represented by a square matrix, the defining equations

suggest $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$

where the 'sum' has definition:

for each x, y, a, b in R , $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \end{bmatrix}$

Composition of mappings

Consider a linear transformation defined by the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ composed with a

linear transformation defined by the matrix $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

The composition consists of the transformation of $\mathbf{A}$ being applied first and then the transformation of $\mathbf{B}$.

The matrix of the resulting composition is the product

$$\mathbf{BA} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$



Exercise 8J

1 Find

a $\begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

b $\begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

Example 18

2 If a linear transformation is defined by the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$, find the image of $(1, 0)$, $(0, 1)$ and $(3, 2)$ under this transformation.

3 Find the images of $(1, 0)$ and $(-1, 2)$ under the linear transformation whose matrix is

a $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

b $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

c $\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$

- 4 a** Find the matrix of the linear transformation which maps $(1, -2) \rightarrow (-4, 5)$ and $(3, 4) \rightarrow (18, 5)$.
- b** The images of two points are given for a linear transformation. Investigate whether this is sufficient information to determine the matrix of the transformation.
- c** Find the matrix of the linear transformation such that $(1, 0) \rightarrow (1, 1)$ and $(0, 1) \rightarrow (2, 2)$. What is the range of this transformation?
- 5** By finding the images of $(1, 0)$ and $(0, 1)$, write down the matrix of each of the following transformations.
- a** reflection in the line $x = 0$
 - b** reflection in the line $y = x$
 - c** reflection in the line $y = -x$
 - d** dilation of factor 2 from the x axis
 - e** expansion of factor 3 from the origin
 - f** dilation of factor 3 from the y axis



Chapter summary

- A **transformation** T is a mapping from R^2 to R^2 such that if $T(a, b) = T(c, d)$ then $a = c$ and $b = d$
- A **translation** is a transformation for which each point in the plane is moved the same distance in the same direction. A translation of 5 units in the positive direction of the x axis can be represented by the rule $(x, y) \rightarrow (x + 5, y)$
- A vector will mean a column of two numbers. The first number indicates a 'move' in the positive or negative direction of the x axis and the second indicates a 'move' in the positive or negative direction of the y axis. Directed line segments are used to illustrate vectors.

For example, the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the vector '2 to the right and 3 up'.

The rule for the translation described by this vector is $(x, y) \rightarrow (x + 2, y + 3)$

- For reflection in the x axis the rule is $(x, y) \rightarrow (x, -y)$
For reflection in the y axis the rule is $(x, y) \rightarrow (-x, y)$
For reflection in the line $y = x$ the rule is $(x, y) \rightarrow (y, x)$
- Lengths and angles are preserved by reflections and translations.
- The transformation '**dilation** from the y axis of factor k ' is defined by the rule $(x, y) \rightarrow (kx, y)$; $k \in R^+$
The transformation '**dilation** from the x axis of factor k ' is defined by the rule $(x, y) \rightarrow (x, ky)$; $k \in R^+$
- Lengths and angles are not preserved by dilations.
- A point (a, b) is said to be invariant under a transformation if $(a, b) \rightarrow (a, b)$ under that transformation.
- The composition of two transformations is defined by applying one transformation, and then the second, to the images of the first.

$$\begin{array}{ccc} \text{e.g., } (x, y) \rightarrow (x + 2, y + 3) \rightarrow (-(x + 2), y + 3) \\ \text{translation} & & \text{reflection} \end{array}$$

■ Applying transformations to graphs of functions

e.g., let $(x, y) \rightarrow (2x + 1, y + 3)$

Let (x', y') be the image of (x, y)

Then $x' = 2x + 1$ and $y' = y + 3$

and $x = \frac{x' - 1}{2}$ and $y = y' - 3$

$\therefore y = f(x)$ is mapped to $y' - 3 = f\left(\frac{x' - 1}{2}\right)$

If $f(x) = x^2$, $y = f(x)$ is mapped to $y = \left(\frac{x - 1}{2}\right)^2 + 3$

■ Determining transformations

For example, if $y = f\left(\frac{x+3}{2}\right) - 4$

then $y + 4 = f\left(\frac{x+3}{2}\right)$

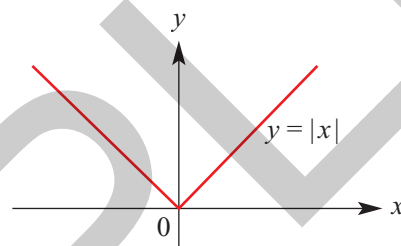
Let $(x, y) \rightarrow (x', y')$

Then $y' + 4 = y$ and $\frac{x' + 3}{2} = x$

Hence $y' = y - 4$ and $x' = 2x - 3$

The transformation is given by the rule $(x, y) \rightarrow (2x - 3, y - 4)$

■ $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

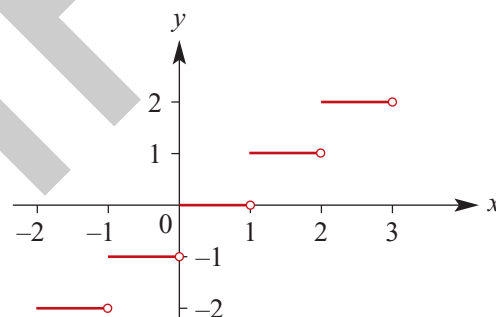


This is called the **absolute value function**.

■ The integer value function is defined by

$$I(x) = [x]$$

where $[x]$ is the greatest integer not exceeding x .



■ Function notation

The image of the graph of $y = f(x)$ under the translation

$(x, y) \rightarrow (x + h, y + k)$ is the graph of $y = f(x - h) + k$

The image of the graph of $y = f(x)$ under the reflection

$(x, y) \rightarrow (x, -y)$ is the graph of $y = -f(x)$

The image of the graph of $y = f(x)$ under the reflection

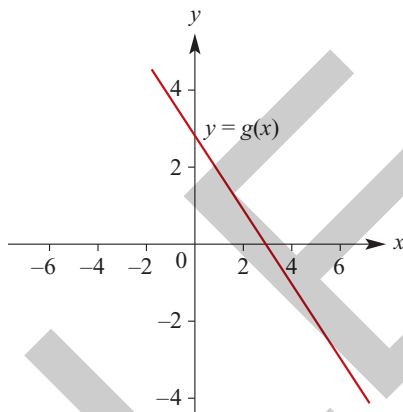
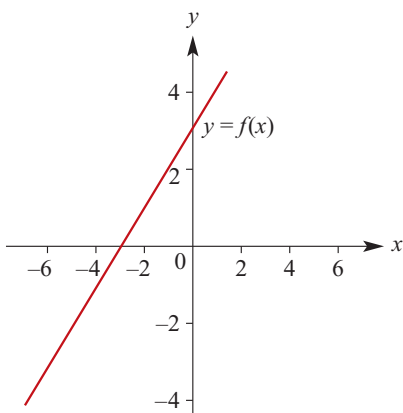
$(x, y) \rightarrow (-x, y)$ is the graph of $y = f(-x)$

The image of the graph of $y = f(x)$ under the dilation

$(x, y) \rightarrow (kx, y)$ is the graph of $y = f\left(\frac{x}{k}\right)$

Multiple-choice questions

1

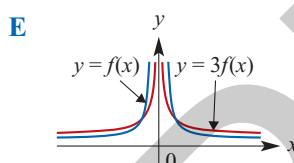
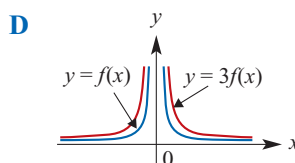
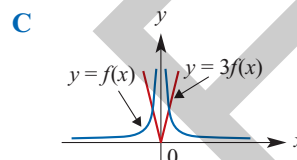
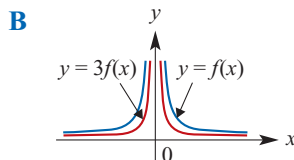
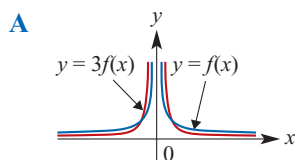


The transformation which maps the graph of $y = f(x)$ to the graph of $y = g(x)$ is

- A** rotation of 180° about the origin **B** rotation of 270° about the origin
C reflection in the y axis **D** reflection in the x axis
E reflection in the line $y = x$
- 2 If the graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x axis, the equation relating $f(x)$ and $g(x)$ is
- A** $f(x) = g(x)$ **B** $f(x) = -g(x)$ **C** $f(x) = g(-x)$
D $f(x) = \frac{1}{g(x)}$ **E** $f(x) = 2g(x)$
- 3 The translation that maps the graph of $y = x^2$ to the graph of $y = (x - 5)^2 - 2$ can be described as
- A** 2 units in the negative direction of the x axis and 5 units in the positive direction of the y axis
B 2 units in the positive direction of the x axis and 5 units in the positive direction of the y axis
C 2 units in the negative direction of the x axis and 5 units in the negative direction of the y axis
D 5 units in the positive direction of the x axis and 2 units in the negative direction of the y axis
E 5 units in the negative direction of the x axis and 2 units in the negative direction of the y axis
- 4 The translation that maps the graph of $f(x) = (x + 2)^2 + 8$ to the graph of $g(x) = x^2$ is
- A** 8 units in the negative direction of the x axis and 2 units in the positive direction of the y axis
B 2 units in the positive direction of the x axis and 8 units in the positive direction of the y axis
C 2 units in the negative direction of the x axis and 8 units in the positive direction of the y axis

- D** 2 units in the positive direction of the x axis and 8 units in the negative direction of the y axis
- E** 2 units in the negative direction of the x axis and 8 units in the negative direction of the y axis

5 If $y = f(x) = \frac{1}{x^2}$, the graph of $y = 3f(x)$ is as shown in



6 The equation of the image of the graph of $y = \sqrt{x}$ after a reflection in the y axis followed by a translation of 2 units in the positive direction of the x axis and 3 units in the positive direction of the y axis is

A $y = \sqrt{-x} - 2 - 3$

B $y = \sqrt{-x - 2} - 3$

C $y = \sqrt{x + 2} - 3$

D $y = \sqrt{x - 2} + 3$

E $y = \sqrt{-x + 2} + 3$

7 The equation of the image of the graph of $y = \frac{1}{x^2}$ after a reflection in the x axis followed by a dilation of factor 2 from the x axis is

A $y = -\frac{2}{x^2}$

B $y = -\frac{1}{2x^2}$

C $y = \frac{2}{x^2}$

D $y = \frac{1}{2x^2}$

E $y = \frac{2}{(-x)^2}$

8 The equation of the image of the graph of $y = |x|$ after a dilation of factor $\frac{1}{3}$ from the y axis followed by a translation of 5 units in the negative direction of the x axis and 2 units in the positive direction of the y axis is

A $y = 3|x - 5| + 2$

B $y = 3|x + 5| + 2$

C $y = \left| \frac{x - 5}{3} \right| + 2$

D $y = \left| \frac{x + 5}{3} \right| + 2$

E $y = \left| \frac{x + 15}{3} \right| + 2$

9 The value of $[-4.6] + [7.2] + [8.7]$ is

A 9

B 10

C 8

D 13

E 11

10 The equation of the image of the graph of $y = [x]$ after a reflection in the x axis followed by a translation of 5 units in the positive direction of the x axis and 4 units in the negative direction of the y axis is

A $y = [x] + 1$

B $y = -[x - 5] - 4$

C $y = [x + 5] - 4$

D $y = [x] + 9$

E $y = [x + 5] + 4$

Short-answer questions (technology-free)

- Find the image of the point $(3, -1)$ under each of the following transformations.
 - a dilation of factor 3 from the y axis
 - a dilation of factor 2 from the x axis
 - a translation determined by the vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$
 - a reflection in the line $x = 0$
 - a reflection in the line $y = 0$
 - a reflection in the line $y = x$
- Find the equation of the image of the graph of $y = x^2$ under each of the following transformations.
 - a translation determined by the vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$
 - a reflection in the line $y = x$
 - a reflection in the line $y = 0$
- Find the rule for each of the following compositions of transformations.
 - a reflection in the line $y = x$ followed by a translation determined by the vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$
 - a dilation of factor 5 from the x axis followed by a reflection in the x axis
 - a dilation of factor 4 from the y axis followed by a translation determined by the vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$
 - a translation determined by the vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ followed by a dilation of factor 4 from the x axis
- Find the image of $y = 2x - 1$ under each of the transformations defined in 1.
- Find the image of $y = 2 - x^2$ under each of the transformations defined in 3.
- Sketch the graph of each of the following. Indicate the coordinates of the vertex and at least one other point.
 - $y = -|x|$
 - $y = |2x| + 3$
 - $y = 4 - |2x|$
- Give the rule for the transformations which would map
 - $y = \frac{1}{x}$ to $y = \frac{2}{x-3} + 4$
 - $y = x^2$ to $y = 2(x-4)^2 + 3$
 - $y = |x|$ to $y = |3x| - 4$
 - $y = \frac{1}{x}$ to $y = \frac{x+1}{x-1}$
 - $y = x^2$ to $y = -(x-2)^2 + 3$
- Sketch the graph of each of the following using transformations.
 - $x^2 = y - 1$
 - $(x-1)^2 = y + 2$
 - $(x-2)^2 = 3(y-2)$
 - $y = [4x]$
 - $y = -3|2x-1|$
 - $y = 2 - 3|x-2|$

Extended-response questions

- Find the image of the point with coordinates $(2, 6)$ under reflection in the line $x = 3$.
 - Find the rule for the following sequence of transformations.

- a translation determined by the vector $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$

- a reflection in the y axis

- a translation determined by the vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

- c** Hence give the rule for the transformation, reflection in the line $x = 3$.
- d i** Give a sequence of three transformations which would determine a reflection in the line $x = m$.
- ii** Give the rule for the transformation, reflection in the line $x = m$.
- e i** Give a sequence of three transformations which would determine a reflection in the line $y = n$.
- ii** Give the rule for the transformation, reflection in the line $y = n$.
- f** Find the image of each of the relations under the transformation, reflection in the line $x = 3$.

- i** $y = x - 3$ **ii** $y = x$ **iii** $y = x^2$ **iv** $y = (x - 3)^2$

- 2 a** Find the coordinates of the image of the point $A(3, 1)$ under a rotation of 90° in an anticlockwise direction around the origin.

- b i** State the gradient of line OA .

- ii** State the gradient of line OA' .

- c** A point A has coordinates (p, q) .

- i** State the gradient of line OA .

- ii** The point is rotated about the origin by 90° in an anticlockwise direction. Find the coordinates of A' , the image of A .

- d** Find the rule for the transformation rotation about the origin by 90° in an anticlockwise direction.

- e** Find the equation of the image of each of the following curves under this transformation.

- i** $y = x$ **ii** $y = x^2$ **iii** $x^2 + y^2 = 1$ **iv** $y = \frac{1}{x}$

- 3 a** Find the image of the point $A(1, 3)$ under a rotation of 180° about the origin.

- b** Find the image of the point (a, b) under a rotation of 180° about the origin.

- c** If the curve with equation $y = f(x)$ is rotated 180° about the origin, find the rule for its image in terms of f .

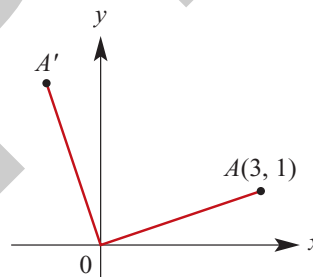
- d i** Find the rule for the following sequence of transformations.

- a translation determined by the vector $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$

- a rotation of 180° about the origin

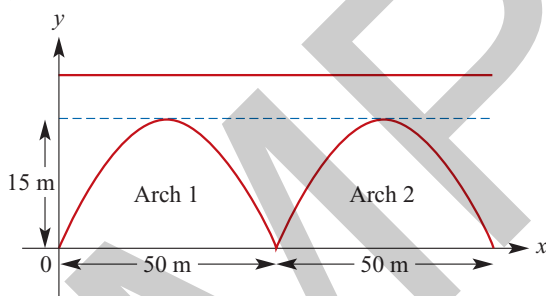
- a translation determined by the vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

- ii** Find the image of the line with equation $y = 3x + 1$ under the transformation with rule determined in **d i**.



(cont'd.)

- e Find the rule for rotation of 180° in an anticlockwise direction about the point (m, n) .
- f Find the rule for rotation of 90° in a clockwise direction about the point (m, n) .
- g Find the rule for rotation of 90° in an anticlockwise direction about the point (m, n) .
- h i Find the image of the curve with rule $y = x^2$ after a rotation of 90° in a clockwise direction about the point $(0, 1)$.
- ii Sketch the graph of the curves on one set of axes.
- 4 a i Find the dilation from the x axis which takes $y = x^2$ to the parabola with vertex at the origin and which passes through the point $(25, 15)$.
- ii State the rule which reflects this dilated parabola in the x axis.
- iii State the rule which takes the reflected parabola of ii to a parabola with x intercepts $(0, 0)$ and $(50, 0)$ and vertex $(25, 15)$.
- iv State the rule which takes the curve of $y = x^2$ to the parabola defined in iii.
- b The plans for the entrance of a new building involve twin parabolic arches as shown in the diagram.



- i From the results of a give the equation for the curve of Arch 1.
- ii Find the translation which maps the curve of Arch 1 to the curve of Arch 2.
- iii Find the equation of the curve of Arch 2.
- c The architect wishes to have flexibility in his planning and so wants to develop an algorithm for determining the equations of the curves given arch width m metres and height n metres.
 - i Find the rule for the transformation which takes $y = x^2$ to Arch 1 with these dimensions.
 - ii Find the equation for the curve of Arch 1.
 - iii Find the equation for the curve of Arch 2.
- 5 a Let $f(x) = [x]$
 - i Find the equation of the image of the graph of $y = f(x)$ under the transformation with rule $(x, y) \rightarrow (2x - 1, y + 3)$
 - ii Sketch the graph of the image of $y = f(x)$ under this transformation.
 - iii Describe a sequence of transformations which takes the graph of $y = f(x)$ to $y = -f(x - 3) - 2$
- b Let $g(x) = |x|$
 - i Sketch the graph of $y = -g(x - 3) + 3$ and the graph of $y = g(2x - 1)$
 - ii Solve the equation $-g(x - 3) + 3 = g(2x - 1)$