CHAPTER 4

Variation

Objectives

- To recognise relationships involving **direct variation**
- To evaluate the **constant of variation** in cases involving direct variation
- To solve problems involving direct variation
- To recognise relationships involving **inverse variation**
- To evaluate the **constant of variation** in cases involving inverse variation
- To solve problems involving inverse variation
- To establish the relationship that exists between variables from given data
- To recognise relationships involving **joint variation**
- To solve problems involving joint variation
- To solve problems involving **part variation**

4.1 Direct variation

Emily sets out to drive from her home in Appleton to visit her friend Kim who lives 600 km away in Brownsville. She drives at a constant speed and notes how far she has travelled every hour. The distance and times are represented in the table below.

Time (t hours)	1	2	3	4	5	6
Distance (d km)	100	200	300	400	500	600

It can be seen that as t increases, d also increases. The rule relating time to distance is $d = 100t$. This is an example of **direct variation** and 100 is the **constant of variation**. In this case d **varies directly** as t or the distance travelled is **proportional** to the time spent travelling. The graph of d against t is a straight line passing through the origin.

A metal ball is dropped from the top of a tall building and the distance it has fallen is recorded each second.

Time (t s)	0	1	2	3	4	5
Distance (d km)	0	4.91	19.64	44.19	78.56	122.75

It can be seen that as t increases, d also increases. This time the rule relating time and distance is $d = 4.91t^2$. This is another example of **direct variation**. In this case, d **varies directly** as the square of t or the distance travelled is **proportional** to t^2. The graph of d against t^2 is a straight line passing through the origin.

The symbol used for 'varies as' or 'is proportional to' is $\propto$. For example, d varies as t can be written as $d \propto t$, and d varies as t^2 can be written as $d \propto t^2$.

In the following, a proportional to a positive power of b is considered,
i.e. a varies directly as b^n, $n \in R^+$

If $a \propto b^n$ then $a = kb^n$ where k is a **constant of variation**.

For all examples of direct variation (where k is positive), as one variable increases the other will also increase. The graph of a against b will show an upwards trend. It should be noted that not all increasing trends will be examples of direct variation.

If $a \propto b^n$ then the graph of a against b^n is a straight line passing through the origin.

Example 1

Use the tables of values below to determine the constant of variation, k, in each case and hence complete each of the tables.

a $y \propto x^2$

x	2	4	6	
y	12		108	192

b $y \propto \sqrt{x}$ (i.e. $y \propto x^{\frac{1}{2}}$)

x	2	4	6	
y		1	1.225	1.414

Solution

a If $y \propto x^2$

then $y = kx^2$

When $x = 2, y = 12$

$\therefore$ $12 = k(2^2)$

$k = 3$

Check:

When $x = 6, y = 3(6^2)$

$= 108$

$\therefore$ $y = 3x^2$

In order to complete the table, consider the following.

When $x = 4,$ $y = 3(4^2)$

$y = 48$

When $y = 192, 192 = 3x^2$

$64 = x^2$

$x = 8$

b If $y \propto \sqrt{x}$

then $y = k\sqrt{x}$

When $x = 4, y = 1$

$\therefore$ $1 = k(\sqrt{4})$

$k = 0.5$

Check:

When $x = 6, y = 0.5(\sqrt{6})$

≈ 1.225

$\therefore$ $y = 0.5\sqrt{x}$

In order to complete the table, consider the following.

When $x = 2, y = 0.5(\sqrt{2})$

$y \approx 0.7071$

When $y = 1.414, 1.414 \approx 0.5(\sqrt{x})$

$2.828 \approx \sqrt{x}$

$x \approx 8$

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x	2	4	6	8
y	12	48	108	192

x	2	4	6	8
y	0.707	1	1.225	1.414

Example 2

In an electrical wire, the resistance (R ohms) varies directly with the length (L m) of the wire.

a If a wire 6 m long has a resistance of 5 ohms, what would be the resistance in a wire of length 4.5 m?

b How long is a wire for which the resistance is 3.8 ohms?

Solution

The constant of variation is determined first.

$$R \propto L$$

$$\therefore \quad R = kL$$

When $L = 6,\ R = 5$

$$\therefore \quad 5 = k(6)$$

$$k = \frac{5}{6}$$

i.e. the constant of variation is $\frac{5}{6}$

Hence $R = \dfrac{5L}{6}$

a When $L = 4.5,\ R = \dfrac{5 \times 4.5}{6}$

$R = 3.75$

The resistance of a wire of length 4.5 m is 3.75 ohms.

b When $R = 3.8,\ 3.8 = \dfrac{5L}{6}$

$L = 4.56$

The length of a wire of resistance 3.8 ohms is 4.56 m.

Example 3

The volume of a sphere varies directly as the cube of its radius. By what percentage will the volume increase if the radius is

a doubled **b** increased by 20%?

Solution

$$V \propto r^3$$

i.e. $V = kr^3$

Initially set the radius equal to 1,

then $V = k(1^3) = k$

a If r is doubled, then set $r = 2$

Then $V = k(2^3) = 8k$

$\therefore$ the volume has increased from k to $8k$, an increase of $7k$

$$\therefore \% \text{ increase of volume } = \frac{7k}{k} \times \frac{100}{1} = 700\%$$

b If r is increased by 20%, then set $r = 1.2$.

Then $V = k(1.2^3) = 1.728k$

$\therefore$ % increase of volume $= 72.8\%$

Exercise 4A

Example 1

1 Determine the value of k, the constant of variation, in each of the following and hence complete the table of values.

a $y \propto x^2$

x	2	4	6	
y	8	32		128

b $y \propto x$

x	$\frac{1}{2}$	1	$\frac{3}{2}$	
y	$\frac{1}{6}$		$\frac{1}{2}$	$\frac{2}{3}$

c $y \propto \sqrt{x}$

x	4	9	49	
y	6	9		90

d $y \propto x^{\frac{1}{5}}$

x	$\frac{1}{32}$	1	32	
y	$\frac{1}{5}$	$\frac{2}{5}$		$\frac{8}{5}$

2 If $V \propto r^3$ and $V = 125$ when $r = 2.5$, find

a V when $r = 3.2$ **b** r when $V = 200$

3 If $a \propto b^{\frac{2}{3}}$ and $a = \frac{2}{3}$ when $b = 1$, find

a a when $b = 2$ **b** b when $a = 2$

Example 2

4 The area (A) of a triangle of fixed base length varies directly as its perpendicular height (h). If the area of the triangle is 60 cm^2 when its height is 10 cm, find

a the area when its height is 12 cm **b** the height when its area is 120 cm^2.

5 The extension in a spring (E) varies directly with the weight (w) suspended from it. If a weight of 452 g produces an extension of 3.2 cm, find

a the extension produced by a weight of 810 g

b the weight that would produce an extension of 10 cm.

6 The weight (W) of a square sheet of lead varies directly with the square of its side length (L). If a sheet of side length 20 cm weighs 18 kg, find the weight of a sheet that has an area of 225 cm^2.

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7 The volume (V) of a sphere varies directly with the cube of its radius (r). A sphere whose radius is 10 cm has a volume of 4188.8 cm^3. Find the radius of a sphere whose volume is 1 cubic metre.

Example 3

8 The time taken for one complete oscillation of a pendulum is called its period. The period (T) of a pendulum varies directly with the square root of the length (L) of the pendulum. A pendulum of length 60 cm has a period of 1.55 seconds. Find the period of a pendulum that is one and a half times as long.

9 The distance (d) to the visible horizon varies directly with the square root of the height (h) of the observer above sea level. An observer 1.8 m tall can see 4.8 km out to sea when standing on the shoreline.

a How far could the person see if they climbed a 4 m tower?

b If the top of a 10 m mast on a yacht is just visible to the observer in the tower, how far out to sea is the yacht?

10 In each of the following calculate the percentage change in y when x is

a doubled **b** halved **c** reduced by 20%

d increased by 40%

i $y \propto x^2$ **ii** $y \propto \sqrt{x}$ **iii** $y \propto x^3$

4.2 Inverse variation

A builder employs a number of bricklayers to build a brick wall. Three bricklayers will complete the wall in eight hours but if he employs six bricklayers the wall will be complete in half the time. The more bricklayers he employs, the shorter the time taken to complete the wall. The time taken (t) decreases as the number of bricklayers (b) increases.

This is an example of **inverse variation**. The time taken to complete the wall **varies inversely** as the number of bricklayers employed.

t **varies inversely** as b or t is **inversely proportional** to b

i.e. $t \propto \frac{1}{b}$

In general, inverse variation exists if $a \propto \frac{1}{b^n}$ where n is some positive number

i.e. a varies inversely as b^n.

If $a \propto \frac{1}{b^n}$

then $a = \frac{k}{b^n}$ where k is a positive constant called the **constant of variation**.

For all examples of inverse variation, as one variable increases the other will decrease and vice versa. The graph of a against b will show a downward trend. It should be noted, however, that any graph showing a decreasing trend will not necessarily be an example of inverse variation.

If $a \propto \frac{1}{b^n}$ then the graph of a against $\frac{1}{b^n}$ will be a straight line.

However, since if $b = 0$, $\frac{1}{b^n}$ is undefined, the line will not be defined at the origin.

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Example 4

Use the tables of values below to determine the value of the constant of variation, k, in each case and hence complete each of the tables.

a $y \propto \frac{1}{x^2}$

x	2	5	10	
y	0.1	0.016		0.001

b $y \propto \frac{1}{\sqrt{x}}$

x	1		25	100
y	10	5		1

Solution

a

$$y \propto \frac{1}{x^2}$$

$$\therefore \quad y = \frac{k}{x^2}$$

When $x = 2, y = 0.1$

$$\therefore \quad 0.1 = \frac{k}{2^2}$$

$$k = 0.4$$

i.e. the constant of variation is 0.4

Check:

When $x = 5, y = \frac{0.4}{5^2}$

$$= 0.16$$

$$\therefore \quad y = \frac{0.4}{x^2}$$

In order to complete the table, consider the following.

When $x = 10, y = \frac{0.4}{10^2}$

$$y = 0.004$$

When $y = 0.001, 0.001 = \frac{0.4}{x^2}$

$$0.001x^2 = 0.4$$

$$x^2 = \frac{0.4}{0.001}$$

$$\therefore \quad x = 20$$

x	2	5	10	20
y	0.1	0.016	0.004	0.001

b

$$y \propto \frac{1}{\sqrt{x}}$$

$$\therefore \quad y = \frac{k}{\sqrt{x}}$$

When $x = 1, y = 10$

$$\therefore \quad 10 = \frac{k}{\sqrt{1}}$$

$$k = 10$$

Check:

When $x = 100, y = \frac{10}{\sqrt{100}}$

$$= 1$$

$$\therefore \quad y = \frac{10}{\sqrt{x}}$$

In order to complete the table, consider the following.

When $x = 4, y = \frac{10}{\sqrt{4}}$

$$y = 5$$

When $y = 2, 2 = \frac{10}{\sqrt{x}}$

$$2\sqrt{x} = 10$$

$$x = 25$$

x	1	4	25	100
y	10	5	2	1

Example 5

For a cylinder of fixed volume, the height (h cm) is inversely proportional to the square of the radius (r cm).

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a What percentage change in the height would result if its radius were reduced by 25%?

b If a cylinder 15 cm high has a base radius of 4.2 cm, how high would a cylinder of equivalent volume be if its radius were 3.5 cm?

Solution

a

$$h \propto \frac{1}{r^2}$$

i.e. $h = \dfrac{k}{r^2}$

If $r = 1$, then $h = \dfrac{k}{(1)^2} = k$

If r is reduced by 25%, then set $r = 0.75$

Then

$$\begin{aligned} h &= \frac{k}{(0.75)^2} \\ &= \frac{k}{0.5625} \\ &\approx 1.778k \text{ (correct to three decimal places)} \end{aligned}$$

$\therefore h$ is increased by 77.8%

b

$$h = \frac{k}{r^2}$$

When $h = 15$, $r = 4.2$

$$\begin{aligned} \therefore \quad 15 &= \frac{k}{(4.2)^2} \\ k &= 15(4.2)^2 \\ &= 264.6 \\ \therefore \quad h &= \frac{264.6}{r^2} \end{aligned}$$

Consider a cylinder of radius 3.5 cm.

If $r = 3.5$, then $h = \dfrac{264.6}{(3.5)^2}$

$$h = 21.6$$

The height of the cylinder is 21.6 cm.

Skillsheet TEACHER

Exercise 4B

Example 4

1 Determine the value of k, the constant of variation, in each of the following and hence complete the tables of values.

a $y \propto \dfrac{1}{x}$

x	2	4	6	
y	1	$\frac{1}{2}$		$\frac{1}{16}$

b $y \propto \dfrac{1}{\sqrt{x}}$

x	$\frac{1}{4}$	1		9
y	1	$\frac{1}{2}$	$\frac{1}{4}$	

c $y \propto \dfrac{1}{x^2}$

x	1	2	3	
y	3	$\frac{3}{4}$		$\frac{1}{12}$

d $y \propto \dfrac{1}{x^{\frac{1}{3}}}$

x	$\frac{1}{8}$	1		125
y	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	

2 If $a \propto \dfrac{1}{b^3}$ and $a = 4$ when $b = \sqrt{2}$, find

a a when $b = 2\sqrt{2}$

b b when $a = \dfrac{1}{16}$

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3 If $a \propto \frac{1}{b^4}$ and $a = 5$ when $b = 2$, find

a a when $b = 4$ **b** b when $a = 20$.

4 The gas in a cylindrical canister occupies a volume of 22.5 cm^3 and exerts a pressure of 1.9 kg/cm^2. If the volume (V) varies inversely as the pressure (P), find the pressure if the volume is reduced to 15 cm^3.

Example 5

5 The current (I amperes) that flows in an electrical appliance varies inversely as the resistance (R ohms) of the appliance. If the current is 3 amperes when the resistance is 80 ohms, find

a the current when the resistance is 100 ohms

b the increase in resistance required to reduce the current to 80% of its original value.

6 The intensity of illumination (I) of a light varies inversely as the square of the distance (d) from the light. At a distance of 20 m a light has an intensity of 100 candela. Find the intensity of the light at a distance of 25 m.

7 The radius (r) of a cylinder of fixed volume varies inversely as the square root of its height (h). If the height is 10 cm when the radius is 5.64 cm, find the radius if the height is 12 cm.

8 In each of the following, calculate the percentage change in y when x is

a doubled **b** halved **c** reduced by 20%

d increased by 40%.

i $y \propto \frac{1}{x^2}$ **ii** $y \propto \frac{1}{\sqrt{x}}$ **iii** $y \propto \frac{1}{x^3}$

4.3 Fitting data

Sometimes the relationship that exists between two variables a and b is not known. By inspection of a table of values, it is sometimes possible to ascertain whether the relationship between the variables is direct or inverse proportion. Analysis is required to establish the rule that best fits the given data. This may involve graphing the data.

Example 6

Establish the relationship between the two variables for each of the following tables of values.

a

b	0	2	4	6	8
a	0	12	48	108	192

b

x	1	3	6	12	15
y	30	10	5	2.5	2

Solution

a By inspection it can be conjectured that some type of direct variation exists. As b increases, a also increases and when $a = 0$, $b = 0$.

Assume	$a \propto b^n$ for some positive number n
$\therefore$	$a = kb^n$
i.e.	$k = \dfrac{a}{b^n}$

Select a value for n (it must be a positive number) and test each of the pairs of values given in the table (do not use (0, 0)). If the value of k for each pair of values is the same then the choice of n is correct.

Let $n = 1 \quad \therefore k = \dfrac{a}{b}$

Consider $\dfrac{a}{b}$ for the values given in the table.

Testing: $\dfrac{12}{2} = 6$, $\dfrac{48}{4} = 12$, $\dfrac{108}{6} = 18$, $\dfrac{192}{8} = 24$

Since the quotients differ, $n \neq 1$.

Let $n = 2 \quad \therefore k = \dfrac{a}{b^2}$

Consider $\dfrac{a}{b^2}$ for the values given in the table.

Testing: $\dfrac{12}{4} = 3$, $\dfrac{48}{16} = 3$, $\dfrac{108}{36} = 3$, $\dfrac{192}{64} = 3$

The quotients are all equal to 3.

$\therefore \quad k = 3$ and $n = 2$

i.e. $\quad a = 3b^2$

b By inspection it can be conjectured that some type of inverse variation exists. As x increases, y decreases.

Assume	$y \propto \dfrac{1}{x^n}$ for some positive number n
$\therefore$	$y = \dfrac{k}{x^n}$
i.e.	$k = yx^n$

Let $n = 1 \quad \therefore k = yx$

Consider the product yx for the values given in the table.

Testing:
$30 \times 1 = 30$
$10 \times 3 = 30$
$5 \times 6 = 30$
$2.5 \times 12 = 30$
$2 \times 15 = 30$

$\therefore \quad k = 30$ and $n = 1$

i.e. $\quad y = \dfrac{30}{x}$

The type of variation can also be investigated by graphical analysis. By plotting the graph of a against b, an upward trend *may* indicate direct variation or a downward trend *may* indicate inverse variation.

To find the specific type of variation that exists, the following can be used as a guide.

- If direct variation exists ($a \propto b^n$), then the graph of a against b^n will be a straight line through the origin. The gradient of this line will be the constant of variation k.
- If inverse variation exists $\left(a \propto \frac{1}{b^n}\right)$, then the graph of a against $\frac{1}{b^n}$ will be a straight line. This line will not be defined at the origin. The gradient of this line will be the constant of variation k.

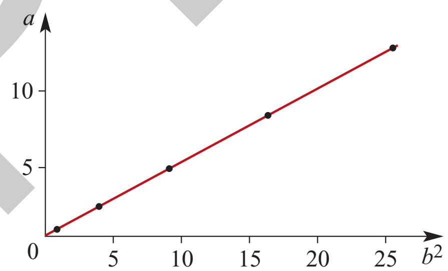
Example 7

For the table of values below, plot the graph of a against b^2 and hence establish the rule relating a to b.

b	1	2	3	4	5
a	0.5	2	4.5	8	12.5

Solution

b^2	1	4	9	16	25
a	0.5	2	4.5	8	12.5



Since this is a straight line, it can be conjectured that the relationship is $a = kb^2$ where k corresponds to the gradient of the graph.

From the graph it can be seen that $a = \frac{1}{2}b^2$.

If it is known that the relationship between two variables x and y is of the form $y = kx^n$ where $k \in R^+$ and $n \in Q\backslash\{0\}$ then a CAS calculator can be used to find n and k is sufficient information is given.

Example 8

The following data was collected recording N, the number of calls to a company, D days after the commencement of an advertising campaign.

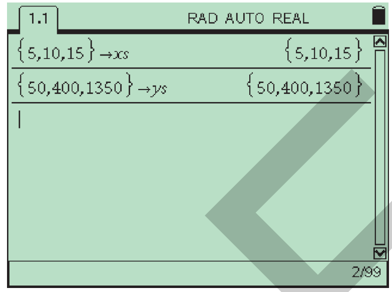
Days (D)	5	10	15
Number of calls (N)	50	400	1350

Find a relationship between N and D using the graphics calculator.

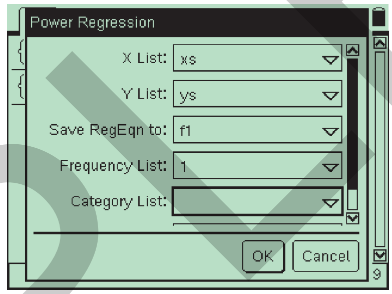
Solution

Using the TI-Nspire

Store the x-values and y-values as shown.

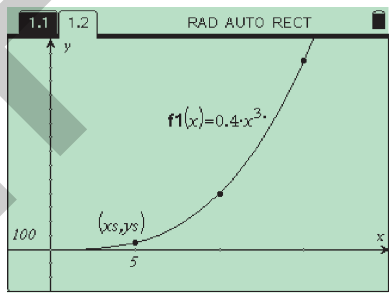


Select **Power Regression** from **Stat Calculations** submenu of the **Statistics** menu (menu 6 1 9) and complete as shown. Press **enter**, and the result is given as $y = a * x^{\wedge}b$, $a = 0.4$, $b = 3$.



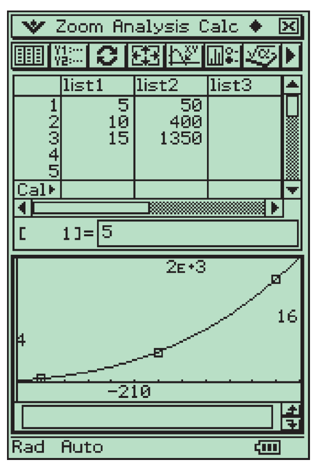
Hence $y = 0.4x^3$, so the required relationship is $N = 0.4D^3$.

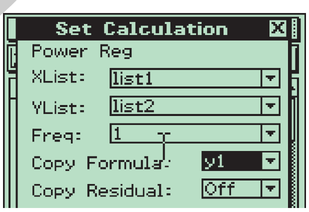
The graph and the data can be graphed in a **Graphs & Geometry** application (home 2) as a **Function** (menu 3 1) and a **Scatter Plot** (menu 3 4) respectively.

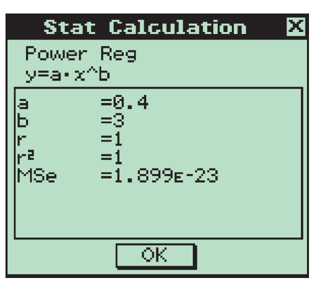


Using the Casio ClassPad

In the Statistics program area, enter the data into list 1 and list 2 then tap **Calc, Power Reg** and ensure the settings are as shown. Note that selecting $y1$ will copy the formula to graph $y1$ in the Graph&Tab program area. Note the formula from the **Stat Calculation** screen before tapping OK. The required relationship is $N = 0.4D^3$.







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Example 9

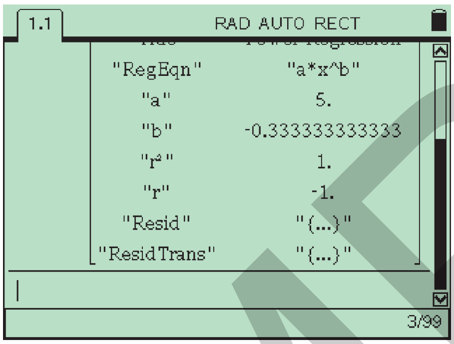
Establish a rule connecting y and x given the following data.

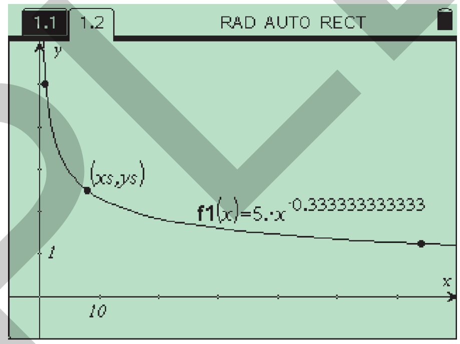
x	1	8	64
y	5	2.5	1.25

Solution

The solution is given in the screens below.

Using the TI-Nspire

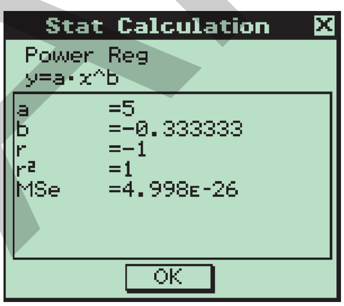


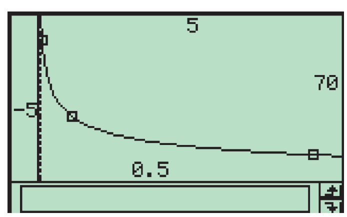


Note that $y = 5x^{-\frac{1}{3}} = \frac{5}{x^{\frac{1}{3}}}$

Using the Casio ClassPad

The solution is given in the screens. Note that $y = 5x^{-\frac{1}{3}} = \frac{5}{x^{-\frac{1}{3}}}$





Exercise 4C

Example 6

1 Each of the tables in parts **a** to **e** fits one of the following types of variation:

direct $y \propto x$ inverse $y \propto \frac{1}{x}$ direct square $y \propto x^2$ inverse square $y \propto \frac{1}{x^2}$
direct square root $y \propto \sqrt{x}$

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Establish the relationship between x and y in each case.

a

x	0	3	6	9	12
y	0	2	4	6	8

b

x	1	2	3	4	5
y	4	16	36	64	100

c

x	20	15	10	5	1
y	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	5

d

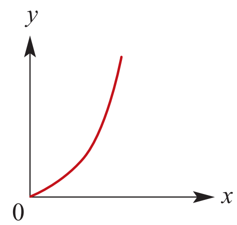
x	1	2	3	4	5
y	2	2.828	3.464	4	4.472

e

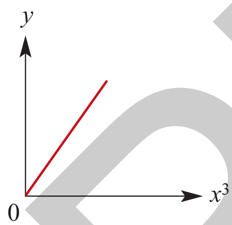
x	1	1.5	2	2.5	3
y	4	1.78	1	0.64	0.444

2 Which of the following graphs could represent examples of direct variation?

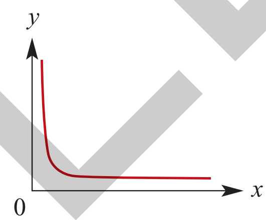
a



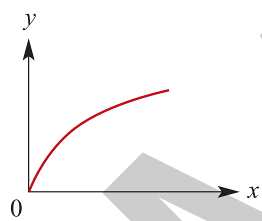
b



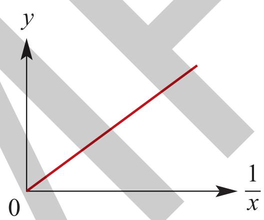
c



d



e



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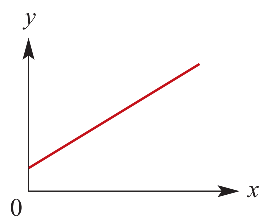
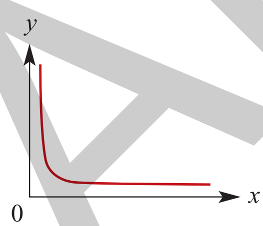


3 Which of the following graphs could represent examples of inverse variation?

a



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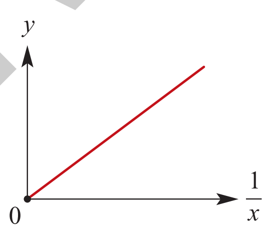

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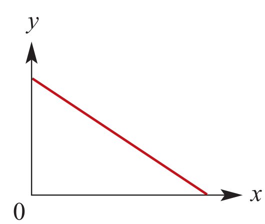

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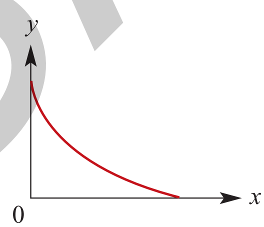

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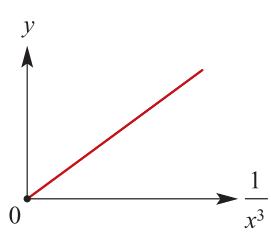

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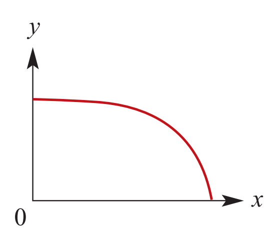

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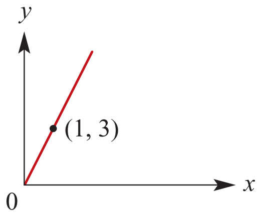
4 Give the rule connecting y and x for each of the following.

a



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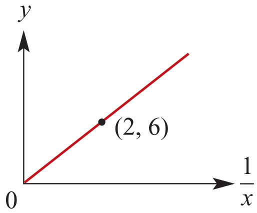
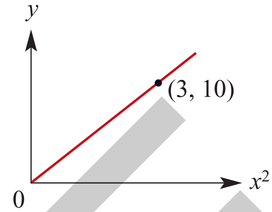
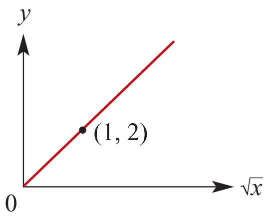


c



d



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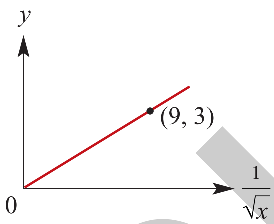

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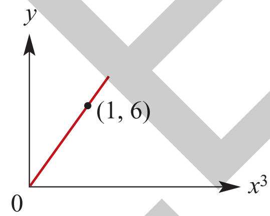

Example 7

5 Plot the graph of y against x^2 and hence establish the relationship between x and y.

x	2	2.5	3	3.5	4
y	9.6	15	21.6	29.4	38.4

6 Plot the graph of y against $\sqrt{x}$ and hence establish the relationship between x and y.

x	1	4	9	16	25
y	1.5	3	4.5	6	7.5

7 Plot the graph of y against $\frac{1}{x^2}$ and hence establish the relationship between x and y.

x	0.2	0.3	0.4	0.5	1
y	50	22.2	12.5	8	2

Example 8

8 Given that for each of the following $y \propto ax^b$ use your graphics calculator's **PwrReg** function to establish the values of a and b.

a

x	4.00	8.00	12.00	16.00
y	0.50	0.71	0.87	1.00

b

x	1	5	10	15
y	2.00	14.95	35.57	59.04

c

x	1	10	100	1000
y	3.50	8.79	22.08	55.47

d

x	10	20	30	40
y	46.42	73.68	96.55	116.96

e

x	1	2	3	4
y	2.00	0.35	0.13	0.06

f

x	1	3	5	7
y	3.20	2.06	1.68	1.47

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Example 9

9 The concentration of antibodies (C) in an animal's bloodstream is directly proportional to time (t hours) after which the animal is injected with an antigen (i.e., $C = at^b$). The following data is collected.

t	1	2	3	4
C	100	114.87	124.57	131.95

a Find values for a and b. **b** Find the concentration after 10 hours.

10 The level of infestation (I) of a pest in a crop is proportional to the time (t days) after which the crop is sprayed with an insecticide. The relationship can be modelled using the rule $I = at^b, t \geq 1$.
The following data is collected.

t	1	2	3	4
I	1500	1061	866	750

a Find values for a and b. **b** Find the level of infestation after 10 days.

4.4 Joint variation

There are many situations where one variable depends on more than one other variable. The variable is said to vary **jointly** as the other variables. For example, the volume of a cylinder varies jointly as the square of the radius and the height.

i.e. $V \propto r^2h$

or $V = kr^2h$ (the value of k is known to be π)

Example 10

Given that $y \propto \frac{x^2}{z}$, use this table of values to determine the value of the constant of variation k and hence complete the table.

x	2	4		10
z	10	8	50	
y	2		2.5	4

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Solution

$$y \propto \frac{x^2}{z}$$

$\therefore \quad y = \frac{kx^2}{z}$

When $x = 2$ and $z = 10$, $y = 2$

$$2 = \frac{k(2^2)}{10}$$

$\therefore \quad k = 5$

i.e. $\quad y = \frac{5x^2}{z}$

When $x = 4, z = 8$

$\therefore \quad y = \frac{5(4^2)}{8}$

$$y = 10$$

When $z = 50, y = 2.5$

$\therefore \quad 2.5 = \frac{5(x^2)}{50}$

$$25 = x^2$$

$$x = 5$$

When $x = 10, y = 4$

$\therefore \quad 4 = \frac{5(10^2)}{z}$

$$4z = 500$$

$$z = 125$$

x	2	4	5	10
z	10	8	50	125
y	2	10	2.5	4

Example 11

The speed (s) of a conveyor belt varies jointly as the diameter (d) of the cog around which it passes and the number of revolutions per second (n) the cog makes. The speed of a belt that passes round a cog of diameter 0.3 m, revolving 20 times per second, is 18.85 m/s. Find the value of

a the constant of variation

b the speed of a belt passing around a cog half as big revolving 30 times per second.

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Solution

a $s \propto dn$

i.e. $s = kdn$

When $n = 20$ and $d = 0.3$, $s = 18.85$

$18.85 = k(0.3)(20)$

$\therefore$ $k = 3.142$ (correct to three decimal places)

$\therefore$ $s = 3.142dn$

b When $d = 0.15$ and $n = 30$

$s = 3.142\,(0.15)\,(30)$

$s = 14.14$ m/s (correct to two decimal places)

Exercise 4D

Example 10

1 Given that $y \propto \frac{x}{z}$, use this table of values to determine the value of the constant of variation k and hence complete the table.

x	2	4		10
z	10	2	60	
y	1	10	0.5	4

2 Given that $y \propto xz$, use this table of values to determine the value of the constant of variation k and hence complete the table.

x	2	4		10
z	10	8	50	
y	10		25	15

3 Given that $y \propto \frac{z}{x^2}$, use this table of values to determine the value of the constant of variation k and hence complete the table.

x	2	3		10
z	10	4	50	
y	$\frac{15}{2}$	$\frac{4}{3}$	6	4

4 a varies directly as b^2 and inversely as c. If $a = 0.54$ when $b = 1.2$ and $c = 2$, find a when $b = 2.6$ and $c = 3.5$.

5 z varies as the square root of x and inversely as the cube of y. If $z = 1.46$ when $x = 5$ and $y = 1.5$, find z when $x = 4.8$ and $y = 2.3$.

Example 11

6 The simple interest (I) earned on an investment varies jointly as the interest rate (r) and the time (t) for which it is invested. If a sum of money invested at 6.5% per annum for two years earns \$130, how much interest would the same amount of money earn if it were invested at 5.8% for three years?

7 The kinetic energy (E) of an object varies directly as its mass (m) and the square of its velocity (v). If the kinetic energy of an object with a mass of 2.5 kg moving at 15 m/s is 281.25 joules, find the energy of an object with a mass of 1.8 kg moving at 20 m/s.

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8 The resistance (R) in an electrical wire varies directly as its length (l) and inversely as the square of its diameter (d). Find the percentage change in R if

- **a** l is increased by 50% and d is reduced by 50%
- **b** l is decreased by 50% and d is increased by 50%.

9 The weight (W) that can be supported by a wooden beam varies directly as the square of its diameter (d) and inversely as its length (l).

- **a** What percentage increase in the diameter would be necessary for a beam twice as long to support an equivalent weight?
- **b** What percentage change in the weight would be capable of being supported by a beam three times as long with twice the diameter?

10 If p varies as the square of q and inversely as the square root of r, what is the effect on p if

- **a** both q and r are doubled
- **b** q is doubled and r is halved?

11
- **a** The tension in a spring (T) varies directly with the extension (x) and inversely with the natural length (l) of the spring. Compare the tension in a spring with a natural length of 3 m that is extended by 1 m with the tension in a second spring with a natural length of 2.7 m that is extended by 0.9 m.
- **b** The work done (W) in stretching a spring varies directly with the square of the extension (x) and inversely with the natural length of the spring (l). Compare the work done on the two springs in part **a**.

4.5 Part variation

The total cost (\$$C$) of printing cards is made up of a fixed overhead charge (\$$b$) plus an amount that varies directly as the number printed (n).

i.e. $C = b + kn$

The total surface area (A) of a closed cylinder of fixed height is made up of two parts. The area of the curved surface ($2\pi rh$), which varies as the radius, and the area of the two ends ($2\pi r^2$), which varies as the square of the radius.
i.e. $A = k_1 r + k_2 r^2$ where $k_1 = 2\pi h$ and $k_2 = 2\pi$ are the constants of variation. These are examples of **part variation**.

Part variation exists when the value of one variable is the sum of two or more quantities each of which varies independently in some way. In some cases, as in the first example above, one of those quantities may be constant.

Example 12

A monthly telephone account (A) is made up of a fixed charge (c) for rental and servicing plus an amount that is proportional to the number of calls made (n). In January, 220 calls were made and the account was for \$98.20. In February, 310 calls were made and the account was for \$120.70. Find the fixed charge and the cost per call.

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Solution

$$A = c + kn, \text{ where } c \text{ equals the fixed charge and } k \text{ equals cost per call}$$

$$98.20 = c + 220k \quad \ldots \boxed{1}$$

$$120.70 = c + 310k \quad \ldots \boxed{2}$$

Solving simultaneously, subtract $\boxed{1}$ from $\boxed{2}$

$$22.5 = 90k$$

$$k = 0.25$$

Substitute in $\boxed{1}$

$$98.20 = c + 220\,(0.25)$$

$$= c + 55$$

$$c = 43.2$$

The fixed charge is \$43.20 and the cost per call is \$0.25, i.e. 25 cents.

Example 13

The stopping distance of a tram (d) (i.e. the distance travelled by the tram after its brakes are applied) varies partly with the speed of the tram (s) and partly with the square of its speed. A tram travelling at 15 km/h can stop in 57 m and at 20 km/h in 96 m. Find the formula that relates s to d and hence the stopping distance of a tram travelling at 18 km/h.

Solution

$$d = k_1 s + k_2 s^2$$

$$57 = 15k_1 + 225k_2 \quad \ldots \boxed{1}$$

$$96 = 20k_1 + 400k_2 \quad \ldots \boxed{2}$$

Multiply $\boxed{1}$ by 4 and $\boxed{2}$ by 3

$$228 = 60k_1 + 900k_2 \quad \ldots \boxed{3}$$

$$288 = 60k_1 + 1200k_2 \quad \ldots \boxed{4}$$

Subtract $\boxed{3}$ from $\boxed{4}$

$$60 = 300k_2$$

$$k_2 = \frac{1}{5}$$

Substitute in $\boxed{1}$

$$57 = 15k_1 + 225\left(\frac{1}{5}\right)$$

$$57 = 15k_1 + 45$$

$$k_1 = \frac{12}{15}$$

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$$k_1 = \frac{4}{5}$$

$$\therefore \quad d = \frac{4}{5}s + \frac{1}{5}s^2$$

When $s = 18$

$$d = \frac{4}{5}(18) + \frac{1}{5}(18)^2$$

$$d = 79.2$$

The stopping distance of the tram will be 79.2 m.

Skillsheet TEACHER

Exercise 4E

Example 12

1 The cost of a taxi ride (C) is partly constant (b) and partly varies with the distance travelled (d). A ride of 22 km costs \$42.40 and a ride of 25 km costs \$47.80. Find the cost of a journey of 17 km.

2 The cost of a wedding reception at Hillview Reception Centre includes a fixed overhead charge and an amount per guest.

a If a reception for 50 people costs \$2625 and a reception for 70 people costs \$3575, find the fixed overhead charge and the cost per guest.

b Hence find the total cost of a reception for 100 guests.

Example 13

3 p is the sum of two numbers, one of which varies as x and the other, as the square of y. If $p = 14$ when $x = 3$ and $y = 4$, and $p = 14.5$ when $x = 5$ and $y = 3$, find p when $x = 4$ and $y = 5$.

4 The cost of running a ferris wheel in an amusement park varies partly as the number of people who ride it and partly as the inverse of the number of people who ride it. If the running cost is \$32 when 200 people ride it and \$61 if 400 people ride it, find the running cost on a day when 360 people ride it.

5 The distance travelled (s) by a particle varies partly with time and partly with the square of time. If it travels 142.5 m in 3 s and 262.5 m in 5 s, find

a how far it would travel in 6 s **b** how far it would travel in the sixth second.

6 The time taken (t) to load boxes onto a truck varies partly with the number of boxes (b) and partly with the inverse of the number of men (m) loading the boxes. If it takes one man 45 minutes to load ten boxes and two men 30 minutes to load eight boxes, how long would it take four men to load sixteen boxes?

MC TEST D&D TEST

Chapter summary

- **Direct variation**

  $a \propto b^n$, i.e. a varies directly as $b^n (n \in R^+)$

  This implies $a = kb^n$ where k is the constant of variation ($k \in R^+$).
  As b increases, a will also increase.
  If $a \propto b^n$, the graph of a against b^n is a straight line through the origin.

- **Inverse variation**

  $a \propto \frac{1}{b^n}$, i.e. a varies inversely as $b^n (n \in R^+)$

  This implies $a = \frac{k}{b^n}$ where k is the constant of variation ($k \in R^+$).
  As b increases, a will decrease.
  If $a \propto \frac{1}{b^n}$, the graph of a against $\frac{1}{b^n}$ is a straight line but is undefined at the origin.

- **Joint variation**
  One quantity varies with more than one other variable. This may be a combination of direct and/or inverse variation.

  e.g. $V \propto r^2 h$ implies $V = kr^2 h$
  $a \propto \frac{c}{\sqrt{b}}$ implies $a = \frac{kc}{\sqrt{b}}$

- **Part variation**
  The value of one variable is the sum of two or more quantities each of which is determined by a variation. In some cases, one of those quantities may be constant.
  e.g. $A = k_1 r + k_2 r^2$ where k_1, k_2 are constants of variation.

Multiple-choice questions

1 For the values in the table shown, it is known that $y \propto x^2$. The value of k, the constant of variation, is equal to

x	2	3	6
y	$\frac{4}{3}$	3	12

A 3 **B** 9 **C** $\frac{1}{3}$ **D** 2 **E** $\frac{4}{3}$

2 For the values in the table shown, it is known that $y \propto \frac{1}{x}$. The value of k, the constant of variation, is equal to

x	2	4	8
y	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

A $\frac{1}{2}$ **B** 1 **C** 4 **D** 2 **E** $\frac{1}{4}$

3 $a \propto b^3$ and $a = 32$ when $b = 2$. Find a when $b = 4$.

A 64 **B** 256 **C** 4 **D** 16 **E** 128

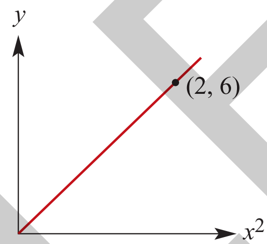
4 $p \propto \frac{1}{q^2}$ and $p = \frac{1}{3}$ when $q = 3$. Find q when $p = 1$

A 3 **B** -3 **C** $\sqrt{3}$ **D** 1 **E** $\frac{1}{3}$

5 The rule connecting y and x as shown in the graph is

A $y = 3x$ **B** $y = 3x^2$ **C** $y = 3\sqrt{x}$

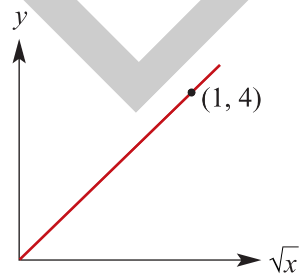
D $y = \frac{1}{3}x$ **E** $y = x^2 + 2$



6 The rule connecting y and x as shown in the graph is

A $y = \frac{1}{4}x$ **B** $y = 4x$ **C** $y = \sqrt{x}$

D $y = 4\sqrt{x}$ **E** $y = x$



7 For the values in the table shown, it is known that $y \propto \frac{x}{z^2}$. The value of k, the constant of variation, is equal to

x	2	4	8
z	2	2	2
y	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$

A 2 **B** $\frac{4}{3}$ **C** $\frac{1}{3}$ **D** 3 **E** $\frac{2}{3}$

8 a varies directly as the square of p and inversely as q and $a = 8$ when $p = 2$ and $q = 5$. If $p = 3$ and $q = 6$ then $a =$

A $\frac{1}{2}$ **B** 12 **C** 120 **D** 15 **E** 5

9 If $p \propto q^2$ and q is increased by 10%, p would be

A Increased by 10% **B** Increased by 20% **C** Increased by 100%

D Increased by 21% **E** Remain the same

10 If $p \propto \frac{1}{q}$ and q is decreased by 20%, p would be

A Decreased by 25% **B** Increased by 25% **C** Decreased by 20%

D Increased by 20% **E** Unchanged

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Review

Short-answer questions (technology-free)

1 **a** If $a \propto b^2$ and $a = \frac{3}{2}$ when $b = 2$, find a when $b = 4$ and b when $a = 8$.

b If $y \propto x^{\frac{1}{3}}$ and $y = 10$ when $x = 2$, find y when $x = 27$ and x when $y = \frac{1}{8}$.

c If $y \propto \frac{1}{x^2}$ and $y = \frac{1}{3}$ when $x = 2$, find y when $x = \frac{1}{2}$ and x when $y = \frac{4}{27}$.

d a varies directly as b and inversely as $\sqrt{c}$. If $a = \frac{1}{4}$ when $b = 1$ and $c = 4$, find a when $b = \frac{4}{9}$ and $c = \frac{16}{9}$.

2 The distance, d metres, which a body falls varies directly as the square of the time, t seconds, for which it has been falling. If a body falls 78.56 m in 4 s, find

a the formula connecting d and t **b** the distance fallen in 10 s

c the time taken to fall 19.64 m.

3 The velocity of a falling body (v metres per second) varies directly as the square root of the distance (s metres) through which it has fallen. A body has a velocity of 7 metres per second after falling 2.5 m.

a Find its velocity after falling 10 m.

b Find the distance through which it falls to attain a velocity of 28 metres per second.

c What variables would be plotted on the axis to obtain a straight line graph?

4 The time taken for a journey is inversely proportional to the average speed of travel. If it takes 4 hours travelling at 30 km/h, how long will it take travelling at 50 km/h?

5 If y varies inversely as x, what is the effect on

a y if x is doubled **b** x if y is doubled **c** y if x is halved **d** x if y is halved?

6 The cost of running an electric appliance varies jointly as the time it is run, the electrical resistance and the square of the current. It costs 9 cents to use an appliance of resistance 60 ohms, which draws 4 amps of current for 2.5 hours. How much will it cost to use an appliance of resistance 80 ohms, which draws 3 amps of current for 1.5 hours?

7 The cost of printing is made up of two parts: a fixed charge and a charge proportional to the number of copies. If the cost of printing 100 copies is \$20 and the cost of printing 500 copies is \$30, what would be the cost of printing 700 copies?

8 For a constant resistance, the voltage (v volts) of an electric circuit varies directly as the current (I amps). If the voltage is 24 volts when the current is 6 amps, find the current when the voltage is 72 volts.

9 The intensity of sound varies inversely as the square of the distance of the observer from the source. If the observer moves to twice the distance from the source, compare the second intensity I_2 with the first intensity I_1.

10 If y varies directly as x^2 and inversely as z, find the percentage change in y when x is increased by 10% and z is decreased by 10%.

Extended-response questions

1 A certain type of hollow sphere is designed in such a way that the mass varies as the square of the diameter. Three spheres of this type are made. One has mass 0.10 kg and diameter 9 cm, the second has diameter 14 cm and the third has mass 0.15 kg. Find

a the mass of the second sphere **b** the diameter of the third sphere.

2 The height (h m) to which a centrifugal pump raises water is proportional to the square of its speed of rotation (n revs/min). If the pump raises water to a height of 13.5 m when it is rotating at 200 revs/min, find

a the formula connecting h and n

b the height that the water can be raised to when it is rotating at 225 revs/min

c the speed required to raise the water to a height of 16 m.

3 The maximum speed of yachts of normal dimensions varies as the square root of their length. If a yacht 20 m long can maintain a maximum speed of 15 knots, find the maximum speed of a yacht 15 m long.

4 **a** The air in a tube occupies 43.5 cm^3 and the pressure is 2.8 kg/cm^2. If the volume ($V\,\text{cm}^3$) varies inversely as the pressure (P), find the formula connecting V and P.

b Calculate the pressure when the volume is decreased to 12.7 cm^3.

5 The weight (w kg) which a beam supported at each end will carry without breaking, varies inversely as the distance (d m) between supports. A beam which measures 6 m between supports will just carry a load of 500 kg.

a Find the formula connecting w and d.

b What weight would a similar beam carry if the distance between the supports were 5 m?

c What weight would a similar beam carry if the distance between the supports were 9 m?

6 The relationship between pressure and volume of a fixed mass of gas when the temperature is constant is shown by the following table.

Pressure (p)	12	16	18
Volume (v)	12	9	8

a What is a possible equation relating p and v?

b Using this equation, find

i the volume when the pressure is 72 units

ii the pressure when the volume is 3 units.

c Sketch the graph relating v and $\frac{1}{p}$.

7 The time taken to manufacture particular items of scientific equipment varies partly as the diameter of the item and partly as the number of parts required in the item. If it takes

30 minutes to make a 3 cm diameter item with eight parts and 38 minutes to make a 5 cm diameter item with ten parts, how long does it take to make a 4 cm diameter item with twelve parts?

8 The cost of decorative wrought iron is the sum of two parts which vary as the length and the square of the length respectively. When the length is 2 m, the cost is \$18.40 and when the length is 3 m, the cost is \$33.60. Find the cost when the length is 5 m.

9 The sum of the first n natural numbers is equal to the sum of two quantities, the first of which is proportional to n and the second to n^2. Work out the sums of the first three and four natural numbers and hence find the formula for the sum of the first n natural numbers.

10 Data about the number of pies sold at football matches and the size of the crowds attending has been recorded as follows

Attendance ($N \times 1000$)	20	30	60
Number of pies sold (P)	15 650	19 170	27 110

a Use a graphics calculator to find an approximate relationship between N and P of the form $P = aN^b$.

b The crowd predicted for a forthcoming match is 55 000. Assuming the model found in part **a** applies, how many pies would the caterers anticipate selling on that day?

c The caterers have only 25 000 pies available for sale. Again assuming the model found in part **a** applies, what is the maximum crowd the caterers might be hoping for if they are able to satisfy all customers wanting pies?

11 The effectiveness of an anaesthetic drug is being tested by varying the dose (d mL) given to patients and recording both the time (t min) for the patient to lose consciousness and the time (T min) for the patient to regain consciousness. The following data was recorded:

Dosage (d mL)	10	30	60
Time to lose consciousness (t min)	36	4	1
Time to regain consciousness (T min)	14	126	504

a Establish the relationship between d and t (assume t is proportional to a power of d).

b Establish the relationship between d and T (assume T is proportional to a power of d).

c If it is desirable to have a particular patient unconscious for no longer than 80 minutes, what is the maximum dosage of the drug that should be given?

d How long would it take that patient to lose consciousness?

e Another patient is given a dose of 20 mL. How long will it take for the patient to lose consciousness and how long will they remain unconscious?

12 The German astronomer Johannes Kepler collected data on the mean distance from the Sun to the planets ($R \times 10^6$ km) and the period of the orbit (T years). He was able to establish a relationship between R and T.

a Using the data below (approximations only)

i establish the relationship between R and T (assume T is proportional to a power of R)

ii complete the table of values showing the period of orbit of the remaining planets

Planet	Approximate radius of orbit ($R \times 10^6$ km)	Period of orbit (T years)
Mercury	58	0.24
Venus	108	0.61
Earth	150	1
Mars	228	
Jupiter	779	
Saturn	1427	
Uranus	2870	
Neptune	4497	
Pluto	5900	

b A comet orbits the sun every 70 years. What is its radius of orbit?

13 To test the effectiveness of an advertising campaign for cheap flights to Hawaii, a travel agent keeps a record of the number of enquiries she receives. It is estimated that the number of enquiries, E, is proportional to the number of times, n, that the advertisement is shown on television. The following data is collected.

Number of advertisements (n)	10	20	30
Number of enquiries (E)	30	40	47

a Assuming that the relationship between the number of enquiries and the number of advertisements is modelled by the rule $E = an^b$, use your graphics calculator to find values for a and b.

b Predict the number of enquiries received if the advertisement is shown 100 times.

After two weeks the advertisement has been shown 50 times and the advertising campaign is stopped. The travel agent continues to get enquiries and continues to record them. It is now estimated that the number of enquiries, E, is proportional to the number of days, d, since the advertising campaign stopped. The following data is recorded.

Number of days (d)	3	5	7	10
Number of enquiries (E)	45	25	17	11

c Assuming that the relationship between the number of enquiries and the number of days is modelled by the rule $E = kd^p$, use your graphics calculator to find values for k and p.

d Predict the number of enquiries received on the 14th day after the advertising campaign has finished.