CHAPTER 3
Number systems and sets

Objectives

To understand and use the notation of sets including the symbols \( \in, \subseteq, \cap, \cup, \emptyset, \xi \)
To be able to identify sets of numbers including natural numbers, integers, rational numbers, irrational numbers, real numbers
To know and be able to apply the rules for:
- simplification of surds
- operations on surds
- rationalisation of surds
To know the definition of factor, prime, highest common factor
To be able to solve problems with sets

Introduction
This chapter introduces set notation and discusses sets of numbers and their properties. Set notation is used widely in mathematics and in this book it is employed where appropriate.

A set is a general name for any collection of things or numbers. There must be a way of deciding whether any particular object is a member of the set or not. This may be done by referring to a list of the members of the set or a statement describing them.

For example, \( A = \{-3, 3\} = \{x : x^2 = 9\} \)

Note: \( \{x : \ldots\} \) is read as ‘the set of objects \( x \) such that \( \ldots \)’.

Number systems
Recall that the elements of \( \{1, 2, 3, 4, \ldots\} \) are called natural numbers and the elements \( \{\ldots, -2, -1, 0, 1, 2, \ldots\} \) are called integers.

The numbers of the form \( \frac{p}{q} \) with \( p \) and \( q \) integers, \( q \neq 0 \), are called rational numbers.
The real numbers which are not rational are called irrational, e.g. \( \pi \) and \( \sqrt{2} \).
The set of real numbers will be denoted by \( R \).
The set of rational numbers will be denoted by \( Q \).
The set of integers will be denoted by \( Z \).
The set of natural numbers will be denoted by \( N \).

\[ R^2 = \{ (x, y) : x, y \in R \} \]

These sets of numbers will be discussed further in Sections 3.2 and 3.3.

3.1 Set notation

The symbol \( \in \) means ‘is a member of’ or ‘is an element of’. For example, \( 3 \in \{ \text{prime numbers} \} \) is read ‘3 is a member of the set of prime numbers’.

The symbol \( \notin \) means ‘is not a member of’ or ‘is not an element of’. For example, \( 4 \notin \{ \text{prime numbers} \} \) is read ‘4 is not a member of the set of prime numbers’.

Two sets are equal if they contain exactly the same elements, not necessarily in the same order. For example, if \( A = \{ \text{prime numbers less than 10} \} \) and \( B = \{ 2, 3, 5, 7 \} \) then \( A = B \).

Two sets \( A \) and \( B \) are equivalent if they have the same number of elements. For example, \( \{1, 2, 3\} \leftrightarrow \{a, b, c\} \).

A set which has no elements is called the empty or null set and is denoted by \( \emptyset \).

The universal set will be denoted by \( \xi \). The universal set is the set of all elements which are being considered.

If all the elements of a set \( B \) are also members of a set \( A \), then the set \( B \) is called a subset of \( A \). This is written \( B \subseteq A \). For example, \( \{1, 2, 3\} \subseteq Z \) where \( Z \) is the set of integers, and \( \{3, 9, 27\} \subseteq \{ \text{multiples of 3} \} \). We note also \( A \subseteq A \) and \( \emptyset \subseteq A \).

Venn diagrams are used to illustrate sets. For example, let \( \xi \) denote the set of all real numbers less than 100, \( A \) denote the set of real numbers less than 50 and \( B \) the set of real numbers between 90 and 100 (non-inclusive).

This may be illustrated by a Venn diagram. \( A \) and \( B \) have no elements in common.

Two such sets are called disjoint sets.

The union of two sets

The set of elements that are in either set \( A \) or set \( B \) (or both) is the union of sets \( A \) and \( B \). The union of \( A \) and \( B \) is written \( A \cup B \).
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Example 1

Let \( \xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), \( A = \{1, 2, 3\} \) and \( B = \{1, 2, 9, 10\} \). Find \( A \cup B \) and illustrate on a Venn diagram.

Solution

\[ A \cup B = \{1, 2, 3, 9, 10\} \]

The shaded area illustrates \( A \cup B \).

The intersection of two sets

The set of all the elements that are members both of set \( A \) and of set \( B \) is called the intersection of \( A \) and \( B \). The intersection of \( A \) and \( B \) is written \( A \cap B \).

Example 2

Let \( \xi = \{\text{prime numbers less than 40}\} \). If \( A = \{3, 5, 7, 11\} \) and \( B = \{3, 7, 29, 37\} \), find \( A \cap B \).

Solution

\[ A \cap B = \{3, 7\} \]

Complement

If \( \xi = \{\text{students at Highland Secondary College}\} \) and \( A = \{\text{students with blue eyes}\} \), then the complement of \( A \) is the set of all members of \( \xi \) that are not members of \( A \). In this case the complement is the set of all students of Highland Secondary College that do not have blue eyes. The complement of \( A \) is denoted by \( A' \).

Similarly, if \( \xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \) and \( A = \{1, 3, 5, 7, 9\} \) then \( A' = \{2, 4, 6, 8, 10\} \).

Finite and infinite sets

When all the elements of a set may be counted the set is called a finite set, e.g. \( A = \{\text{months of the year}\} \). The number of elements of a set \( A \) will be denoted \( n(A) \). In this example \( n(A) = 12 \). For the set \( C = \{\text{letters of the alphabet}\} \), \( n(C) = 26 \).

Sets which are not finite are infinite sets. For example, \( R \), the set of real numbers, and \( Z \), the set of integers, are infinite sets.
Example 3

Given \( \xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)

\( A = \{\text{odd numbers}\} = \{1, 3, 5, 7, 9\} \)

\( B = \{\text{multiples of } 3\} = \{3, 6, 9\} \)

show these sets on a Venn diagram.

Use the diagram to list the following sets.

a. \( A' \)

b. \( B' \)

c. \( A \cup B \)

d. the complement of \( A \cup B \) i.e. \( (A \cup B)' \)

e. \( A' \cap B' \)

Solution

(Each of the numbers in the given sets is placed in the correct position on this Venn diagram.)

From the diagram:

a. \( A' = \{2, 4, 6, 8, 10\} \)

b. \( B' = \{1, 2, 4, 5, 7, 8, 10\} \)

C. \( A \cup B = \{1, 3, 5, 6, 7, 9\} \)

d. \( (A \cup B)' = \{2, 4, 8, 10\} \)

Exercise 3A

1. \( \xi = \{1, 2, 3, 4, 5\}, A = \{1, 2, 3, 5\}, B = \{2, 4\} \)

Show these sets on a Venn diagram and use the diagram to find

a. \( A' \)

b. \( B' \)

c. \( A \cup B \)

d. \( (A \cup B)' \)

e. \( A' \cap B' \)

2. \( \xi = \{\text{natural numbers less than } 17\}, P = \{\text{multiples of } 3\}, Q = \{\text{even numbers}\} \)

Show these sets on a Venn diagram and use it to find

a. \( P' \)

b. \( Q' \)

c. \( P \cup Q \)

d. \( (P \cup Q)' \)

e. \( P' \cap Q' \)

3. \( \xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}, A = \{\text{multiples of } 4\}, B = \{\text{even numbers}\} \)

Show these sets on a Venn diagram and use this diagram to list the sets

a. \( A' \)

b. \( B' \)

c. \( A \cup B \)

d. \( (A \cup B)' \)

e. \( A' \cap B' \)

4. \( \xi = \{\text{natural numbers from } 10 \text{ to } 25\}, P = \{\text{multiples of } 4\}, Q = \{\text{multiples of } 5\} \)

Show these sets on a Venn diagram and use this diagram to list the sets

a. \( P' \)

b. \( Q' \)

c. \( P \cup Q \)

d. \( (P \cup Q)' \)

e. \( P' \cap Q' \)

5. \( \xi = \{\text{different letters in the word } \text{GENERAL}\}, \)

\( A = \{\text{different letters in the word } \text{ANGEL}\}, \)

\( B = \{\text{different letters in the word } \text{LEAN}\} \)

Show these sets on a Venn diagram and use this diagram to list the sets

a. \( A' \)

b. \( B' \)

c. \( A \cap B \)

d. \( A \cup B \)

e. \( (A \cup B)' \)

f. \( A' \cap B' \)
6 \( \xi = \{ p, q, r, s, t, u, v, w \} \), \( X = \{ r, s, t, w \} \), \( Y = \{ q, s, t, u, v \} \)

Show \( \xi \), \( X \) and \( Y \) on a Venn diagram, entering all members. Hence list the sets

\begin{itemize}
  \item [a] \( X' \)
  \item [b] \( Y' \)
  \item [c] \( X' \cap Y' \)
  \item [d] \( X' \cup Y' \)
  \item [e] \( X \cup Y \)
  \item [f] \( (X \cup Y)' \)
\end{itemize}

Which two sets are equal?

7 \( \xi = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \} \), \( X = \{ \text{factors of 12} \} \),

\( Y = \{ \text{even numbers} \} \)

Show \( \xi \), \( X \) and \( Y \) on a Venn diagram entering all members. Hence list the sets

\begin{itemize}
  \item [a] \( X' \)
  \item [b] \( Y' \)
  \item [c] \( X' \cup Y' \)
  \item [d] \( X' \cap Y' \)
  \item [e] \( X \cup Y \)
  \item [f] \( (X \cup Y)' \)
\end{itemize}

Which two sets are equal?

8 Draw this diagram six times. Use shading to illustrate each of the following sets.

\begin{itemize}
  \item [a] \( A' \)
  \item [b] \( B' \)
  \item [c] \( A' \cap B' \)
  \item [d] \( A' \cup B' \)
  \item [e] \( A \cup B \)
  \item [f] \( (A \cup B) \)
\end{itemize}

9 \( \xi = \{ \text{different letters in the word MATHEMATICS} \} \)

\( A = \{ \text{different letters in the word ATTIC} \} \)

\( B = \{ \text{different letters in the word TASTE} \} \)

Show \( \xi \), \( A \) and \( B \) on a Venn diagram entering all the elements. Hence list the sets

\begin{itemize}
  \item [a] \( A' \)
  \item [b] \( B' \)
  \item [c] \( A \cap B \)
  \item [d] \( (A \cup B)' \)
  \item [e] \( A' \cup B' \)
  \item [f] \( A' \cap B' \)
\end{itemize}

3.2 Sets of numbers

The following notation was introduced earlier in this chapter.

\( R \) denotes the set of real numbers.

\( Q \) denotes the set of rational numbers.

\( Z \) denotes the set of integers.

\( N \) denotes the set of natural numbers.

A geometric construction of a line segment of length \( \frac{m}{n} \) where \( m \) and \( n \) are non-zero integers is shown in Chapter 9. Constructions of products and quotients are also shown in that chapter.

It is clear that \( N \subseteq Z \subseteq Q \subseteq R \) and this may be represented by the diagram shown.

The set of all \( x \) such that \( \ldots \) is denoted by \( \{ x : \ldots \} \), where \( \ldots \) stands for some condition. Thus

\( \{ x : 0 < x < 1 \} \) is the set of all real numbers between 0 and 1.

\( \{ x : x > 0, x \in Q \} \) is the set of all positive rational numbers.

\( \{ 2n : n = 1, 2, 3, \ldots \} \) is the set of all even numbers.
Examples of irrational numbers are \( \sqrt{3}, \sqrt{2}, \pi, \pi + 2, \sqrt{7} + \sqrt{6} \). These numbers cannot be written in the form \( \frac{p}{q} \) where \( p \) and \( q \) are integers. The decimal representation of these numbers does not terminate or repeat.

### Rational numbers

Every rational number can be expressed as a terminating or recurring decimal. For example

\[
\frac{1}{2} = 0.5, \quad \frac{1}{5} = 0.2, \quad \frac{1}{10} = 0.1, \quad \frac{1}{3} = 0.\bar{3}, \quad \frac{1}{7} = 0.\overline{142857}
\]

Numbers of the form \( \frac{m}{n} \), where \( m \) and \( n \in \mathbb{N} \) have a terminating decimal representation if and only if \( n = 2^a \times 5^b \), where \( a \) and \( b \) are members of the set \( \mathbb{N} \cup \{0\} \).

In order to find the decimal representation of a rational number \( \frac{m}{n} \), the division \( m \div n \) is undertaken. For example

\[
\begin{align*}
\text{a} & \quad \frac{1}{20} = 0.05 \\
\text{b} & \quad \frac{3}{7} = 0.\overline{428571}
\end{align*}
\]

Therefore

\[
\begin{align*}
\text{a} & \quad \frac{1}{20} = 0.05 \\
\text{b} & \quad \frac{3}{7} = 0.\overline{428571}
\end{align*}
\]

The method to find a rational number \( \frac{m}{n} \) from its decimal representation is demonstrated in the following example.

### Example 4

Write each of the following in the form \( \frac{m}{n} \).

\[
\begin{align*}
\text{a} & \quad 0.05 \\
\text{b} & \quad 0.428571
\end{align*}
\]

**Solution**

\[
\begin{align*}
\text{a} & \quad 0.05 = \frac{5}{100} = \frac{1}{20} \quad \text{Therefore} \quad \frac{1}{20} = 0.05 \\
\text{b} & \quad 0.428571 = 0.428571428571\ldots \quad \text{Therefore} \quad \frac{3}{7} = 0.\overline{428571}
\end{align*}
\]

Multiply both sides by 10^6

\[
0.428571 \times 10^6 = 428571.428571428571\ldots
\]

Subtract 1 from 2

\[
0.428571 \times (10^6 - 1) = 428571
\]

\[
\therefore \quad 0.428571 = \frac{428571}{10^6 - 1} = \frac{3}{7}
\]

### Irrational numbers

The set of irrational numbers has two important subsets, **algebraic numbers** and **transcendental numbers**.

Algebraic numbers are those which are the solutions of an equation of the form

\[
a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0,
\]

where \( a_0, a_1, \ldots, a_n \) are integers.
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For example, $\sqrt{2}$ is an algebraic number, as it is a solution of the equation

$$x^2 - 2 = 0$$

$\pi$ is not an algebraic number, it is a transcendental number.

The proof that $\sqrt{2}$ is irrational is presented here. The proof is by contradiction.

Assume $\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{N}$

and $\frac{a}{b}$ is a fraction in simplest form.

Then $\frac{a^2}{b^2} = 2$

$\therefore a^2 = 2b^2$ which implies $a^2$ is even which implies $a$ is even

$\therefore a = 2k$ for some $k \in \mathbb{N}$

$\therefore a^2 = 4k^2$

$\therefore 4k^2 = 2b^2$

$\therefore b^2 = 2k^2$ which implies $b^2$ is even which implies $b$ is even

But this contradicts the assumption that $\frac{a}{b}$ is a fraction in simplest form, as $a$ and $b$ are both divisible by 2.

**Exercise 3B**

1. Is the
   a sum   b product   c quotient (if defined)
   of two irrational numbers rational?

2. Is the
   a sum   b product   c quotient
   of two irrational numbers irrational?

3. Write each of the following in the form $\frac{m}{n}$ where $m$ and $n$ are integers.
   a $0.\overline{2}$   b $0.12$   c $0.285714$   d $0.3\overline{6}$   e $0.\overline{2}$   f $0.45$

4. Give the decimal representation of each of the following rational numbers.
   a $\frac{2}{7}$   b $\frac{5}{11}$   c $\frac{7}{20}$   d $\frac{4}{13}$   e $\frac{1}{17}$

5. Prove that $\sqrt{3}$ is not a rational number.

### 3.3 Surds

A number of the form $\sqrt{a}$ where $a$ is a rational number which is not a square of another rational number is called a **quadratic surd**.

**Note:** $\sqrt{a}$ is taken to mean the positive square root.
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If \( a \) is a rational which is not a perfect \( n \)th power, \( \sqrt[n]{a} \) is called a **surd of the \( n \)th order**.

\[
\sqrt{2}, \sqrt{7}, \sqrt{24}, \sqrt[9]{\frac{9}{7}}, \sqrt[2]{\frac{1}{2}} \text{ are quadratic surds}
\]

\[
\sqrt{9}, \sqrt{16}, \sqrt{\frac{9}{4}} \text{ are not surds}
\]

\[
\sqrt[3]{7}, \sqrt[3]{15} \text{ are surds of the third order}
\]

\[
\sqrt[4]{100}, \sqrt[4]{26} \text{ are surds of the fourth order}
\]

Quadratic surds hold a prominent position in school mathematics. For example, the solutions of quadratic equations often involve surds, e.g.

\[
x = \frac{1}{2} + \frac{1}{2} \sqrt{5} \text{ is a solution of the quadratic equation } x^2 - x - 1 = 0
\]

Values of trigonometric functions sometimes involve surds, e.g.

\[
\sin 60^\circ = \frac{\sqrt{3}}{2}, \sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}
\]

In Mathematical Methods Units 3 and 4 and Specialist Mathematics Units 3 and 4, exact solutions are often required.

Lengths such as \( \sqrt{2}, \sqrt{3} \) or \( \sqrt{6} \) can be constructed geometrically, using a straight edge and a compass. For example, from the right-angled isosceles triangle \( ABC \):

![Diagram of right-angled isosceles triangle with sides labeled](image)

The length \( AB = \sqrt{2} \)

Any quadratic surd (of a natural number) may be constructed in this way. This makes it possible to construct a line segment of the length determined by the solutions of many quadratic equations.

For example, one solution of \( x^2 - x - 1 = 0 \) is \( x = \frac{1}{2} + \frac{1}{2} \sqrt{5} \).

The construction of a line segment of this length involves the right angled triangle \( XYZ \).

![Diagram of right-angled triangle with sides labeled](image)

The length is now achieved by bisecting the line segment.
Properties of square roots
The following properties of square roots are often used.
\[
\sqrt{ab} = \sqrt{a} \sqrt{b}, \text{ e.g. } \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2} \\
\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, \text{ e.g. } \sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{\sqrt{9}} = \frac{\sqrt{7}}{3}
\]

Properties of surds

Simplest form
If possible, a factor is taken ‘out of a square root’. When the number under the square root has no factors which are squares of a rational number, then the surd is said to be in simplest form.

Example 5
Write each of the following in simplest form.

\[a \sqrt{72} \quad b \sqrt{28} \quad c \sqrt{700} \quad d \sqrt{99}\]

Solution
\[a \sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2} \quad b \sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7} \]
\[c \sqrt{\frac{700}{117}} = \sqrt{\frac{700}{117}} = \sqrt{\frac{7 \times 100}{9 \times 13}} = \frac{10}{3} \sqrt{\frac{7}{13}} \]
\[d \sqrt{\frac{99}{64}} = \sqrt{\frac{99}{64}} = \sqrt{\frac{9 \times 11}{8}} = \frac{3\sqrt{11}}{8} \]

Surd which have the same ‘irrational factor’ are called like surds. For example, \(3\sqrt{7}, 2\sqrt{7}\) and \(\sqrt{7}\) are like surds.
The sum or difference of two like surds can be found.
i.e. \(m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p}\) and \(m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}\)

Example 6
Express each of the following as a single surd in simplest form.

\[a \sqrt{147} + \sqrt{108} - \sqrt{363} \quad b \sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48} \]
\[c \sqrt{\frac{1}{8}} - \sqrt{\frac{1}{18}} - 5\sqrt{\frac{1}{72}} \quad d \sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}\]

Solution
\[a \sqrt{147} + \sqrt{108} - \sqrt{363} = \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3} - \sqrt{11^2 \times 3} = 7\sqrt{3} + 6\sqrt{3} - 11\sqrt{3} = 2\sqrt{3} \]

\[b \sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48} = \sqrt{3} + \sqrt{5} + \sqrt{4 \times 5} + \sqrt{9 \times 3} - \sqrt{9 \times 5} - \sqrt{16 \times 3} = \sqrt{3} + \sqrt{5} + 2\sqrt{5} + 3\sqrt{3} - 3\sqrt{5} - 4\sqrt{3} = 2\sqrt{3} \]

\[c \sqrt{\frac{1}{8}} - \sqrt{\frac{1}{18}} - 5\sqrt{\frac{1}{72}} = \sqrt{\frac{1}{8}} - \sqrt{\frac{1}{18}} - 5\sqrt{\frac{1}{72}} = \frac{1}{2\sqrt{2}} - \frac{1}{3\sqrt{2}} - 5\frac{1}{6\sqrt{2}} = \frac{1}{2\sqrt{2}} - \frac{1}{3\sqrt{2}} - \frac{5}{6\sqrt{2}} = \frac{3 - 2 - 5}{6\sqrt{2}} = -\frac{4}{6\sqrt{2}} = -\frac{2}{3\sqrt{2}} = -\frac{2\sqrt{2}}{6} = -\frac{\sqrt{2}}{3} \]

\[d \sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8} = \sqrt{25 \times 2} + \sqrt{2} - 2\sqrt{9 \times 2} + \sqrt{4 \times 2} = 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} + \sqrt{2} - 6\sqrt{2} + 2\sqrt{2} = 0 \]
\[
\begin{align*}
\text{b} & \quad \sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48} \\
& \quad = \sqrt{3} + \sqrt{5} + 2\sqrt{5} + 3\sqrt{3} - 3\sqrt{5} - 4\sqrt{3} \\
& \quad = 0\sqrt{3} + 0\sqrt{5} \\
& \quad = 0 \\
\text{c} & \quad \sqrt{\frac{1}{8}} - \sqrt{\frac{1}{18}} - 5\sqrt{\frac{1}{72}} \\
& \quad = \sqrt{\frac{1}{4 \times 2}} - \sqrt{\frac{1}{9 \times 2}} - 5\sqrt{\frac{1}{36 \times 2}} \\
& \quad = \frac{1}{2\sqrt{2}} - \frac{1}{3\sqrt{2}} - 5\sqrt{\frac{1}{2}} \\
& \quad = \frac{3}{6\sqrt{2}} - \frac{2}{6\sqrt{2}} - 5\sqrt{\frac{1}{2}} \\
& \quad = -\frac{4}{6\sqrt{2}} \\
& \quad = -\frac{2}{3\sqrt{2}} \\
\text{d} & \quad \sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8} \\
& \quad = 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2} \\
& \quad = 8\sqrt{2} - 6\sqrt{2} \\
& \quad = 2\sqrt{2}
\end{align*}
\]

Rationalising the denominator

In the past, a labour saving procedure with surds was to rationalise any surds in the denominator of an expression. It is still considered to be a neat way of expressing final answers.

For \(\sqrt{5}\) a rationalising factor is \(\sqrt{5}\) as \(\sqrt{5} \times \sqrt{5} = 5\)

For \(\sqrt{2} + 1\) a rationalising factor is \(1 - \sqrt{2}\) as \((1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1\)

For \(\sqrt{3} + \sqrt{6}\) a rationalising factor is \(\sqrt{3} - \sqrt{6}\) as \((\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6}) = 3 - 6 = -3\)

Example 7

Rationalise the denominator of each of the following.

\[
\begin{align*}
a & \quad \frac{1}{2\sqrt{7}} \\
b & \quad \frac{1}{2 - \sqrt{3}} \\
c & \quad \frac{1}{\sqrt{3} - \sqrt{6}} \\
d & \quad \frac{3 + \sqrt{8}}{3 - \sqrt{8}}
\end{align*}
\]

Solution

\[
\begin{align*}
a & \quad \frac{1}{2\sqrt{7}} \times \sqrt{\frac{7}{7}} = \frac{\sqrt{7}}{14} \\
b & \quad \frac{1}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{2 + \sqrt{3}}{4 - 3} = 2 + \sqrt{3} \\
c & \quad \frac{1}{\sqrt{3} - \sqrt{6}} \times \frac{\sqrt{3} + \sqrt{6}}{\sqrt{3} + \sqrt{6}} = \frac{\sqrt{3} + \sqrt{6}}{3 - 6} = -\frac{1}{3}(\sqrt{3} + \sqrt{6}) \\
d & \quad \frac{3 + \sqrt{8}}{3 - \sqrt{8}} = \frac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \times \frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}} = \frac{9 + 12\sqrt{2} + 8}{9 - 8} = 17 + 12\sqrt{2}
\end{align*}
\]
Example 8

Expand the brackets in each of the following and collect like terms, expressing surds in simplest form.

a \((3 - \sqrt{2})^2\)

**Solution**

\[
(3 - \sqrt{2})^2 = (3 - \sqrt{2})(3 - \sqrt{2}) = 3(3 - \sqrt{2}) - \sqrt{2}(3 - \sqrt{2}) = 9 - 3\sqrt{2} - 3\sqrt{2} + 2 = 11 - 6\sqrt{2}
\]

b \((3 - \sqrt{2})(1 + \sqrt{2})\)

\[
(3 - \sqrt{2})(1 + \sqrt{2}) = 3(1 + \sqrt{2}) - \sqrt{2}(1 + \sqrt{2}) = 3 + 3\sqrt{2} - \sqrt{2} - 2 = 1 + 2\sqrt{2}
\]

**Using the TI-Nspire**

A CAS calculator can be used to work with irrational numbers.

Expressions can be reached and selected using the **up arrow** (▲). This returns the expression to the entry line and modifications can be made.

To illustrate this, evaluate \(\frac{2^3 \cdot 2^2}{5} \cdot 2^7\) as shown.

To find the square root of this expression, type \(\sqrt{\text{expression}}\) and move upwards by pressing the **up arrow** (▲) so that the expression is highlighted.

Press **enter** (\(\text{\texttt{\textbf{\textsc{enter}}}}\)) to paste this expression into the square root sign and press **enter** once more to evaluate the square root of this expression.
Exercise 3C

1. Express each of the following in terms of the simplest possible surds.
   - a $\sqrt{8}$
   - b $\sqrt{12}$
   - c $\sqrt{27}$
   - d $\sqrt{50}$
   - e $\sqrt{45}$
   - f $\sqrt{1210}$
   - g $\sqrt{98}$
   - h $\sqrt{108}$
   - i $\sqrt{25}$
   - j $\sqrt{75}$
   - k $\sqrt{512}$

2. Simplify each of the following.
   - a $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$
   - b $\sqrt{75} + 2\sqrt{12} - \sqrt{27}$
   - c $\sqrt{28} + \sqrt{175} - \sqrt{63}$
   - d $\sqrt{1000} - 40 - \sqrt{96}$
   - e $\sqrt{512} + \sqrt{128} + \sqrt{32}$
   - f $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

3. Simplify each of the following.
   - a $\sqrt{75} + \sqrt{108} + \sqrt{14}$
   - b $\sqrt{847} - \sqrt{567} + \sqrt{63}$
   - c $\sqrt{720} + \sqrt{175} - \sqrt{63}$
   - d $\sqrt{338} - \sqrt{288} + \sqrt{363} - \sqrt{300}$
   - e $\sqrt{12} + \sqrt{8} + \sqrt{18} + \sqrt{27} + \sqrt{300}$
   - f $2\sqrt{18} + 3\sqrt{5} - \sqrt{50} + \sqrt{20} - \sqrt{80}$

4. Express each of the following with rational denominators.
   - a $\frac{1}{\sqrt{5}}$
   - b $\frac{1}{\sqrt{7} + 1}$
   - c $\frac{1}{\sqrt{2}}$
   - d $\frac{2}{\sqrt{3}}$
   - e $\frac{3}{\sqrt{6}}$
   - f $\frac{1}{2\sqrt{2}}$
   - g $\frac{1}{\sqrt{2} + 1}$
   - h $\frac{1}{2 - \sqrt{3}}$
   - i $\frac{1}{4 - \sqrt{10}}$
   - j $\frac{1}{\sqrt{6} + 2}$
   - k $\frac{1}{\sqrt{5} - \sqrt{3}}$
   - l $\frac{1}{\sqrt{6} - \sqrt{5}}$
   - m $\frac{1}{3 - 2\sqrt{2}}$

5. Express each of the following in the form $a + b\sqrt{c}$.
   - a $\frac{2}{3 - 2\sqrt{2}}$
   - b $(\sqrt{5} + 2)^2$
   - c $(1 + \sqrt{2})(3 - 2\sqrt{2})$
   - d $(\sqrt{3} - 1)^2$
   - e $\sqrt{\frac{1}{3} - \frac{1}{\sqrt{27}}}$
   - f $\frac{\sqrt{3} + 2}{2\sqrt{3} - 1}$
   - g $\frac{\sqrt{5} + 1}{\sqrt{5} - 1}$
   - h $\frac{\sqrt{8} + 3}{\sqrt{18} + 2}$

6. Expand and simplify each of the following.
   - a $(2\sqrt{a} - 1)^2$
   - b $(\sqrt{x} + 1 + \sqrt{x} + 2)^2$

7. For real numbers $a$ and $b$, $a > b$ if and only if $a - b > 0$. Use this to state the larger of
   - a $5 - 3\sqrt{2}$ and $6\sqrt{2} - 8$
   - b $2\sqrt{6} - 3$ or $7 - 2\sqrt{6}$

3.4 Natural numbers

Factors and composites

The factors of 8 are 1, 2, 4 and 8.

The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

The factors of 5 are 1 and 5.
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A natural number, $a$, is a factor of a natural number, $b$, if there exists a natural number, $k$, such that $b = ak$.

If a number greater than 1 has only factors 1 and itself, it is said to be prime.

Among the first 100 numbers, the following are prime:


A number, $m$, is called a composite if it can be written as a product, $m = a \times b$ where $a$ and $b$ are numbers greater than 1 and less than $m$.

**Prime decomposition**

$$3000 = 3 \times 5^3 \times 2^3$$
$$2294 = 2 \times 31 \times 37$$

This method of uniquely expressing a composite in terms of a product of powers of prime numbers is called **decomposition**. It is useful for finding factors of numbers.

For example, the prime decomposition of 12 is given by $12 = 2^2 \times 3$

The factors of 12 are 1, 2, 2, 3, 2 23, 22, 3, 22, 2, 3, 6, 12.

**Example 9**

Give the prime decomposition of 17 248 and hence list the factors of this number.

**Solution**

The prime decomposition can be determined by repeated division.

$$
\begin{array}{c|c}
2 & 17248 \\
2 & 8624 \\
2 & 4312 \\
2 & 2156 \\
2 & 1078 \\
7 & 539 \\
7 & 77 \\
11 & 11 \\
& 1 \\
\end{array}
$$

The prime decomposition of 17 248 is $17248 = 2^5 \times 7^2 \times 11$

The factors can systematically be determined in the following way.

$2^5, 2^4, 2^3, 2^2, 2, 1$
$2^5 \times 7, 2^4 \times 7, 2^3 \times 7, 2^2 \times 7, 2 \times 7, 7$
$2^5 \times 7^2, 2^4 \times 7^2, 2^3 \times 7^2, 2^2 \times 7^2, 2 \times 7^2, 7^2$
$2^5 \times 11, 2^4 \times 11, 2^3 \times 11, 2^2 \times 11, 2 \times 11, 11$
$2^5 \times 7 \times 11, 2^4 \times 7 \times 11, 2^3 \times 7 \times 11, 2^2 \times 7 \times 11, 2 \times 7 \times 11, 7 \times 11$
$2^5 \times 7^2 \times 11, 2^4 \times 7^2 \times 11, 2^3 \times 7^2 \times 11, 2^2 \times 7^2 \times 11, 2 \times 7^2 \times 11, 7^2 \times 11$
Highest common factor (greatest common divisor)

The highest common factor of two natural numbers is the largest natural number which is a factor of both the numbers. For example, the highest common factor of 15 and 24 is 3. The prime decomposition can be used to find the highest common factor of two numbers. Consider the numbers 140 and 110. The prime factorisations of these numbers are

\[ 140 = 2^2 \times 5 \times 7 \] and \[ 110 = 2 \times 5 \times 11. \]

The number which is a factor of 140 and 110 must have prime factors which occur in both factorisations. The exponent (power) of each of these prime factors will be the smaller of the two exponents occurring in the factorisation of 140 and 110. Thus the highest common factor = \[ 2 \times 5 = 10. \]

Example 10

Find the highest common factor of 528 and 3168.

**Solution**

\[ 528 = 2^4 \times 3 \times 11 \] and \[ 3168 = 2^5 \times 3^2 \times 11 \]

\[ \therefore \text{highest common factor} = 2^4 \times 3 \times 11 = 528 \]

Example 11

Find the highest common factor of 3696 and 3744.

**Solution**

\[ 3696 = 2^4 \times 3 \times 7 \times 11 \] and \[ 3744 = 2^5 \times 3^2 \times 13 \]

\[ \therefore \text{highest common factor} = 2^4 \times 3 = 48 \]

Using the TI-Nspire

The calculator can be used to factor Natural numbers by using the `factor()` function from the Algebra menu as shown.

The highest common factor (also called greatest common divisor) of two numbers can be found using the command `gcd()` from the Number menu as shown.

Note how “nested” `gcd()` may be used to find the greatest common divisor of several numbers.
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Using the Casio ClassPad

To find the highest common factor, tap Interactive, Calculation, gcd and enter the required numbers in the two lines provided.

Exercise 3D

1 Give the prime decomposition of each of the following numbers.
   a 68 640  b 96 096  c 32 032  d 544 544

2 Find the highest common factor of each of the following pairs of numbers.
   a 4361, 9281  b 999, 2160  c 5255, 716 845
   d 1271, 3875  e 804, 2358

Note: Extended-answer questions 5, 6 and 7 are concerned with natural numbers.

3.5 Problems involving sets

Sets can be used to help sort information, as each of the following examples demonstrates.

Example 12

Two hundred and eighty students either travel by train or tram or both to get to school. One hundred and fifty travel by train and 60 travel by both train and tram.

a Show this information on a Venn diagram.

b Hence find the number of students who travel by
   i tram
   ii train but not tram
   iii just one of these modes of transport.
Example 13
An athletics team consists of 18 members. Each member performs in at least one of three events, sprints (S), jumps (J) and hurdles (H). Every hurdler either jumps or sprints. Also the following information is available.
\[ n(S) = 11, \quad n(J) = 10, \quad n(J \cap H' \cap S') = 5, \quad n(J' \cap H' \cap S) = 5 \quad \text{and} \quad n(J \cap H') = 7 \]

a) Draw a Venn diagram.
b) Find
   i) \( n(H) \)
   ii) \( n(S \cap H \cap J) \)
   iii) \( n(S \cup J) \)
   iv) \( n(S \cap J \cap H') \)

Solution

a)
\[ n(\xi) = 280 \]

b) i) \( n(\text{TRAM}) = 130 + 60 = 190 \)
   ii) \( n(\text{TRAIN} \cap (\text{TRAM})') = 90 \)
   iii) \( n(\text{TRAIN} \cap (\text{TRAM})') + n((\text{TRAIN})' \cap \text{TRAM}) = 90 + 130 = 220 \)

The information given above can be summarised as equations in terms of \( p, q, r, w, x, y, z \).
\[ x + y + z + w = 11 \quad \text{as} \quad n(S) = 11 \]  
\[ p + q + z + w = 10 \quad \text{as} \quad n(J) = 10 \]  
\[ x + y + z + w + p + q + r = 18 \quad \text{as} \quad \text{all members compete} \]  
\[ p = 5 \quad \text{as} \quad n(J' \cap H' \cap S') = 5 \]  
\[ s = 5 \quad \text{as} \quad n(J' \cap H' \cap S) = 5 \]  
\[ r = 0 \quad \text{as} \quad \text{every hurdler either jumps or sprints} \]  
\[ w + p = 7 \quad \text{as} \quad n(J \cap H') = 7 \]  

From \( [4] \) and \( [7] \), \( w = 2 \)

Equation \( [3] \) becomes
\[ 5 + y + z + 2 + 5 + q = 18 \]
i.e. \[ y + z + q = 6 \]
Equation 1 becomes
\[ y + z = 4 \]
Therefore from 8,
\[ q = 2 \]
Equation 2 becomes
\[ 5 + 2 + z + 2 = 10 \]
\[ \therefore z = 1 \]
and
\[ \therefore y = 3 \]
The Venn diagram can be completed.

```
\begin{tabular}{|c|c|c|c|}
\hline
S & 5 & 2 & 3 \\
\hline
H & 1 & 2 & \\
\hline
J & & & 5 \\
\hline
\end{tabular}
```

Exercise 3E

1. There are 28 students in a class all of whom take History or Economics or both. Fourteen take History, five of whom also take Economics.
   a. Show this information on a Venn diagram.
   b. Hence find the number of students who take
      i. Economics
      ii. History but not Economics
      iii. just one of these subjects.

2. a. Draw a Venn diagram to show three sets, A, B and C in a universal set \( \xi \). Enter numbers in the correct parts of the diagram using the following information.
   \[ n(A \cap B \cap C) = 2, n(A \cap B) = 7, n(B \cap C) = 6, n(A \cap C) = 8, n(A) = 16, n(B) = 20, n(C) = 19, n(\xi) = 50 \]
   b. Use the diagram to find
      i. \( n(A' \cap C') \)
      ii. \( n(A \cup B') \)
      iii. \( n(A' \cap B \cap C') \)

3. In a border town in the Balkans, 60% of people speak Bulgarian, 40% speak Greek and 20% speak neither. What percentage of the town speak both Bulgarian and Greek?

4. A survey of a class of 40 students showed that 16 own at least one dog and 25 at least one cat. Six students had neither. How many students own both?
5 At an international conference there were 105 delegates. Seventy spoke English, 50 spoke French and 50 spoke Japanese. Twenty five spoke English and French, 15 spoke French and Japanese and 30 spoke Japanese and English.

a How many delegates spoke all three languages?

b How many spoke Japanese only?

6 A restaurant serves 350 people lunches. It offers three desserts, profiteroles, gelati and fruit. It is found that 40 people have all three desserts, 70 have gelati only, 50 have profiteroles only and 60 have fruit only. Forty five people have fruit and gelati only, 30 people have gelati and profiteroles only and 10 people have fruit and profiteroles only. How many people do not have a dessert?

7 Forty travellers were questioned about the various methods of transport they had used the previous day. Every traveller questioned travelled by at least one of the following: car (C), bus (B), train (T).

Of those questioned, eight had used all three methods of transport.
Four had travelled by bus and car only.
Two had travelled by car and train only.
The number (x) who had travelled by train only was equal to the number who had travelled by bus and train only.

If 20 travellers had used a train and 33 had used a bus, find

a the value of x
b the number who travelled by bus only
c the number who travelled by car only.

8 \( \xi \) is the set of integers and

\[ X = \{ x : 21 < x < 37 \} \]
\[ Y = \{ y : 0 < y \leq 13 \} \]
\[ Z = \{ z^2 : 0 < z < 8 \} \]

a Draw a Venn diagram representing the information.

b Find

i \( X \cap Y \cap Z \)
ii \( n(X \cap Y) \)

9 A number of students bought red, green and black pens. Three bought one of each colour. Of the students who bought two colours, three did not buy red, five not green and two not black. The same number of students bought red only as bought red with other colours. The same number bought black only as bought green only. More students bought red and black but not green than bought black only. More bought only green than bought green and black but not red. How many students were there and how many pens of each colour were sold?
10 For three subsets, $B$, $M$ and $F$ of a universal set $\xi$

\[
\begin{align*}
n(B \cap M) &= 12, \quad n(M \cap F \cap B) = n(F') , \quad n(F \cap B) > n(M \cap F), \\
n(B \cap F' \cap M') &= 5, \quad n(M \cap B' \cap F') = 5, \quad n(F \cap M' \cap B') = 5, \\
n(\xi) &= 28
\end{align*}
\]

Find $n(M \cap F)$.

11 A group of 80 students were interviewed about which sport they played. It was found that 23 did athletics, 22 swim and 18 play football. If 10 people do athletics and swim only and 11 people do athletics and play football only, six people both swim and play football only and 46 people do none of these activities on a regular basis, how many people do all three?

12 At a certain secondary college students have to be proficient in at least one of the languages Italian, French or German. In a particular group of 33 students, two are proficient in all three languages, three in Italian and French only, four in French and German only and five in German and Italian only. The number proficient in Italian only is $x$, in French only is $x$ and in German only is $x + 1$. Find $x$ and the total number proficient in Italian.

13 Two hundred and one students at a certain school studied one or more of Mathematics, Physics and Chemistry. 35 took Chemistry only, 50% more students studied Mathematics only than studied Physics only, four studied all three subjects, 25 studied both Mathematics and Physics but not Chemistry, seven studied both Mathematics and Chemistry but not Physics and 20 studied both Physics and Chemistry but not Mathematics. Find the number of students studying Mathematics.
Chapter summary

- **Set notation**
  - \( \in \) is a member of
  - \( \notin \) is not a member of
  - \( \emptyset \) the empty set
  - \( \xi \) the universal set
  - \( \subseteq \) subset

- **The union of two sets**
The set of elements that are in either set \( A \) or set \( B \) (or both) is the union of set \( A \) and \( B \). The union of \( A \) and \( B \) is written \( A \cup B \).

- **The intersection of two sets**
The set of all the elements that are members both of set \( A \) and of set \( B \) is called the intersection of \( A \) and \( B \). The intersection of \( A \) and \( B \) is written \( A \cap B \).

- **The complement of \( A \)**, written \( A' \), is the set of all members of \( \xi \) that are not members of \( A \).

- **Sets of numbers**
  - \( R \) denotes the set of real numbers
  - \( Q \) denotes the set of rational numbers
  - \( Z \) denotes the set of integers
  - \( N \) denotes the set of natural numbers

  **Note:** \( N \subseteq Z \subseteq Q \subseteq R \)

- Numbers of the form \( \frac{m}{n} \), where \( m, n \in N \), have a terminating decimal representation if, and only if, \( n = 2^a5^b \) where \( a, b \) are members of the set \( N \cup \{0\} \).

- **Algebraic numbers** are those which are the solution(s) of an equation of the form
  \[ a_0x^n + a_1x^{n-1} + \cdots + a_n = 0 \]
  where \( a_0, a_1, \ldots, a_n \) are integers.

- A number of the form \( \sqrt{a} \) where \( a \) is a rational number which is not a square of another rational number is called a quadratic surd.

- If \( a \) is a rational which is not a perfect \( n \)th power \( \sqrt[n]{a} \) is called a **surd of the \( n \)th order**.

- When the number under the square root has no factors which are squares of a rational number, then the surd is said to be in **simplest form**.

- Surds which have the same ‘irrational factor’ are called **like surds**.

- The sum or difference of two like surds can be found
  \[ m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p} \quad \text{and} \quad m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p} \]

- A natural number, \( a \), is a **factor** of a natural number, \( b \), if there exists a natural number, \( k \), such that \( b = ak \).

- If a natural number greater than 1 has only factors 1 and itself, it is said to be **prime**.

- A natural number, \( m \), is called a **composite** if it can be written as a product \( m = a \times b \) where \( a \) and \( b \) are natural numbers greater than 1 and less than \( m \).

- The **highest common factor** of two natural numbers is the largest natural number which is a factor of both numbers.
Multiple-choice questions

1. \( \frac{4}{3 + 2\sqrt{2}} \) expressed in the form \( a + b\sqrt{2} \) is
   \( \text{A} \; 12 - 8\sqrt{2} \quad \text{B} \; 3 + 2\sqrt{2} \quad \text{C} \; \frac{3}{17} - \frac{8}{17}\sqrt{2} \quad \text{D} \; \frac{3}{17} + \frac{8}{17}\sqrt{2} \quad \text{E} \; 12 + 8\sqrt{2} \)

2. The prime decomposition of 86400 is
   \( \text{A} \; 2^5 \times 3^2 \times 5 \quad \text{B} \; 2^6 \times 3^3 \times 5^2 \quad \text{C} \; 2^7 \times 3^3 \times 5^3 \)
   \( \text{D} \; 2^7 \times 3^3 \times 5^2 \quad \text{E} \; 2^6 \times 3^3 \times 5^3 \)

3. \((\sqrt{6} + 3)(\sqrt{6} - 3)\) is equal to
   \( \text{A} \; 3 - 12\sqrt{6} \quad \text{B} \; -3 - 6\sqrt{6} \quad \text{C} \; -3 + 6\sqrt{6} \quad \text{D} \; -3 \quad \text{E} \; 3 \)

4. For the Venn diagram shown, \( \xi \) is the set of positive integers less than 20. \( A \) is the set of positive integers less than 10 and \( B \) is the set of positive integers less than 20 divisible by 3. The set \( B' \cap A \) is
   \( \text{A} \; \{6, 3, 9\} \quad \text{B} \; \{12, 15, 18\} \quad \text{C} \; \{10, 11, 13, 14, 16, 17, 19\} \quad \text{D} \; \{1, 2, 4, 5, 7, 8\} \quad \text{E} \; \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 18\} \)

5. \((3, \infty) \cap (-\infty, 5]\) =
   \( \text{A} \; (-\infty, 3] \quad \text{B} \; (-\infty, 5] \quad \text{C} \; (3, 5] \quad \text{D} \; \text{R} \quad \text{E} \; [3, 5] \)

6. A bell is rung every 6 minutes and a gong is sounded every 14 minutes. If these occur together at a particular time then the smallest number of minutes until the bell and the gong are again heard simultaneously is
   \( \text{A} \; 10 \quad \text{B} \; 20 \quad \text{C} \; 72 \quad \text{D} \; 42 \quad \text{E} \; 84 \)

7. If \( X \) is the set of multiples of 2, \( Y \) the set of multiples of 7 and \( Z \) the set of multiples of 5 then describe \( X \cap Y \cap Z = \)
   \( \text{A} \; \text{the set of multiples of } 2 \quad \text{B} \; \text{the set of multiples of } 70 \quad \text{C} \; \text{the set of multiples of } 35 \quad \text{D} \; \text{the set of multiples of } 14 \quad \text{E} \; \text{the set of multiples of } 10 \)

8. In a class of students, 50% play football, 40% play tennis and 30% play neither. The percentage that plays both is
   \( \text{A} \; 10 \quad \text{B} \; 20 \quad \text{C} \; 30 \quad \text{D} \; 50 \quad \text{E} \; 40 \)

9. \( \sqrt{7} - \sqrt{6} = \)
   \( \text{A} \; 5 + 2\sqrt{7} \quad \text{B} \; 13 + 2\sqrt{6} \quad \text{C} \; 13 - 2\sqrt{42} \quad \text{D} \; 1 + 2\sqrt{42} \quad \text{E} \; 13 - 2\sqrt{13} \)

10. There are 40 students in a class, all of whom take Literature or Economics or both. Twenty take Literature and five of these also take Economics. The number of students who only take Economics is
    \( \text{A} \; 20 \quad \text{B} \; 5 \quad \text{C} \; 10 \quad \text{D} \; 15 \quad \text{E} \; 25 \)
1. Express the following as fractions in their simplest form.
   a. 0.07
   b. 0.45
   c. 0.005
   d. 0.405
   e. 0.26
   f. 0.1714285

2. Express 504 as a product of powers of prime numbers.

3. Express each of the following with a rational denominator.
   a. \[\frac{2\sqrt{3} - 1}{\sqrt{2}}\]
   b. \[\frac{\sqrt{5} + 2}{\sqrt{5} - 2}\]
   c. \[\frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}\]

4. Express \[\frac{3 + 2\sqrt{75}}{3 - \sqrt{12}}\] in the form \(a + b\sqrt{3}\) where \(a, b \in R\setminus\{0\}\).

5. Express each of the following with a rational denominator.
   a. \[\frac{6\sqrt{2}}{3\sqrt{2} - 2\sqrt{3}}\]
   b. \[\frac{\sqrt{a + b} - \sqrt{a - b}}{\sqrt{a + b} + \sqrt{a - b}}\]

6. In a class of 100 students, 55 are girls, 45 have blue eyes, 40 are blond, 25 are blond girls, 15 are blue-eyed blonds, 20 are blue-eyed girls, and 5 are blue-eyed blond girls. Find
   a. the number of blond boys
   b. the number of boys who are not blond or blue-eyed.

7. A group of 30 students received prizes in at least one of the subjects of English, Mathematics, and French. Two students received prizes in all three subjects. Fourteen received prizes in English and Mathematics but not French. Two received prizes in English alone, two in French alone and five in Mathematics alone. Four received prizes in English and French but not Mathematics.
   a. How many received prizes in Mathematics and French but not English?
   b. How many received prizes in Mathematics?
   c. How many received prizes in English?

8. Fifty people are interviewed. Twenty-three say they like Brand X, 25 say they like Brand Y, 19 say they like Brand Z. Eleven say they like X and Z. Eight say they like Y and Z. Five say they like X and Y. Two like all three. How many like none of them?

9. Three rectangles \(A, B\) and \(C\) overlap (intersect). Their areas are 20 cm\(^2\), 10 cm\(^2\) and 16 cm\(^2\) respectively. The area common to \(A\) and \(B\) is 3 cm\(^2\), that common to \(A\) and \(C\) is 6 cm\(^2\), that common to \(B\) and \(C\) is 4 cm\(^2\). How much of the area is common to all three if the total area covered is 35 cm\(^2\)?

10. Express \[\sqrt{112} - \sqrt{63} - \frac{224}{\sqrt{28}}\] in simplest form.

11. If \[\frac{\sqrt{7} - \sqrt{3}}{x} = \frac{x}{\sqrt{7} + \sqrt{3}}\], find the values of \(x\).

12. Express \[\frac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} + \frac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}}\] in the form \(a\sqrt{5} + b\sqrt{6}\).

13. Simplify \[\sqrt{27} - \sqrt{12} + 2\sqrt{75} - \sqrt{\frac{48}{25}}\].
14. A, B and C are three sets and \( \xi = A \cup B \cup C \). The number of elements in the regions of the Venn diagram are as shown. Find

a. the number of elements in \( A \cup B \)
b. the number of elements in \( C \)
c. the number of elements in \( B' \cap A \).

15. Using the result that \((\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}\), determine the square root of \(17 + 6\sqrt{8}\).

Extended-response questions

1. a. Show that \((\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}\).
b. Substitute \(x = 3\) and \(y = 5\) in the identity of a to show \(\sqrt{3} + \sqrt{5} = \sqrt{8} + 2\sqrt{15}\).
c. Use this technique to find the square root of:
   i. \(14 + 2\sqrt{33}\) (Hint: use \(x = 11\) and \(y = 3\))
   ii. \(15 - 2\sqrt{56}\)
   iii. \(51 - 36\sqrt{2}\)

2. In this question, \(\{a + b\sqrt{3} : a, b \in \mathbb{Q}\}\) is considered. Later in this book the set, \(C\), of complex numbers is introduced, where \(C = \{a + b\sqrt{-1} : a, b \in \mathbb{R}\}\).
   a. If \((2 + 3\sqrt{3}) + (4 + 2\sqrt{3}) = a + b\sqrt{3}\), find \(a\) and \(b\).
   b. If \((2 + 3\sqrt{3})(4 + 2\sqrt{3}) = p + q\sqrt{3}\), find \(p\) and \(q\).
   c. If \(\frac{1}{\sqrt{3} + 2\sqrt{3}} = a + b\sqrt{3}\), find \(a\) and \(b\).
   d. Solve each of the following equations for \(x\):
      i. \((2 + 5\sqrt{3})x = 2 - \sqrt{3}\)
      ii. \((x - 3)^2 - 3 = 0\)
      iii. \((2x - 1)^2 - 3 = 0\)
   e. Explain why every rational number is a member of \(\{a + b\sqrt{3} : a, b \in \mathbb{Q}\}\).

3. a. Show that \(\frac{1}{\sqrt{3} + 2\sqrt{3}} = 2 - \sqrt{3}\).
   b. Use the substitution \(t = \left(\sqrt{2} + \sqrt{3}\right)^x\) and the result of a to show that the equation \((\sqrt{2} + \sqrt{3})^x + (\sqrt{2} - \sqrt{3})^x = 4\) can be written as \(t + \frac{1}{t} = 4\).
   c. Show that the solutions of the equation are \(t = 2 - \sqrt{3}\) and \(t = 2 + \sqrt{3}\).
   d. Use this result to solve the equation \((\sqrt{2} + \sqrt{3})^x + (\sqrt{2} - \sqrt{3})^x = 4\).

4. Use Venn diagrams to illustrate
   a. \(n(A \cup B) = n(A) + n(B) - n(A \cap B)\)
   b. \(n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C)\)\
      \(- n(A \cap C) + n(A \cap B \cap C)\)

5. The number \(2 - \sqrt{3}\) is a root of the quadratic equation with integer coefficients \(x^2 + bx + c = 0\).
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a If \( x = 2 - \sqrt{3} \) is a solution to the equation, find the values of \( b \) and \( c \).
(Hint: Use the result that if \( m + n\sqrt{3} = 0 \) then \( m = 0 \) and \( n = 0 \), \( m, n \) rational.)

b Find the other solution to the quadratic.

c If \( x^2 + bx + c = 0 \) and \( m - n\sqrt{q} \) is a solution, show that

\[
\begin{align*}
 i & \quad b = -2m \\
 ii & \quad c = m^2 - n^2q
\end{align*}
\]

and hence that

\[
iii \quad x^2 + bx + c = (x - (m - n\sqrt{q}))(x - (m + n\sqrt{q}))
\]

6 A triple \((x, y, z)\) is said to be Pythagorean if \( x^2 + y^2 = z^2 \), e.g. \((3, 4, 5)\) is a Pythagorean triple, \((5, 12, 13)\) is a Pythagorean triple.

All Pythagorean triples may be generated by the following:

\[
x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2 \text{ where } m, n \in \mathbb{N}
\]

e.g. if \( m = 2, n = 1 \) then \( x = 4, y = 3, z = 5 \)

a Find the Pythagorean triple for \( m = 5, n = 2 \).

b Verify that for \( x = 2mn, y = m^2 - n^2, z = m^2 + n^2 \) where \( m, n \in \mathbb{N}, x^2 + y^2 = z^2 \).

7 The factors of 12 are 1, 2, 3, 4, 6, 12.

a How many factors does each of the following have?

\[
i \quad 2^3 \\
 ii \quad 3^7
\]

b How many factors does \( 2^n \) have?

c How many factors does each of the following have?

\[
i \quad 2^3 \cdot 3^7 \quad ii \quad 2^n \cdot 3^n
\]

d Every natural number may be expressed as a product of powers and primes. This is called prime factorisation, e.g. \( 1080 = 2^3 \times 3 \times 5 \).

For \( x \), a natural number, let \( p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_n^{\alpha_n} \) be the prime factorisation where \( \alpha_i \in \mathbb{N} \) and each \( p_i \) is a prime number.

How many factors does \( x \) have? (Answer to be given in terms of \( \alpha_i \).)

e Find the smallest number which has eight factors.

8 The least common multiple of natural numbers \( m \) and \( n \) is the smallest natural number which is a multiple of both \( m \) and \( n \), e.g. the least common multiple of 4 and 6 is 12.

a Give the prime decomposition of 1080 and 25 200.

b In order to find the least common multiple of two numbers \( a \) and \( b \), take the prime decomposition of each of the numbers, i.e. \( a = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_n^{\alpha_n} \) and \( b = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \cdots p_m^{\beta_m} \) and then the least common multiple of \( a \) and \( b \)

\[
b = p_1^{\max(\alpha_1, \beta_1)} p_2^{\max(\alpha_2, \beta_2)} \cdots p_n^{\max(\alpha_n, \beta_n)} \cdots p_m^{\max(\alpha_m, \beta_m)}
\], where all primes in the prime decomposition of either \( a \) or \( b \) are included in this product, e.g. the least common multiple of 24 = \( 2^3 \times 3 \) and 18 = \( 3^2 \times 2 \) is \( 2^3 \times 3^2 = 72 \). Find the least common multiple of 1080 and 25 200.

c Carefully explain why if \( m \) and \( n \) are integers \( mn = \text{least common multiple of } m \) and \( n \times \text{highest common factor of } m \) and \( n \).
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Chapter 3 — Number systems and sets

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d
(i) Find four consecutive even numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

(ii) Find four consecutive natural numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

9

(a) In the Venn diagram $\xi$ is the set of all students enrolled at Argos Secondary College. Set $R$ is the set of all students with red hair. Set $B$ is the set of all students with blue eyes. Set $F$ is the set of all female students.

The numbers on the diagram indicate the eight different regions.

(i) Identify the region in the Venn diagram which represents male students who have neither red hair nor blue eyes.

(ii) Describe the gender, hair colour and eye colour of students represented in region 1 of the diagram.

(iii) Describe the gender, hair colour and eye colour of students represented in region 2 of the diagram.

(b) It is known that at Argos Secondary College, 250 of the students study French ($F$), Greek ($G$) or Japanese ($J$). Forty-one students do not study French. Twelve students study French and Japanese, but not Greek. Thirteen students study Japanese and Greek, but not French. Thirteen students study only Greek. Twice as many students study French and Greek but not Japanese as study all three. The number studying only Japanese is the same as the total of those studying both French and Greek.

(i) How many students study all three languages?

(ii) How many students study only French?

10

Consider the universal set as the set of all students enrolled at Sounion Secondary College. Let $B$ denote the set of students taller than 180 cm at Sounion Secondary College and $A$ denote the set of female students.

(a) Give a brief description of each of the following sets.

(i) $B'$

(ii) $A \cup B$

(iii) $A' \cap B'$

(b) Use a Venn diagram to show $(A \cup B)' = (A' \cap B')$.

(c) Hence show that $A \cup B \cup C = (A' \cap B' \cap C')'$ where $C$ is the set of students who play sport.
11 In a certain city three Sunday newspapers \((A, B \text{ and } C)\) are available. In a sample of 500 people from this city, it was found that

- nobody regularly reads both \(A\) and \(C\)
- a total of 100 people regularly read \(A\)
- 205 people regularly read only \(B\)
- of those who regularly read \(C\), exactly half of them also regularly read \(B\)
- 35 people regularly read \(A\) and \(B\), but not \(C\)
- 35 people don’t read any of the papers at all.

\(\text{a} \quad \) Draw a set diagram showing the number of regular readers for each possible combination of \(A, B\) and \(C\).

\(\text{b} \quad \) How many people in the sample were regular readers of \(C\)?

\(\text{c} \quad \) How many people in the sample regularly read \(A\) only?

\(\text{d} \quad \) How many people are regular readers of \(A, B\) and \(C\)?