CHAPTER 1

Matrices

Objectives

- To be able to identify when two matrices are **equal**
- To be able to add and subtract matrices of the **same dimensions**
- To be able to perform multiplication of a matrix and a **scalar**
- To be able to identify when the multiplication of two given matrices is possible
- To be able to perform **multiplication** on two suitable matrices
- To be able to find the **inverse** of a **2 × 2** matrix
- To be able to find the **determinant** of a matrix
- To be able to solve **linear simultaneous equations** in two unknowns using an inverse matrix

1.1 Introduction to matrices

A **matrix** is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.

The following are examples of matrices:

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

Matrices vary in size. The size, or **dimension**, of the matrix is described by specifying the number of rows (horizontal lines) and columns (vertical lines) that occur in the matrix.

The dimensions of the above matrices are, in order:

$$3 \times 2, \quad 1 \times 4, \quad 3 \times 3, \quad 1 \times 1.$$

The first number represents the number of rows and the second, the number of columns.

Example 1

Write down the dimensions of the following matrices.

a $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$

Solution

a 2×3 **b** 4×1 **c** 1×3

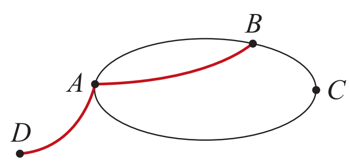
The use of matrices to store information is demonstrated by the following two examples.

Four exporters A, B, C and D sell televisions (t), CD players (c), refrigerators (r) and washing machines (w). The sales in a particular month can be represented by a 4×4 array of numbers. This array of numbers is called a matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} r & c & w & t \end{array} \\ \begin{bmatrix} 120 & 95 & 370 & 250 \\ 430 & 380 & 1000 & 900 \\ 60 & 50 & 150 & 100 \\ 200 & 100 & 470 & 50 \end{bmatrix} \\ \text{column 1} \quad \text{column 2} \quad \text{column 3} \quad \text{column 4} \end{array}\begin{array}{l} \\ \text{row 1} \\ \text{row 2} \\ \text{row 3} \\ \text{row 4} \\ \\ \end{array}$$

From the matrix it can be seen:
Exporter A sold 120 refrigerators, 95 CD players, 370 washing machines and 250 televisions.
Exporter B sold 430 refrigerators, 380 CD players, 1000 washing machines and 900 televisions.
The entries for the sales of refrigerators are made in column 1.
The entries for the sales of exporter A are made in row 1.

The diagram on the right represents a section of a road map. The number of direct connecting roads between towns can be represented in matrix form.



$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

If $\mathbf{A}$ is a matrix, a_{ij} will be used to denote the entry that occurs in row i and column j of $\mathbf{A}$. Thus a 3×4 matrix may be written

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

For **B**, an $m \times n$ matrix

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \; \ldots\ldots & b_{1n} \\ b_{21} & b_{22} \; \ldots\ldots & b_{2n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{m1} & b_{m2} \; \ldots\ldots & b_{mn} \end{bmatrix}$$

Matrices provide a format for the storage of data. In this form the data is easily operated on. Some graphics calculators have a built-in facility to operate on matrices and there are computer packages which allow the manipulation of data in matrix form.

A car dealer sells three models of a certain make and his business operates through two showrooms. Each month he summarises the number of each model sold by a sales matrix **S**:

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \end{bmatrix}$$, where s_{ij} is the number of cars of model j sold by showroom i.

So, for example, s_{12} is the number of sales made by showroom 1, of model 2.

If in January, showroom 1 sold three, six and two cars of models 1, 2 and 3 respectively, and showroom 2 sold four, two and one car(s) of models 1, 2 and 3 (in that order), the sales matrix for January would be:

$$\mathbf{S} = \begin{bmatrix} 3 & 6 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

A matrix is, then, a way of recording a set of numbers, arranged in a particular way. As in Cartesian coordinates, the order of the numbers is significant, so that although the matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

have the same numbers and the same number of elements, they are different matrices (just as (2, 1), (1, 2) are coordinates of different points).

Two matrices **A**, **B**, are **equal**, and can be written as **A** = **B** when

- each has the same number of rows and the same number of columns
- they have the same number or element at corresponding positions.

e.g. $$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 1 & -1 \\ 1-1 & 1 & \frac{6}{2} \end{bmatrix}$$

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Example 2

If matrices **A** and **B** are equal, find the values of x and y.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ x & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -3 & y \end{bmatrix}$$

Solution

$x = -3$ and $y = 4$

Although a matrix is made from a set of numbers, it is important to think of a matrix as a single entity, somewhat like a 'super number'.

Example 3

There are four rows of seats of three seats each in a minibus. If 0 is used to indicate a seat is vacant and 1 is used to indicate a seat is occupied, write down a matrix that represents

a the 1st and 3rd rows are occupied but the 2nd and 4th rows are vacant

b only the seat on the front left corner of the bus is occupied.

Solution

a $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Example 4

There are four clubs in a local football league.

Team A has 2 senior teams and 3 junior teams
Team B has 2 senior teams and 4 junior teams
Team C has 1 senior team and 2 junior teams
Team D has 3 senior teams and 3 junior teams

Represent this information in a matrix.

Solution

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$$

Note: rows represent teams A, B, C, D and columns represent the number of senior and junior teams respectively.

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Exercise 1A

Example 1

1 Write down the dimensions of the following matrices.

a $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ **c** $[a \quad b \quad c \quad d]$ **d** $\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$

Example 3

2 There are 25 seats arranged in five rows and five columns. If 0, 1 respectively are used to indicate whether a seat is vacant or occupied, write down a matrix which represents the situation when

a only seats on the two diagonals are occupied
b all seats are occupied.

3 If seating arrangements (as in **2**) are represented by matrices, consider the matrix in which the i, j element is 1 if $i = j$, but 0 if $i \neq j$. What seating arrangement does this matrix represent?

Example 4

4 At a certain school there are 200 girls and 110 boys in Year 7, 180 girls and 117 boys in Year 8, 135 and 98 respectively in Year 9, 110 and 89 in Year 10, 56 and 53 in Year 11 and 28 and 33 in Year 12. Summarise this information in matrix form.

Example 2

5 From the following, select those pairs of matrices which could be equal, and write down the values of x, y which would make them equal.

a $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix}, [0 \quad x], [0 \quad 4]$

b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}, \begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}, [4 \quad x \quad 1 \quad -2]$

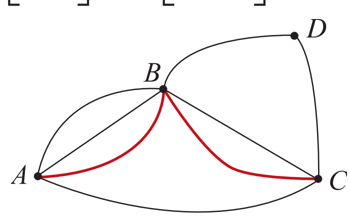
c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix}, \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$

6 In each of the following find the values of the pronumerals so that matrices **A** and **B** are equal.

a $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} x & 1 & -1 \\ 0 & 1 & y \end{bmatrix}$ **b** $\mathbf{A} = \begin{bmatrix} x \\ 2 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 3 \\ y \end{bmatrix}$

c $\mathbf{A} = [-3 \quad x]$ $\mathbf{B} = [y \quad 4]$ **d** $\mathbf{A} = \begin{bmatrix} 1 & y \\ 4 & 3 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$

7 A section of a road map connecting towns A, B, C and D is shown. Construct the 4×4 matrix which shows the number of connecting roads between each pair of towns.



8 The statistics for the five members of a basketball team are recorded as follows.
Player A: points 21, rebounds 5, assists 5
Player B: points 8, rebounds 2, assists 3
Player C: points 4, rebounds 1, assists 1
Player D: points 14, rebounds 8, assists 60
Player E: points 0, rebounds 1, assists 2
Express this data in a 5×3 matrix.

1.2 Addition, subtraction and multiplication by a scalar

Addition will be defined for two matrices **only** when they have the same number of rows and the same number of columns. In this case the sum of two matrices is found by adding corresponding elements. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

Subtraction is defined in a similar way. When the two matrices have the same number of rows and the same number of columns the difference is found by subtracting corresponding elements.

Example 5

Find

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

Solution

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & -1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

It is useful to define **multiplication of a matrix by a real number**. If **A** is an $m \times n$ matrix, and k is a real number, then k**A** is an $m \times n$ matrix whose elements are k times the corresponding elements of **A**. Thus

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

These definitions have the helpful consequence that if a matrix is added to itself, the result is twice the matrix, i.e. $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$. Similarly the sum of n matrices each equal to **A** is $n\,\mathbf{A}$ (where n is a natural number).
The $m \times n$ matrix with all elements equal to zero is called the **zero matrix**.

Example 6

Let $\mathbf{X} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$, $-3\mathbf{A} + \mathbf{B}$.

Solution

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$2\mathbf{X} = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4\mathbf{Y} + \mathbf{X} = 4\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$-3\mathbf{A} = -3\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix}$$

$$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

Example 7

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, find matrices $\mathbf{X}$ such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Solution

If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$

$$\therefore \mathbf{X} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2 \times \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times -1 & 8 - 2 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix}$$

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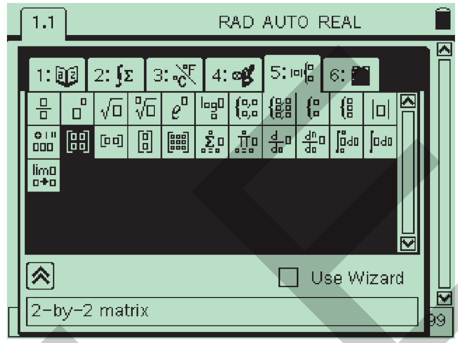
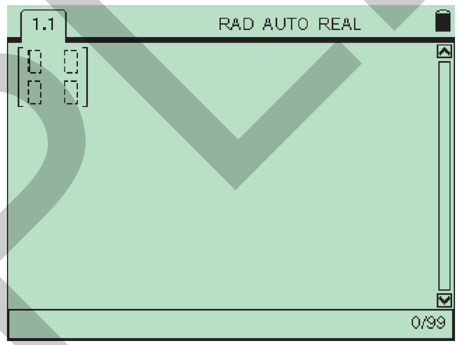
Using the TI-Nspire

2-by-2 matrices are easiest entered using the 2-by-2 matrix template (template key 5) as shown.

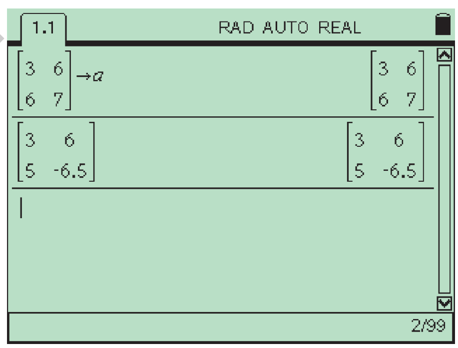
Notice that there is also a template for entering *m*-by-*n* matrices.





To enter the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$, use the NavPad to move between the entries of the 2-by-2 matrix template and **store** (ctrl sto→var) the matrix as *a*.

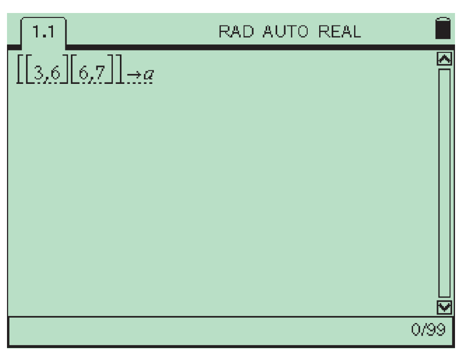
Define the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$ in a similar way.



Entering matrices directly

To enter matrix **A** without using the template, enter the matrix row by row as [[3,6][6,7]] and **store** (ctrl sto→var) the matrix as *a*.

Alternatively, enter the matrix by typing [3,6;6,7]. Semicolon (;) can be obtained by typing ctrl ⋮.



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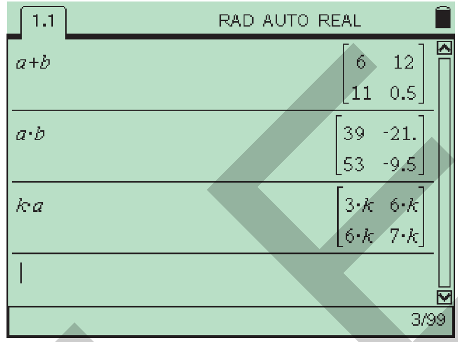
Addition, multiplication and multiplication by a scalar

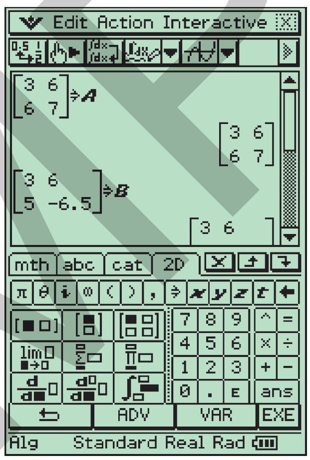
Once **A** and **B** are defined as above, **A** + **B**, **AB** and *k***A** can easily be determined.

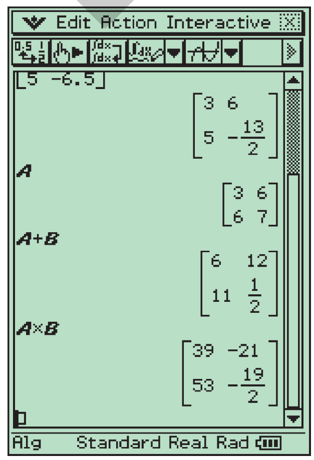


Using the Casio ClassPad

Matrices are entered using the 2D CALC menu on the (Keyboard). Tap [matrix icon], enter the numbers required then store this as a variable (using VAR).

Calculations can be performed as shown in the screen at the far right.





Exercise 1B

Example 6

1 Let $\mathbf{X} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $-3\mathbf{A} + \mathbf{B}$.

2 Each showroom of a car dealer sells exactly twice as many cars of each model in February as in January. (See example in section 1.1.)

a Given that the sales matrix for January is $\begin{bmatrix} 3 & 6 & 2 \\ 4 & 2 & 1 \end{bmatrix}$, write down the sales matrix for February.

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b If the sales matrices for January and March (with twice as many cars of each model sold in February as January) had been $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 0 \\ 6 & 1 & 4 \end{bmatrix}$ respectively, find the sales matrix for the first quarter of the year.

c Find a matrix to represent the average monthly sales for the first three months.

3 Let $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

Find $2\mathbf{A}$, $-3\mathbf{A}$ and $-6\mathbf{A}$.

4 **A**, **B**, **C** are $m \times n$ matrices. Is it true that

a $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ **b** $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$?

5 $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix}$

Calculate

a $2\mathbf{A}$ **b** $3\mathbf{B}$ **c** $2\mathbf{A} + 3\mathbf{B}$ **d** $3\mathbf{B} - 2\mathbf{A}$

6 $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$

Calculate

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} + 3\mathbf{Q}$ **c** $2\mathbf{P} - \mathbf{Q} + \mathbf{R}$

Example 7

7 If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -10 \\ -2 & 17 \end{bmatrix}$, find matrices **X** and **Y** such that $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$ and $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$.

8 Matrices **X** and **Y** show the production of four models a, b, c, d at two automobile factories P, Q in successive weeks.

$$\mathbf{X} = \begin{matrix} P \\ Q \end{matrix} \overset{\begin{matrix} a & \;\; b & \;\; c & \;\; d \end{matrix}}{\begin{bmatrix} 150 & 90 & 100 & 50 \\ 100 & 0 & 75 & 0 \end{bmatrix}} \qquad \mathbf{Y} = \begin{matrix} P \\ Q \end{matrix} \overset{\begin{matrix} a & \;\; b & \;\; c & \;\; d \end{matrix}}{\begin{bmatrix} 160 & 90 & 120 & 40 \\ 100 & 0 & 50 & 0 \end{bmatrix}}$$

week 1 week 2

Find $\mathbf{X} + \mathbf{Y}$ and write what this sum represents.

1.3 Multiplication of matrices

Multiplication of a matrix by a real number has been discussed in the previous section. The definition for multiplication of matrices is less natural. The procedure for multiplying two 2×2 matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$

Then
$$\mathbf{AB} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix}$$
$$= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix}$$

and
$$\mathbf{BA} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.

If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows.

To find the entry in row i and column j of **AB** single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.

Example 8

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ find **AB**.

Solution

A is a 2×2 matrix and **B** is a 2×1 matrix. Therefore **AB** is defined.

The matrix **AB** is a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

Example 9

Matrix **X** shows the number of cars of models a and b bought by four dealers, A, B, C and D.
Matrix **Y** shows the cost in dollars of model a and model b.
Find **XY** and explain what it represents.

$$\mathbf{X} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \qquad \mathbf{Y} = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

Solution

$$\mathbf{XY} = \begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

$$4 \times 2 \qquad\qquad 2 \times 1$$

The matrix **XY** is a 4×1 matrix

$$\mathbf{XY} = \begin{bmatrix} 3 \times 26\,000 + 1 \times 32\,000 \\ 2 \times 26\,000 + 2 \times 32\,000 \\ 1 \times 26\,000 + 4 \times 32\,000 \\ 1 \times 26\,000 + 1 \times 32\,000 \end{bmatrix} = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix}$$

The matrix **XY** shows dealer A spent \$110 000, dealer B spent \$116 000, dealer C spent \$154 000 and dealer D spent \$58 000.

Example 10

For $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$ find **AB**.

Solution

A is a 2×3 matrix and **B** is a 3×2 matrix. Therefore **AB** is a 2×2 matrix.

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 4 + 3 \times 1 + 4 \times 0 & 2 \times 0 + 3 \times 2 + 4 \times 3 \\ 5 \times 4 + 6 \times 1 + 7 \times 0 & 5 \times 0 + 6 \times 2 + 7 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 18 \\ 26 & 33 \end{bmatrix} \end{aligned}$$

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Skillsheet TEACHER

Exercise 1C

Examples 8,10

1 If $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
find the products **AX**, **BX**, **AY**, **IX**, **AC**, **CA**, (**AC**)**X**, **C**(**BX**), **AI**, **IB**, **AB**, **BA**, $\mathbf{A}^2$, $\mathbf{B}^2$, **A**(**CA**) and $\mathbf{A}^2\mathbf{C}$.

2 a Are the following products, of matrices given in **1**, defined?

AY, **YA**, **XY**, $\mathbf{X}^2$, **CI**, **XI**

b If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$, find **AB**.

3 The matrices **A** and **B** are 2×2 matrices, and **O** is the zero 2×2 matrix. Is the following argument correct?
'If $\mathbf{AB} = \mathbf{O}$, and $\mathbf{A} \neq \mathbf{O}$, then $\mathbf{B} = \mathbf{O}$'.

4 If $\mathbf{L} = [2 \quad -1]$, $\mathbf{X} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find **LX** and **XL**.

5 **A** and **B** are both $m \times n$ matrices. Are **AB** and **BA** defined and, if so, how many rows and columns do they have?

6 Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Show that $ad - bc = 1$. What is the product matrix if the order of multiplication on the left-hand side is reversed?

7 Using the result of **6**, write down a pair of matrices **A**, **B** such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

8 Select any three 2×2 matrices **A**, **B** and **C**.
Calculate $\mathbf{A}(\mathbf{B} + \mathbf{C})$, $\mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\,\mathbf{A}$.

Example 9

9 It takes John five minutes to drink a milk shake which costs \$2.50, and twelve minutes to eat a banana split which costs \$3.00.
Calculate the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in milk bar economics.
Suppose two friends join John.
Calculate $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.

10 The reading habits of five students A, B, C, D and E are shown in the first matrix below where the columns p, q, r, and s represent four weekly magazines. The second matrix shows the cost in dollars of each magazine. Find the product of the two matrices and interpret the result.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} p \\ q \\ r \\ s \end{array}\begin{bmatrix} 2.00 \\ 3.00 \\ 2.50 \\ 3.50 \end{bmatrix}$$

11 Let $\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \end{bmatrix}$ be the sales matrix for two showrooms selling three models of cars. Here s_{ij} is the number of cars of model j sold from showroom i. Let the prices of the three models of cars be $\$c_1$, $\$c_2$, $\$c_3$.

Call the 3×1 matrix, $\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ the price matrix.

a Find **SC**. **b** What is the practical meaning of **SC**?

c Suppose the car dealer sells both new and used cars and the price of two-year-old used cars for the three models is $\$u_1$, $\$u_2$ and $\$u_3$, respectively.
Form a new cost matrix

$$\mathbf{C} = \begin{bmatrix} c_1 & u_1 \\ c_2 & u_2 \\ c_3 & u_3 \end{bmatrix}$$

Find **SC** and state its meaning.

d Suppose the car dealer makes 30% profit on his selling of new cars and 25% on used cars.
If $\mathbf{V} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.25 \end{bmatrix}$, what is the meaning of **CV**?

1.4 Identities, inverses and determinants for 2 × 2 matrices

Identities

A matrix with the same number of rows and columns is called a square matrix. For square matrices of a given dimension, e.g. 2×2, a multiplicative identity $\mathbf{I}$ exists.

For example, for 2×2 matrices $\quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and for 3×3 matrices $\quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

If $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$, and this result holds for any square matrix multiplied by the appropriate multiplicative identity.

Inverses

Given a 2×2 matrix $\mathbf{A}$, is there a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$?

Let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Then $\mathbf{AB} = \mathbf{I}$ implies $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. $\begin{bmatrix} 2x + 3u & 2y + 3v \\ x + 4u & y + 4v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore 2x + 3u = 1 \quad \text{and} \quad 2y + 3v = 0$$
$$x + 4u = 0 \qquad\qquad y + 4v = 1$$

These simultaneous equations can be solved to find x, u, y, and v and hence $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

$\mathbf{B}$ is said to be the **inverse** of $\mathbf{A}$ as $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

Let $\mathbf{A}$ be a 2×2 matrix with $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$ where $\mathbf{B}$ is the inverse of $\mathbf{A}$.

Then $\mathbf{AB} = \mathbf{I}$. In full this is written $\begin{bmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence $ax + bu = 1 \quad ay + bv = 0$
$\qquad cx + du = 0 \quad cy + dv = 1$

which form two pairs of simultaneous equations, for x, u and y, v respectively.
Taking the x, u pair and eliminating u, $(ad - bc)x = d$
Similarly, eliminating x, $(bc - ad)u = c$
These two equations can be solved for x and u respectively provided $ad - bc \neq 0$

i.e. $x = \dfrac{d}{ad - bc}$ and $u = \dfrac{c}{cb - ad} = \dfrac{-c}{ad - bc}$

In a similar way it can be found that

$$y = \frac{-b}{ad - bc} \text{ and } v = \frac{-a}{cb - ad} = \frac{a}{ad - bc}$$

Therefore the inverse $= \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$

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The inverse of a square matrix **A**, is denoted by $\mathbf{A}^{-1}$. The inverse is unique. $ad - bc$ has a name, the **determinant** of **A**. This is denoted det(**A**).

i.e. for $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(\mathbf{A}) = ad - bc$

A 2×2 matrix has an inverse only if $\det(\mathbf{A}) \neq 0$

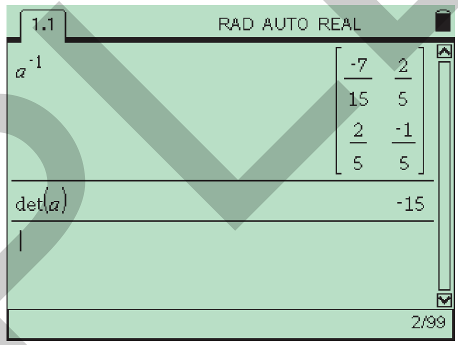
A square matrix is said to be **regular** if its inverse exists. Those square matrices which do not have an inverse are called **singular** matrices; i.e. for a **singular** matrix $\det(\mathbf{A}) = 0$.

Using the TI-Nspire

The operation of matrix inverse is obtained by raising the matrix to the power of −1.

The **Determinant** command is found in the **Matrix and Vector** menu (menu 7 2) and used as shown.

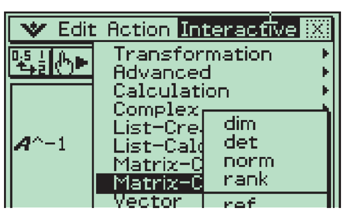
(a is the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$, defined on page 8.)

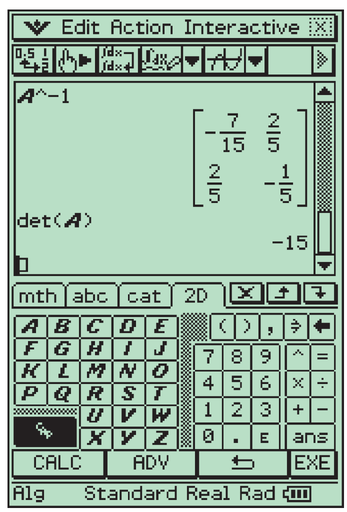


Using the Casio ClassPad

The operation of matrix inverse is obtained by entering A^−1 in the entry line.

The determinant is obtained by entering and highlighting *A* and tapping **Interactive, Matrix-Calc, det**.





Example 11

For the matrix $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$ find

a det(**A**) **b** $\mathbf{A}^{-1}$

Solution

a $\det(\mathbf{A}) = 5 \times 1 - 2 \times 3 = -1$

b $$\mathbf{A}^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$$

Example 12

For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$ find

a det(**A**) **b** $\mathbf{A}^{-1}$ **c** $\mathbf{X}$ if $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$ **d** $\mathbf{Y}$ if $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Solution

a $\det(\mathbf{A}) = 3 \times 6 - 2 = 16$

b $$\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

c $$\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

Multiply both sides (from the left) by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \mathbf{IX} = \mathbf{X} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 16 & 30 \\ 16 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

d $$\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

Multiply both sides (from the right) by $\mathbf{A}^{-1}$

$$\mathbf{YAA}^{-1} = \frac{1}{16}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y} = \frac{1}{16}\begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix}$$

$$\therefore \quad \mathbf{Y} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & \frac{-1}{2} \end{bmatrix}$$

Exercise 1D

Example 11

1 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$ find

a $\det(\mathbf{A})$ **b** $\mathbf{A}^{-1}$ **c** $\det(\mathbf{B})$ **d** $\mathbf{B}^{-1}$

2 Find the inverse of the following regular matrices (θ is any real number, k is any non-zero real number).

a $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ **d** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3 If **A**, **B** are the regular matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, find $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$.

Also find **AB** and hence find, if possible, $(\mathbf{AB})^{-1}$.
Also find from $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, the products $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$. What do you notice?

Example 12

4 For the matrix $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

a find $\mathbf{A}^{-1}$ **b** if $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find **X** **c** if $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find **Y**.

5 If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$, find

a **X** such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$ **b** **Y** such that $\mathbf{YA} + \mathbf{B} = \mathbf{C}$

6 If **A** is a 2×2 matrix, $a_{12} = a_{21} = 0$, $a_{11} \neq 0$, $a_{22} \neq 0$, then show that **A** is regular and find $\mathbf{A}^{-1}$.

7 Let **A** be a regular 2×2 matrix, **B** a 2×2 matrix and $\mathbf{AB} = 0$. Show that $\mathbf{B} = \mathbf{0}$.

8 Find all 2×2 matrices such that $\mathbf{A}^{-1} = \mathbf{A}$.

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1.5 Solution of simultaneous equations using matrices

Inverse matrices can be used to solve certain sets of simultaneous linear equations. Consider the equations

$$3x - 2y = 5$$
$$5x - 3y = 9$$

This can be written as

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

If $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$ the determinant of $\mathbf{A}$ is $3(-3) - 5(-2) = 1$

which is not zero and so $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}$$

Multiplying the matrix equation $\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ on the left hand side by $\mathbf{A}^{-1}$ and using the fact that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ yields the following:

$$\mathbf{A}^{-1}\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\therefore \quad \mathbf{I}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ since } \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This is the solution to the simultaneous equations.
Check by substituting $x = 3$, $y = 2$ in the equations.

When dealing with simultaneous linear equations in two variables which represent parallel straight lines, a singular matrix results.
For example the system

$$x + 2y = 3$$
$$-2x - 4y = 6$$

has associated matrix equation

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Note that the determinant of $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} = 1 \times -4 - (-2 \times 2) = 0$.

There is no unique solution to the system of equations.

Example 13

If $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, solve the system $\mathbf{AX} = \mathbf{K}$ where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution

If $\mathbf{AX} = \mathbf{K}$, then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

$$\mathbf{A}^{-1}\mathbf{K} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 14

Solve the following simultaneous equations.

$$3x - 2y = 6$$
$$7x + 4y = 7$$

Solution

The matrix equation is $\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}$

Then $\mathbf{A}^{-1} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}$

and $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 38 \\ -21 \end{bmatrix}$

Since any linear system of n equations in n unknowns can be written as

$\mathbf{AX} = \mathbf{K}$, where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ k_n \end{bmatrix}$, this method can be applied more generally when **A** is regular. In fact, as shown, an expression for the solution can be written at once. Multiply **AX**, and **K**, on the left by $\mathbf{A}^{-1}$, and $\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{K}$ and $\mathbf{A}^{-1}(\mathbf{AX}) = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$.

Hence $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$, which is a formula for the solution of the system. Of course it depends on the inverse $\mathbf{A}^{-1}$ existing, but once $\mathbf{A}^{-1}$ is found then equations of the form $\mathbf{AX} = \mathbf{K}$ can be solved for all possible $n \times 1$ matrices **K**.

Again this process can be completed using a calculator as long as matrices **A** and **K** have been entered onto the calculator.

Example 15

Consider the system of five equations in five unknowns.

$$2a + 3b - c + d + 2e = 9$$
$$a + b - c = 4$$
$$a + 2d - 3e = 4$$
$$-b + 2c - d + e = -6$$
$$a - b + d - 2e = 0$$

Use a graphics calculator to solve for a, b, c, d and e.

Solution

Enter 5×5 matrix **A** and 5×1 matrix **B** into the graphics calculator.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 & 1 & 2 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 & -3 \\ 0 & -1 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 9 \\ 4 \\ 4 \\ -6 \\ 0 \end{bmatrix}$$

Then $\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{4}{9} \\ \frac{23}{9} \\ -1 \\ \frac{7}{9} \\ -\frac{2}{3} \end{bmatrix}$ $\quad \therefore\ a = \frac{4}{9}, b = \frac{23}{9}, c = -1, d = \frac{7}{9}$ and $e = -\frac{2}{3}$

It should be noted that just as for two equations in two unknowns, there is a geometric interpretation for three equations in three unknowns. There is only a unique solution if the equations represent three planes intersecting at a point.

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Skillsheet TEACHER

Exercise 1E

Example 13

1 If $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$, solve the system $\mathbf{AX} = \mathbf{K}$ where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, and

a $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **b** $\mathbf{K} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

2 If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$, solve the system $\mathbf{AX} = \mathbf{K}$ where

a $\mathbf{K} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ **b** $\mathbf{K} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Example 14

3 Use matrices to solve the following pairs of simultaneous equations.

a $-2x + 4y = 6$
$3x + y = 1$

b $-x + 2y = -1$
$-x + 4y = 2$

c $2x + 5y = -10$
$y = x + 4$

d $1.3x + 2.7y = -1.2$
$4.6y - 3.5x = 11.4$

4 Use matrices to find the point of intersection of the lines given by the equations $2x - 3y = 7$ and $3x + y = 5$.

5 Two children spend their pocket money buying some books and some CDs. One child spends \$120 and buys four books and four CDs. The other child buys three CDs and five books and spends \$114. Set up a system of simultaneous equations and use matrices to find the cost of a single book and a single CD.

6 Consider the system
$$2x - 3y = 3$$
$$4x - 6y = 6$$

a Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.
b Is $\mathbf{A}$ a regular matrix?
c Can any solutions be found for this system?
d How many pairs does the solution set contain?

Example 15

7 Consider the system of four equations in four unknowns.

$$p + q - r - s = 5$$
$$r + s = 1$$
$$2p - q + 2r = -2$$
$$p - q + s = 0$$

Use a graphics calculator to solve for p, q, r and s.

MC TEST D&D TEST

Chapter summary

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - each has the same number of rows and the same number of columns, and
  - they have the same number or element at corresponding positions.
- The size or **dimension** of a matrix is described by specifying the number of rows (m) and the number of columns (n). The dimension is written $m \times n$.
- Addition will be defined for two matrices only when they have the same dimension. The sum is found by adding corresponding elements.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

  Subtraction is defined in a similar way.
- If **A** is an $m \times n$ matrix and k is a real number, k**A** is defined to be an $m \times n$ matrix whose elements are k times the corresponding element of **A**.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows.
  To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.
  The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.
- If **A** and **B** are square matrices of the same dimension and **AB** = **BA** = **I** then **A** is said to the inverse of **B** and **B** is said to be the inverse of **A**.
- If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\mathbf{A}^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$

  $\det(\mathbf{A}) = ad - bc$ is the **determinant** of matrix **A**.
- A square matrix is said to be **regular** if its inverse exists. Those square matrices which do not have an inverse are called **singular** matrices.
- Simultaneous equations can be solved using inverse matrices, for example

$$ax + by = c$$
$$dx + ey = f$$

  can be written as $\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}\begin{bmatrix} c \\ f \end{bmatrix}$

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Review

Multiple-choice questions

1 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -2 & 3 \\ 3 & 0 \end{bmatrix}$ has dimension

A 8 **B** 4×2 **C** 2×4 **D** 1×4 **E** 3×4

2 If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & -3 & -1 \end{bmatrix}$ then $\mathbf{A} + \mathbf{B} =$

A $\begin{bmatrix} 3 & -3 \\ -2 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ **E** Cannot be determined

3 If $\mathbf{C} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ then $\mathbf{D} - \mathbf{C} =$

A $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -6 & 4 \\ -2 & 0 & -4 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

D $\begin{bmatrix} 1 & -6 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ **E** Cannot be determined

4 If $\mathbf{M} = \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$ then $-\mathbf{M} =$

A $\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -4 \\ -6 & -2 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 0 \\ -2 & -6 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 6 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$

5 If $\mathbf{M} = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$ then $2\mathbf{M} - 2\mathbf{N} =$

A $\begin{bmatrix} 0 & 0 \\ -9 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -2 \\ -6 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 12 & -2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 \\ 6 & -1 \end{bmatrix}$

6 If **A** and **B** are both $m \times n$ matrices, where $m \neq n$, then $\mathbf{A} + \mathbf{B}$ is an

A $m \times n$ matrix **B** $m \times m$ matrix **C** $n \times n$ matrix

D $2m \times 2n$ matrix **E** Cannot be determined

7 If **P** is an $m \times n$ matrix, and **Q** is a $n \times p$ matrix, the dimension of matrix **QP** is

A $n \times n$ **B** $m \times p$ **C** $n \times p$ **D** $m \times n$ **E** Cannot be determined

8 The determinant of matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ is

A 4 **B** 0 **C** -4 **D** 1 **E** 2

9 The inverse of matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ is

A -1 **B** $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

10 If $\mathbf{M} = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$ then $\mathbf{NM} =$

A $\begin{bmatrix} 0 & -4 \\ -9 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} -4 & -2 \\ 2 & -8 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 4 \\ 9 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} -6 & 2 \\ -3 & -5 \end{bmatrix}$ **E** $\begin{bmatrix} 6 & -2 \\ -3 & -5 \end{bmatrix}$

Short-answer questions (technology-free)

1 If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, find

a $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **b** $\mathbf{A}^2 - \mathbf{B}^2$

2 Find all possible matrices **A** which satisfy the equation $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$.

3 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $\mathbf{B} = [3 \quad -1 \quad 2]$, $\mathbf{C} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $\mathbf{D} = [2 \quad 4]$ and $\mathbf{E} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

a State whether or not each of the following products exist: **AB**, **AC**, **CD**, **BE**

b Evaluate **DA** and $\mathbf{A}^{-1}$.

4 If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, evaluate **AB** and $\mathbf{C}^{-1}$.

5 Find the 2×2 matrix **A** such that $\mathbf{A} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$

6 If $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, find $\mathbf{A}^2$ and hence $\mathbf{A}^{-1}$.

7 If $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ is a singular matrix, find the value of x.

8 **a** If $\mathbf{M} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, find the value of

i $\mathbf{MM} = \mathbf{M}^2$ **ii** $\mathbf{MMM} = \mathbf{M}^3$ **iii** $\mathbf{M}^{-1}$

b Find x and y given that $\mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Extended-response questions

1 $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & -4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix}$

a Find

i $\mathbf{A} + \mathbf{B}$ **ii** $\mathbf{A} - \mathbf{B}$ **iii** $2\mathbf{A} + 3\mathbf{B}$ **iv** $\mathbf{C}$ such that $3\mathbf{A} + 2\mathbf{C} = \mathbf{B}$

b Find

i $\mathbf{AB}$ **ii** $\mathbf{A}^{-1}$ **iii** $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$ **iv** $\mathbf{Y}$ such that $\mathbf{YA} = \mathbf{B}$

2 If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 0 & -1 \\ 1 & 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 0 & 1 \\ 4 & 2 & -2 \\ 1 & 3 & 3 \end{bmatrix}$ and $\mathbf{C}$ $\begin{bmatrix} 2 & 0 & 2 \\ 3 & 0 & -1 \\ 1 & 3 & 1 \end{bmatrix}$, find

a $\mathbf{AB}$ **b** $\mathbf{AC}$ **c** $\mathbf{BC}$

d $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{C}$ **e** $\mathbf{Y}$ such that $\mathbf{YA} = \mathbf{B}$

f $\mathbf{X}$ such that $\mathbf{AXC} = \mathbf{CB}$ **g** $\mathbf{Y}$ such that $\mathbf{CYA} = \mathbf{BA}$

3 **a** Consider the system of equations

$$2x - 3y = 3$$
$$4x + y = 5$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Find det$\mathbf{A}$ and $\mathbf{A}^{-1}$.

iii Solve the system of equations.

iv Interpret your solution geometrically.

b Consider the system of equations

$$2x + y = 3$$
$$4x + 2y = 8$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Find det$\mathbf{A}$ and explain why $\mathbf{A}^{-1}$ does not exist.

c Interpret your findings in part **b** geometrically.

4 The final grades for Physics and Chemistry are made up of three components, tests, practical work and exams. Marks out of 100 are awarded in each component each semester.

Wendy scored the following marks in each of the three components for Physics.

Semester 1: tests 79, practical work 78, exam 80

Semester 2: tests 80, practical work 78, exam 82

a Represent this information in a 2×3 matrix.

To calculate the final grade for each semester the three components are weighted so that tests are worth 20%. Practical work is worth 30% and the exam is worth 50%.

(cont'd)

b Represent this information in a 3×1 matrix.

c Calculate Wendy's final grade for Physics in each semester.

Wendy also scored the following marks in each of the three components for Chemistry.

Semester 1: tests 86, practical work 82, exam 84

Semester 2: tests 81, practical work 80, exam 70

d Calculate Wendy's final grade for Chemistry in each semester.

Students who gain an aggregate score for Physics and Chemistry of 320 or more over the two semesters are awarded a Certificate of Merit in Science.

e Will Wendy be awarded a Certificate of Merit in Science?

She asks her teacher to remark her Semester 2 Chemistry Exam hoping that she will gain the necessary marks to be awarded a Certificate of Merit.

f How many extra marks does she need?

5 A company runs Computing classes and employs full-time and part-time teaching staff as well as technical support staff, cleaners and catering staff. The number of staff employed depends on demand from term to term.

In one year they employed the following teaching staff:

Term 1: full-time 10, part-time 2

Term 2: full-time 8, part-time 4

Term 3: full-time 8, part-time 8

Term 4: full-time 6, part-time 10

a Represent this information in a 4×2 matrix.

Full-time teachers are paid \$70 per hour and part-time teachers are paid \$60 per hour.

b Represent this information in a 2×1 matrix.

c Calculate the cost per hour to the company for teaching staff for each term.

In the same year they also employed the following support staff

Term 1: technical staff 2, catering staff 2, cleaning staff 1.

Term 2: technical staff 2, catering staff 2, cleaning staff 1.

Term 3: technical staff 3, catering staff 4, cleaning staff 2.

Term 4: technical staff 3, catering staff 4, cleaning staff 2.

d Represent this information in a 4×3 matrix.

Technical staff are paid \$60 per hour, catering staff \$55 per hour and cleaners are paid \$40 per hour.

e Represent this information in a 3×1 matrix.

f Calculate the cost per hour to the company for support staff for each term.

g Calculate the total cost per hour to the company for teaching and support staff for each term.