CHAPTER 17

Loci

Objectives

To find the cartesian equation of a locus, where each point \( P \) of the locus satisfies the following properties:

- that \( P \) is equidistant from two given points \( A \) and \( B \), i.e., \( PA = PB \)
- that \( P \) is a fixed distance from a given point \( A \), i.e., there is a positive number \( k \) such that \( PA = k \)
- that \( PA = kPB \), i.e., the distance from a fixed point \( A \) is \( k \) times its distance from a fixed point \( B \)
- that the sum of the distances from two points \( A \) and \( B \) is always a constant, i.e. \( PA + PB = k \)
- that the difference of the distances from two points \( A \) and \( B \) is always a constant, i.e. \( PA - PB = k \)

To sketch the graphs of circles, ellipses and hyperbolas

To consider the asymptotes of hyperbolas

17.1 Introduction and parabolas

Introduction

In order to find the equation of a curve, some condition must be given which establishes which points are on a curve. Up to now in this book, curves have been described through a relationship between the \( x \) and \( y \) coordinates (and in Chapter 16, between the polar coordinates). For example, \( y = 2x \) is the straight line through the origin with gradient 2 (and for polar coordinates, \( r = 5 \) is the circle with center the origin and radius 5).

Many curves can also be described through a geometric description. For example, the set of points equidistant from the points \( A(4, 0) \) and \( B(2, 0) \) lie on the line with equation \( x = 3 \).

A locus is a set of points which satisfy a condition. For the above example, the locus of the points which are equidistant from \( A \) and \( B \) is the straight line with equation \( x = 3 \). Note that every point which lies on the line \( x = 3 \) satisfies this condition. This is an important observation which should be thought about with every locus problem.
**Example 1**

Find the equation of the locus of points $P$ satisfying $PA = PB$, where $A$ is the point with coordinates $(3, 0)$ and $B$ is the point with coordinates $(6, 4)$.

**Solution**

Let $(x, y)$ be the coordinates of point $P$.

If $PA = PB$

Then $\sqrt{(x - 3)^2 + (y - 0)^2} = \sqrt{(x - 6)^2 + (y - 4)^2}$

Squaring both sides and expanding

$x^2 - 6x + 9 + y^2 = x^2 - 12x + 36 + y^2 - 8y + 16$

$-6x + 9 = -12x - 8y + 52$

$8y + 6x = 43$

The locus is a straight line as shown.

Every point $P$ on the line also satisfies the property that $PA = PB$.

**Example 2**

Find the equation of the locus of points $P$ satisfying $PA = 3$, where $A$ is the point with coordinates $(2, 1)$.

**Solution**

Let $(x, y)$ be the coordinates of point $P$.

If $PA = 3$

Then $\sqrt{(x - 2)^2 + (y - 1)^2} = 3$

Squaring both sides

$(x - 2)^2 + (y - 1)^2 = 9$

The locus is a circle with centre $(2, 1)$ and radius 3.

Every point $P$ on the circle satisfies the property that $PA = 3$.

**Example 3**

Find the equation of the locus of points $P$ satisfying $PO = 2PA$, where $A$ is the point with coordinates $(4, 0)$ and $O$ is the origin.

**Solution**

Let $(x, y)$ be the coordinates of point $P$.

If $PO = 2PA$

Then $\sqrt{x^2 + y^2} = 2\sqrt{(x - 4)^2 + y^2}$
Squaring both sides
\[ x^2 + y^2 = 4[(x - 4)^2 + y^2] \]

Expanding
\[ x^2 + y^2 = 4[x^2 - 8x + 16 + y^2] \]
\[ x^2 + y^2 = 4x^2 - 32x + 64 + 4y^2 \]
\[ 0 = 3x^2 - 32x + 64 + 3y^2 \]
\[ 0 = 3 \left( x^2 - \frac{32}{3}x + \frac{64}{3} \right) + 3y^2 \]

Completing the square
\[ 0 = 3 \left( x^2 - \frac{32}{3}x + \frac{256}{9} + \frac{64}{3} - \frac{256}{9} \right) + 3y^2 \]
\[ 0 = 3 \left( x - \frac{16}{3} \right)^2 + 3y^2 - \frac{64}{3} \]
\[ \frac{64}{9} = \left( x - \frac{16}{3} \right)^2 + y^2 \]

The locus is a circle with centre \( \left( \frac{16}{3}, 0 \right) \) and radius \( \frac{8}{3} \).
Every point \( P \) on this circle satisfies the property that \( PO = 2 PA \).

**Parabolas**

**Example 4**

Find the equation of the locus of points \( P \) satisfying \( PM = PF \), where \( F \) is the point with coordinates \( (3, 0) \) and \( PM \) is the perpendicular distance from \( P \) to the line with equation \( x = -3 \).

**Solution**

Let \((x, y)\) be the coordinates of point \( P \).

If \( PF = PM \)

Then \( \sqrt{(x - 3)^2 + y^2} = \sqrt{(x + 3)^2} \)

Squaring both sides
\[ (x - 3)^2 + y^2 = (x + 3)^2 \]

Hence
\[ x^2 - 6x + 9 + y^2 = x^2 + 6x + 9 \]

Therefore
\[ y^2 = 12x \]

This is a parabola.
Chapter 17 — Loci

Exercise 17A

1 Sketch the locus of points \(P(x, y)\) for each of the following and hence write down its cartesian equation.

- **Example 1**
  - a \(P\) is equidistant from the points \(A(3, 0)\) and \(B(6, 0)\)
  - b \(P\) is equidistant from the points \(A(0, 8)\) and \(B(0, 12)\)
  - c \(P\) is always three units from the origin
  - d A triangle \(OAP\) has vertices \(O(0, 0)\), \(A(4, 0)\) and \(P(x, y)\). The triangle has area 12 square units. Find the locus of \(P\) as it moves.

2 Find the locus of a point \(P(x, y)\) which moves so that it is equidistant from the origin and the point \((-2, 5)\).

3 Find the locus of a point \(P(x, y)\) which moves so that it is equidistant from the points \((0, 6)\) and \((-2, 4)\).

4 Find the locus of a point \(P(x, y)\) which moves so that the sum of the squares of its distances from the points \((-2, 0)\) and \((2, 0)\) is 26 units.

5 A point \(P(x, y)\) moves so that its distance from the point \(K(2, 5)\) is twice its distance from the line \(x = 1\). Find its locus.

**Example 3**

6 A point \(P\) moves so that its distance from the point \((0, 20)\) is twice its distance from \(B(-4, 5)\). What is the locus of \(P\)?

7 Find the locus of a point \(P(x, y)\) which moves so that it is equidistant from the points \((1, 2)\) and \((-2, -1)\).

8 A point \(P(x, y)\) moves so that its distance from the point \(K(4, -2)\) is twice its distance from the origin. Find its locus.

9 Determine the locus of a point \(P\) which moves so that the difference of the squares of its distances from the points \(A(4, 0)\) and \(B(-4, 0)\) is 16.

10 Determine the locus of a point \(P\) which moves so that the square of its distance from the origin is equal to the sum of its coordinates.

11 \(A(0, 0)\) and \(B(4, 0)\) are two of the vertices of a triangle \(ABP\). The third vertex \(P\) is such that \(PA : PB = 2\). Find the locus of \(P\).

12 Find the locus of the point \(P\) which moves so that it is always equidistant from two fixed points \(A(1, 2)\) and \(B(-1, 0)\).

13 Given two fixed points \(A(0, 1)\) and \(B(2, 5)\) find the locus of \(P\) if the slope of \(AB\) equals that of \(BP\).

14 \(P\) moves so that its distance from the line \(y = 3\) is always 2 units. Find the locus of \(P\).

**Example 4**

15 Find the equation of the locus of points \(P(x, y)\) which satisfy the property that the distance of \(P\) to the point \(F(2, 0)\) is equal to the distance \(PM\), the perpendicular distance to the line with equation \(x = -4\). That is, \(PF = PM\).
Find the equation of the locus of points \( P(x, y) \) which satisfy the property that the distance from \( P \) to the point \( F(0, -4) \) is equal to the distance \( PM \), the perpendicular distance to the line with equation \( y = 2 \). That is, \( PF = PM \).

Describe the locus, in terms of equal distance from a line and a point, of a parabola with equation \( y^2 = 3x \).

### 17.2 Ellipses

The equation for an ellipse can be found in a similar way to those loci considered in Section 17.1.

#### Example 5

Find the equation of the locus of points \( P \) satisfying \( PA + PB = 8 \), where \( A \) is the point with coordinates \((-2, 0)\) and \( B \) is the point with coordinates \((2, 0)\).

**Solution**

Let \((x, y)\) be the coordinates of point \( P \).

If \( PA + PB = 8 \)

Then \( \sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 8 \)

Then \( \sqrt{(x+2)^2 + y^2} = 8 - \sqrt{(x-2)^2 + y^2} \)

Squaring both sides

\( (x+2)^2 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2 \)

Expanding and simplifying

\( x^2 + 4x + 4 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2 \)

and

\( x - 8 = -2\sqrt{(x-2)^2 + y^2} \)

Squaring both sides and expanding

\( x^2 - 16x + 64 = 4(x^2 - 4x + 4 + y^2) \)

Simplifying yields

\( 48 = 3x^2 + 4y^2 \) or \( \frac{x^2}{16} + \frac{y^2}{12} = 1 \)

This is an ellipse with centre the origin, \( x \) axis intercepts 4 and \(-4\) and \( y \) axis intercepts \( 2\sqrt{3} \) and \(-2\sqrt{3} \).

Every point on the ellipse satisfies the property that \( PA + PB = 8 \).
In general, an ellipse can be defined as the locus of the point \( P \) so that, as it moves, 
\[ PA + PB = k \]
for some \( k \) greater than the distance between \( A \) and \( B \). This is shown in the diagram.

This can be pictured as a string of length \( P_1F_1 + P_1F_2 \) being attached by nails to a board at \( F_1 \) and \( F_2 \) and, considering the path mapped out by a pencil, extending the string so that it is taut, and moving ‘around’ the two points.

**Example 6**

Find the image of the circle \( x^2 + y^2 = 1 \) under each of the following transformations.

a. a dilation of factor 4 from the \( x \) axis followed by a dilation of factor 5 from the \( y \) axis

b. a dilation of factor 4 from the \( x \) axis followed by a dilation of factor 5 from the \( y \) axis and then a translation of 4 units in the positive direction of the \( x \) axis and 3 units in the negative direction of the \( y \) axis

**Solution**

a. The transformation is defined by the rule \((x, y) \rightarrow (5x, 4y)\)

Therefore let \( x' = 5x \) and \( y' = 4y \) where \((x', y')\) is the image of \((x, y)\) under the transformation.

Hence \( x = \frac{x'}{5} \) and \( y = \frac{y'}{4} \). The image is

\[
\frac{(x')^2}{25} + \frac{(y')^2}{16} = 1.
\]

This is an ellipse with centre the origin, \( x \) axis intercepts 5 and \(-5\) and \( y \) axis intercepts 4 and \(-4\).

b. The transformation is defined by the rule

\((x, y) \rightarrow (5x + 4, 4y - 3)\)

Therefore let \( x' = 5x + 4 \) and \( y' = 4y - 3 \) where \((x', y')\) is the image of \((x, y)\) under the transformation.

Hence \( x = \frac{x' - 4}{5} \) and \( y = \frac{y' + 3}{4} \). The image is

\[
\frac{(x' - 4)^2}{25} + \frac{(y' + 3)^2}{16} = 1.
\]

This an ellipse with centre \((4, -3)\).

**Example 7**

Find the equation of the locus of points \( P(x, y) \) which satisfy the property that the distance of \( P \) to the point \( F(1, 0) \) is half the distance \( PM \), the perpendicular distance to the line with equation \( x = -2 \). That is, \( PF = \frac{1}{2}PM \).
Solution

Let \((x, y)\) be the coordinates of point \(P\).

If \(PF = \frac{1}{2} PM\)

\[\sqrt{(x - 1)^2 + y^2} = \frac{1}{2}\sqrt{(x + 2)^2}\]

Squaring both sides

\[(x - 1)^2 + y^2 = \frac{1}{4}(x + 2)^2\]

\[4(x^2 - 2x + 1) + 4y^2 = x^2 + 4x + 4\]

\[4x^2 - 8x + 4 + 4y^2 = x^2 + 4x + 4\]

\[3x^2 - 12x + 4y^2 = 0\]

Completing the square

\[3[x^2 - 4x + 4] + 4y^2 = 12\]

\[3(x - 2)^2 + 4y^2 = 12\]

or equivalently

\[\frac{(x - 2)^2}{4} + \frac{y^2}{3} = 1\]

This is an ellipse with centre \((2, 0)\).

It can be shown that the locus of points \(P(x, y)\) satisfying \(PF = ePM\), where \(0 < e < 1\), \(F\) is a fixed point and \(PM\) is the perpendicular distance from \(P\) to a fixed line \(l\), is an ellipse. From the symmetry of the ellipse it is clear that there is a second point \(F'\) and a second line \(l'\) such that \(PF' = ePM'\), where \(PM'\) is the perpendicular distance from \(P\) to \(l'\), that defines the same locus.

Exercise 17B

1 Sketch the graph of each of the following ellipses. Label axes intercepts.

\[\text{a} \quad \frac{x^2}{9} + \frac{y^2}{64} = 1\]

\[\text{b} \quad \frac{x^2}{25} + \frac{y^2}{100} = 1\]

\[\text{c} \quad \frac{y^2}{9} + \frac{x^2}{64} = 1\]

\[\text{d} \quad 25x^2 + 9y^2 = 225\]
2 Sketch the graph of each of the following ellipses. State the centre and label the axes intercepts.

\[ \frac{(x - 3)^2}{9} + \frac{(y - 4)^2}{64} = 1 \quad \text{a} \]
\[ \frac{(x + 3)^2}{9} + \frac{(y + 4)^2}{25} = 1 \quad \text{b} \]
\[ \frac{(y - 3)^2}{16} + \frac{(x - 2)^2}{4} = 1 \quad \text{c} \]
\[ 25(x - 5)^2 + 9y^2 = 225 \quad \text{d} \]

3 Sketch the graph of the image of the circle with equation \( x^2 + y^2 = 1 \) transformed by a dilation of factor 3 from the \( x \) axis and a dilation of factor 5 from the \( y \) axis. Give the equation of this image.

4 Find the locus of the point \( P \) as it moves such that the sum of its distances from two fixed points \( F(4, 0) \) and \( F'(-4, 0) \) is 10 units.

5 Sketch the graph of the image of the circle with equation \( x^2 + y^2 = 1 \) under the transformation, dilation of factor 4 from the \( x \) axis and a dilation of factor 8 from the \( y \) axis. Give the equation of this image.

6 Find the equation of the locus of points \( P(x, y) \) which satisfy the property that the distance of \( P \) to the point \( F(2, 0) \) is half the distance \( PM \), the perpendicular distance to the line with equation \( x = -4 \). That is, \( PF = \frac{1}{2}PM \).

7 Find the equation of the locus of points \( P(x, y) \) which satisfy the property that the distance of \( P \) to the point \( F(0, 8) \) is half the distance \( PM \), the perpendicular distance to the line with equation \( y = 4 \). That is, \( PF = \frac{1}{2}PM \).

17.3 Hyperbolas

The curve with equation \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) is a hyperbola with centre at the origin. The axis intercepts are \((a, 0)\) and \((-a, 0)\). The hyperbola has asymptotes \( y = \frac{b}{a}x \) and \( y = -\frac{b}{a}x \). An argument for this is as follows.

The equation

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

may be rearranged

\[ \frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 \]
\[ \therefore y^2 = \frac{b^2x^2}{a^2} \left( 1 - \frac{a^2}{x^2} \right) \]

But as \( x \to \pm \infty, \frac{a^2}{x^2} \to 0 \)

\[ \therefore y^2 \to \frac{b^2x^2}{a^2} \]
\[ \text{i.e.} \quad y \to \pm \frac{bx}{a} \]
The general equation for a hyperbola is formed by suitable translations. The curve with equation
\[ \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \]
is a hyperbola with centre \((h, k)\). The asymptotes are
\[ y - k = \pm \frac{b}{a}(x - h) \]
This hyperbola is obtained from the hyperbola with equation \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) by the translation defined by \((x, y) \to (x + h, y + k)\).

**Example 8**

For each of the following equations, sketch the graph of the corresponding hyperbola, give the coordinates of the centre, the axes intercepts and the equations of the asymptotes.

\[ \frac{x^2}{9} - \frac{y^2}{4} = 1 \]
\[ \frac{y^2}{9} - \frac{x^2}{4} = 1 \]
\[ (x - 1)^2 - (y + 2)^2 = 1 \]
\[ \frac{(y - 1)^2}{4} - \frac{(x + 2)^2}{9} = 1 \]

**Solution**

\[ \frac{x^2}{9} - \frac{y^2}{4} = 1 \]
\[ \therefore y^2 = \frac{4x^2}{9}(1 - \frac{9}{x^2}) \]
Equations of asymptotes
\[ y = \pm \frac{2}{3}x \]
When \(y = 0\), \(x^2 = 9\) and therefore \(x = \pm 3\)
Axes intercepts \((3, 0)\) and \((-3, 0)\), centre \((0, 0)\)

\[ \frac{y^2}{9} - \frac{x^2}{4} = 1 \] is the reflection of \(\frac{x^2}{9} - \frac{y^2}{4} = 1\) in the line \(y = x\)
\[ \therefore \text{asymptotes are} \]
\[ x = \pm \frac{2}{3}y \]
i.e. \(y = \pm \frac{3}{2}x\)
The \(y\) axis intercepts are \((0, 3)\) and \((0, -3)\)
c \((x - 1)^2 - (y + 2)^2 = 1\). The graph of \(x^2 - y^2 = 1\) is sketched first. The asymptotes are \(y = x\) and \(y = -x\).

This hyperbola is called a **rectangular hyperbola** as its asymptotes are perpendicular.

The centre is \((0, 0)\) and the axes intercepts are at \((1, 0)\) and \((-1, 0)\).

A translation of \((x, y) \rightarrow (x + 1, y - 2)\) is applied. The new centre is \((1, -2)\) and the asymptotes have equations \(y + 2 = \pm(x - 1)\), i.e., \(y = x - 3\) and \(y = -x - 1\).

When \(x = 0\), \(y = -2\) and when \(y = 0\),

\[
(x - 1)^2 = 5 \quad \Rightarrow \quad x = 1 \pm \sqrt{5}
\]

d \[
\frac{(y - 1)^2}{4} - \frac{(x + 2)^2}{9} = 1
\]

This is obtained by translating the hyperbola \(\frac{y^2}{4} - \frac{x^2}{9} = 1\) through the translation defined by \((x, y) \rightarrow (x - 2, y + 1)\)

**Note:** the asymptotes for \(\frac{y^2}{4} - \frac{x^2}{9} = 1\) are the same as for those of the hyperbola \(\frac{x^2}{9} - \frac{y^2}{4} = 1\). The two hyperbolae are called **conjugate hyperbolae**.

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**Example 9**

Find the equation of the locus of points \(P(x, y)\) which satisfy the property that the distance of \(P\) to the point \(F(1, 0)\) is twice the distance \(PM\), the perpendicular distance to the line with equation \(x = -2\). That is, \(PF = 2PM\).
Solution

Let \((x, y)\) be the coordinates of point \(P\).

If \(PF = 2PM\)

\[
\sqrt{(x - 1)^2 + y^2} = 2\sqrt{(x + 2)^2}
\]

Squaring both sides

\[
(x - 1)^2 + y^2 = 4(x + 2)^2
\]

\[
x^2 - 2x + 1 + y^2 = 4x^2 + 8x + 16
\]

\[
x^2 - 2x + 1 + y^2 = 4x^2 + 16x + 16
\]

\[
0 = 3x^2 + 18x - y^2 + 15
\]

Completing the square

\[
0 = 3[x^2 + 6x + 9] - y^2 + 15 - 27
\]

\[
3(x + 3)^2 - y^2 = 12
\]

or equivalently

\[
\frac{(x + 3)^2}{4} - \frac{y^2}{12} = 1
\]

This is a hyperbola with centre \((-3, 0)\)

It can be shown that the locus of points \(P(x, y)\) satisfying \(PF = ePM\), where \(e > 1\), \(F\) is a fixed point and \(PM\) is the perpendicular distance from \(P\) to a fixed line \(l\), is a hyperbola. From the symmetry of the hyperbola it is clear that there is a second point \(F'\) and a second line \(l'\) such that \(PF' = ePM'\), where \(PM'\) is the perpendicular distance from \(P\) to \(l'\), that defines the same locus.

Hyperbolas may be defined in a manner similar to the methods discussed earlier in this section for circles and ellipses.

Consider the set of all points, \(P\), such that \(PF_1 - PF_2 = k\) where \(k\) is a suitable constant and \(F_1\) and \(F_2\) are points with coordinates \((m, 0)\) and \((-m, 0)\) respectively. Then the equation of the curve defined in this way is

\[
\frac{x^2}{a^2} - \frac{y^2}{m^2 - a^2} = 1, \quad k = 2a
\]
Example 10

Find the equation of the locus of points \( P \) satisfying \( PA - PB = 3 \) where \( A \) is the point with coordinates \((-2, 0)\) and \( B \) is the point with coordinates \((2, 0)\).

Solution

Let \((x, y)\) be the coordinates of point \( P \).

If \( PA - PB = 3 \)

Then \( \sqrt{(x + 2)^2 + y^2} - \sqrt{(x - 2)^2 + y^2} = 3 \)

Then \( \sqrt{(x + 2)^2 + y^2} = 3 + \sqrt{(x - 2)^2 + y^2} \)

Squaring both sides

\((x + 2)^2 + y^2 = 9 + 6\sqrt{(x - 2)^2 + y^2} + (x - 2)^2 + y^2 \)

Expanding and simplifying

\( x^2 + 4x + 4 + y^2 = 9 + 6\sqrt{x^2 - 4x + 4 + y^2 + x^2 - 4x + 4 + y^2} \)

and \( 8x - 9 = 6\sqrt{x^2 - 2y^2} \). Note that this only holds if \( x > \frac{9}{8} \)

Squaring both sides

\( 64x^2 - 144x + 81 = 36[x^2 - 4x + 4 + y^2] \)

Simplifying yields

\[ \frac{28x^2 - 36y^2}{9} = 63 \]

\[ \frac{4x^2}{9} - \frac{4y^2}{7} = 1 \quad x \geq \frac{3}{2} \]

This is the right branch of a hyperbola with centre the origin, \( x \) axis intercept \( \frac{3}{2} \).

The equations of the asymptotes are \( y = \pm \frac{\sqrt{7}x}{3} \).

Exercise 17C

1 Sketch the graph of each of the following hyperbolas. Label axes intercepts and give the equation of the asymptotes.

\[
\begin{align*}
\text{a} & \quad \frac{x^2}{9} - \frac{y^2}{64} = 1 \\
\text{b} & \quad \frac{x^2}{25} - \frac{y^2}{100} = 1 \\
\text{c} & \quad \frac{y^2}{9} - \frac{x^2}{64} = 1 \\
\text{d} & \quad 25x^2 - 9y^2 = 225
\end{align*}
\]
2 Sketch the graph of each of the following hyperbolas. State the centre and label axes
intercepts and asymptotes.

\[ \frac{(x - 3)^2}{9} - \frac{(y - 4)^2}{64} = 1 \quad \text{b} \quad \frac{(x + 3)^2}{9} - \frac{(y + 4)^2}{25} = 1 \]
\[ \frac{(y - 3)^2}{16} - \frac{(x - 2)^2}{4} = 1 \quad \text{d} \quad 25(x - 5)^2 - 9y^2 = 225 \]
\[ x^2 - y^2 = 4 \quad \text{e} \quad 2x^2 - y^2 = 4 \]
\[ x^2 - 4y^2 - 4x - 8y - 16 = 0 \quad \text{f} \quad 9x^2 - 25y^2 - 90x + 150y = 225 \]

3 Find the locus of the point \( P \) as it moves such that the difference of its distances from two
fixed points \( F(4, 0) \) and \( F'(-4, 0) \) is 6 units.

4 Find the equation of the locus of points \( P(x, y) \) which satisfy the property that the distance
of \( P \) to the point \( F(2, 0) \) is twice the distance \( PM \), the perpendicular distance to the line
with equation \( x = -4 \). That is, \( PF = 2PM \).

5 Find the equation of the locus of points \( P(x, y) \) which satisfy the property that the distance
of \( P \) to the point \( F(0, 8) \) is four times the distance \( PM \), the perpendicular distance to the line
with equation \( y = 4 \). That is, \( PF = 4PM \).

6 Find the equation of the locus of points \( P(x, y) \) satisfying \( PA - PB = 4 \) where \( A \) is the
point with coordinates \((-3, 0)\) and \( B \) is the point with coordinates \((3, 0)\).
Chapter summary

- **Lines**
  The general equation of a straight line may be written as $ax + by = c$.
  For fixed points $A$ and $B$, the locus of $P(x, y)$, as $P$ moves such that $PA = PB$, is a straight line.

- **Circles**
  The circle with centre the origin and radius $a$ is the graph of the equation $x^2 + y^2 = a^2$.
  The circle with centre $(h, k)$ and radius $a$ is the graph of the equation $(x - h)^2 + (y - k)^2 = a^2$. For a fixed point $A$, the locus of $P(x, y)$ as $P$ moves such that $PA = k$, where $k > 0$, is a circle.

- **Ellipses**
  The curve with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse with centre the origin, $x$ axis intercepts $(-a, 0)$ and $(a, 0)$, and $y$ axis intercepts $(0, -b)$ and $(0, b)$. For $a > b$, the ellipse will appear as shown in the diagram to the left. If $b > a$, the ellipse is as shown in the diagram to the right.

- **Hyperbolas**
  The curve with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola with centre the origin. The axis intercepts are $(a, 0)$ and $(-a, 0)$. The hyperbola has asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

  The curve with equation $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ is a hyperbola with centre $(h, k)$. The hyperbola has asymptotes $y - k = \frac{b}{a}(x - h)$ and $y - k = -\frac{b}{a}(x - h)$.

  For fixed points $A$ and $B$, the locus of $P(x, y)$ as $P$ moves such that $|PA - PB| = k$, where $k$ is a suitable constant, is a hyperbola.
Multiple-choice questions

1. The equation of the ellipse shown is
   A. $5x^2 + y^2 = 5$
   B. $5x^2 + y^2 = 25$
   C. $x^2 + 5y^2 = 25$
   D. $x^2 + 5y^2 = 5$
   E. $\left(\frac{x}{5}\right)^2 + y^2 = 1$

2. The coordinates of the $x$ axis intercepts of the graph of the ellipse with equation
   $\frac{x^2}{25} + \frac{y^2}{9} = 1$ are
   A. $(-5, 0)$ and $(-3, 0)$
   B. $(-3, 0)$ and $(3, 0)$
   C. $(0, -5)$ and $(0, 5)$
   D. $(-5, 0)$ and $(5, 0)$
   E. $(5, 0)$ and $(3, 0)$

3. The graph of the ellipse with equation $\left(\frac{x}{9}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$ is
   A. 
   B. 
   C. 
   D. 
   E. 

4. The coordinates of the $y$ axis intercepts of the graph of the ellipse with equation
   $\left(\frac{x^2}{9}\right) + \left(\frac{(y + 2)^2}{4}\right) = 1$ are
   A. $(-2, 0)$ and $(2, 0)$
   B. $(-4, 0)$ and $(4, 0)$
   C. $(0, -4)$ and $(0, 4)$
   D. $(0, 0)$ and $(0, -4)$
   E. $(3, 0)$ and $(0, 2)$
5. The graph of the equation $ax^2 + by^2 = 8$ has $y$ axis intercept 2 and passes through the point with coordinates $\left(1, \frac{\sqrt{10}}{2}\right)$. Then

A. $a = 2$ and $b = 3$

B. $a = 4$ and $b = 3$

C. $a = \sqrt{3}$ and $b = 2$

D. $a = 3$ and $b = 2$

E. $a = 2$ and $b = 2$

6. The circle with equation $(x - a)^2 + (y - b)^2 = 16$ has its centre on the $y$ axis and passes through the point with coordinates $(4, 4)$. Then

A. $a = 0$ and $b = 4$

B. $a = 0$ and $b = 0$

C. $a = 2$ and $b = 0$

D. $a = -4$ and $b = 0$

E. $a = 4$ and $b = 0$

7. The circle with equation $x^2 + y^2 = 1$ is transformed to an ellipse through the following sequence of transformations:

- dilation of factor 4 from the $x$ axis
- dilation of factor 3 from the $y$ axis
- translation of 4 units in the positive direction of the $x$ axis
- translation of 3 units in the positive direction of the $y$ axis

The equation of the resulting ellipse is

A. $\frac{(x - 4)^2}{16} + \frac{(y - 3)^2}{9} = 1$

B. $\frac{(x - 3)^2}{9} + \frac{(y - 3)^2}{16} = 1$

C. $\frac{(x + 4)^2}{3} + \frac{(y + 3)^2}{4} = 1$

D. $\frac{(x - 4)^2}{36} + \frac{(y - 3)^2}{16} = 1$

E. $\frac{(x + 4)^2}{20} + \frac{(y + 3)^2}{48} = 1$

8. The equation of the graph shown is

A. $\frac{(x + 2)^2}{27} - \frac{y^2}{108} = 1$

B. $\frac{(x - 2)^2}{9} - \frac{y^2}{34} = 1$

C. $\frac{(x + 2)^2}{81} - \frac{y^2}{324} = 1$

D. $\frac{(x - 2)^2}{81} - \frac{y^2}{324} = 1$

E. $\frac{(x + 2)^2}{9} - \frac{y^2}{36} = 1$

9. The locus of points $P(x, y)$ which satisfy the property that $PA = PB$ where $A$ is the point with coordinates $(2, -5)$ and $B$ is the point with coordinates $(-4, 1)$ is described by the equation

A. $y = x - 1$

B. $y = x - 6$

C. $y = -x - 3$

D. $y = x + 1$

E. $y = 3 - x$

10. The locus of points $P(x, y)$ which satisfy the property that $PA = 2PB$ where $A$ is the point with coordinates $(2, -5)$ and $B$ is the point with coordinates $(-4, 1)$ is

A. a straight line

B. an ellipse

C. a circle

D. a parabola

E. a hyperbola
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### Short-answer questions (technology-free)

1. A circle has equation \(x^2 + 4x + y^2 + 8y = 0\). Find the coordinates of the centre and radius of the circle.
2. An ellipse has equation \(x^2 + 4x + 2y^2 = 0\). Find the coordinates of the centre and the axes intercepts of the ellipse.
3. Find the locus of the point \(P(x, y)\) such that \(PA = PB\), where \(A\) is the point with coordinates \((0, 2)\) and \(B\) is the point with coordinates \((6, 0)\).
4. Find the locus of the point \(P(x, y)\) such that \(PA = 6\), where \(A\) is the point with coordinates \((3, 2)\).
5. State the equations of the asymptotes of the hyperbola with equation \(\frac{x^2}{9} - \frac{y^2}{4} = 1\).
6. Find the locus of the point \(P(x, y)\) such that \(PA = 2PB\), where \(A\) is the point with coordinates \((0, 2)\) and \(B\) is the point with coordinates \((6, 0)\).
7. Sketch the graph of the ellipse with equation \(\frac{(x - 2)^2}{9} + \frac{y^2}{4} = 1\) and state its centre.
8. Determine the locus of a point \(P\) which moves so that the difference of the squares of its distances from two fixed points \(P_1(4, 0)\) and \(P_2(-4, 0)\) is constant.

### Extended-response questions

1. Let \(A\), \(B\) and \(C\) be points with coordinates \((6, 0)\), \((-6, 0)\) and \((0, 6)\) respectively. Find the locus of the points \(P\) which satisfy each of the following.
   a. \(PA = PC\)
   b. \(PA = 6\)
   c. \(PA = 2PC\)
   d. \(PA = 2PB\)
   e. \(PA = \frac{1}{2}PB\)
   f. \(PA + PB = 20\)
   g. \(PA + PB = 12\)
   h. \(PA - PB = 5\)
   i. \(PB - PA = 5\)

2. Find the equation of the locus of points \(P(x, y)\) which satisfy the property that the distance of \(P\) to the point \(F(0, 0)\) is
   a. equal to \(PM\), the perpendicular distance to the line with equation \(y = 2\)
   b. half the distance \(PM\), the perpendicular distance to the line with equation \(y = 2\)
   c. twice the distance \(PM\), the perpendicular distance to the line with equation \(y = 2\).

3. a. The base of a triangle is fixed and the distance from one end of the base to the midpoint of the opposite side is a constant. Find the locus of the vertex joining the other two sides.
   b. The base of a triangle is fixed and the ratio of the lengths of the other two sides is a constant. Find the locus of the vertex joining the other two sides.
   c. Three vertices of a convex quadrilateral are fixed. Find the locus of the fourth vertex if the area of the quadrilateral is a constant.