CHAPTER 18

The normal distribution

Objectives

- To introduce the standard normal distribution.
- To introduce the family of normal distributions as transformations of the standard normal distribution.
- To consider the effect of variation in the values of the parameters defining the normal distribution on the graph of the probability density function.
- To recognise the mean, median, mode, variance and standard deviation of a normal distribution.
- To use technology to determine probabilities for intervals in the solution of problems where the normal distribution is appropriate.

18.1 The normal distribution

The most useful continuous distribution, and one that occurs frequently, is the normal distribution. The probability density functions of normal random variables are symmetric, single peaked, bell-shaped density curves. Data sets occurring in nature will often have such a bell-shaped distribution, as measurements on many random variables are closely approximated by a normal probability distribution.

The simplest form of the normal distribution is a random variable with probability density function $f$ with rule:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

This random variable has a mean $\mu = 0$ and a standard deviation $\sigma = 1$. Because it is the simplest form of the normal distribution it is given a special name, the standard normal distribution. The graph of the standard normal distribution is as shown.

The probability density function is symmetric around zero as $f(-x) = f(x)$, i.e. it is an even function.
The line \( y = 0 \) is an asymptote; that is, as \( x \to \pm \infty, \ y \to 0 \). Almost all of the area under the probability density function lies between \( \pm 3 \). It can be seen from the graph that the mean, median and mode of this distribution are the same, and are equal to 0. While the probability density function for the standard normal distribution cannot be integrated exactly, the value of the mean can be verified using the calculator.

\[
E(X) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx
\]

Entering the function into the calculator and integrating (between \(-5\) and \(+5\) is adequate) gives the screen at right.

That is, the mean \( E(X) \) of the standard normal distribution is 0.

What can be said about the standard deviation of this distribution? Again the calculator must be used.

\[
E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx
\]

Entering the function into the calculator and integrating (again, between \(-5\) and \(+5\) is adequate) gives the screen at right.

That is, \( E(X^2) = 1 \).

Thus, \( \text{Var}(X) = E(X^2) - [E(X)]^2 = 1 - 0 = 1 \) and \( \text{sd}(X) = \sqrt{\text{Var}(X)} = 1 \)

A random variable with a standard normal distribution has probability density function

\[
f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}
\]

The standard normal distribution has a mean \( \mu = 0 \) and a standard deviation \( \sigma = 1 \)

Henceforth, we will denote the random variable of the standard normal distribution by \( Z \).

The normal distribution does not apply just to the special circumstances where the mean is zero and the standard deviation is 1. The curve of the probability density function for a normal distribution with mean \( \mu \) and standard deviation \( \sigma \) may be obtained from the curve of the standard normal distribution by a transformation with rule:

\[
(x, y) \rightarrow \left( \sigma x + \mu, \frac{y}{\sigma} \right)
\]

i.e. a dilation of factor \( \sigma \) from the \( y \)-axis and a dilation of factor \( \frac{1}{\sigma} \) from the \( x \)-axis followed by a translation of \( \mu \) units in the positive direction of the \( x \)-axis, for \( \frac{\mu}{\sigma} > 0 \). This was discussed for general continuous probability distributions in the previous chapter.
The transformation which maps the curve of the density function of a normal distribution with mean \( \mu \) and standard deviation \( \sigma \) to the curve of the standard normal distribution is:

\[
(x, y) \rightarrow \left( \frac{x - \mu}{\sigma}, \sigma y \right)
\]

That is a translation of \( \mu \) units in the negative direction of the x-axis, a dilation of factor \( \frac{1}{\sigma} \) from the y-axis and a dilation of factor \( \sigma \) from the x-axis.

For the example with \( \mu = 100 \) and \( \sigma = 15 \):

\[
(x, y) \rightarrow \left( \frac{x - 100}{15}, 15y \right)
\]

These transformations are area-preserving. For example, the rectangle \( ABCD \) is mapped to \( A'B'C'D' \). The area of both rectangles is 180 square units.

This property enables the probabilities of any normal distribution to be determined by the probabilities of the standard normal distribution.

The rule for the general normal probability density function is:

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2}, \quad \mu \in \mathbb{R}, \sigma > 0
\]

If \( X \) is a normal probability distribution with mean \( \mu \) and standard deviation \( \sigma \),

\[
\Pr(X \leq a) = \Pr \left( Z \leq \frac{a - \mu}{\sigma} \right), \text{ where } Z \text{ is the random variable of the standard normal.}
\]
The general form of the normal density function includes two parameters, \( \mu \) and \( \sigma \), which are the mean (\( \mu \)) and the standard deviation (\( \sigma \)) of that particular distribution.

When a random variable has a distribution described by a normal probability density function the random variable is said to have a **normal distribution**.

The graph below shows a normal density function.

As with all continuous probability density functions, the normal density function has the fundamental properties that:

- probability corresponds to an area under the curve
- the total area under the curve is 1.

However, it has some additional special properties.

The normal density function is symmetric and bell-shaped with its **centre** determined by the mean of the distribution it represents and its **width** determined by the standard deviation. It may be seen from the graph on the right that the graph of \( y = f(x) \) is symmetric about the line \( x = \mu \), and has a maximum value of \( \frac{1}{\sigma \sqrt{2\pi}} \) which occurs when \( x = \mu \).

Thus the location of the curve on the graph is determined by the value of \( \mu \) and the steepness of the curve by the value of \( \sigma \).

Irrespective of the values of the mean and standard deviation of a particular normal function the percentage of values that lie within a given number of standard deviations from the mean is always the same.

**Example 1**

On the same set of axes sketch (a calculator can be used to help) the graphs of the standard normal probability density function and graph of the probability density function of:

- **a** the normal distribution with mean 1 and standard deviation 1
- **b** the normal distribution with mean 1 and standard deviation 2.
Solution

a  The graph has been translated 1 unit in the positive direction of the x-axis.

The rules of the probability density functions are

\[ y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \]  
\[ y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (x - 1)^2} \]

b  The graph has been dilated from the y-axis by a factor of 2 and from the x-axis by factor \( \frac{1}{2} \), and translated 1 unit in the positive direction of the x-axis.

The rules of the probability density functions are

\[ y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \]  
\[ y = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2} (\frac{x-1}{2})^2} \]

Exercise 18A

1  Which of the following data distributions appear to be approximately normally distributed?

a  

b  


c

2  The random variables \( X_1 \) and \( X_2 \) are both normally distributed with means \( \mu_1 \) and \( \mu_2 \), and standard deviations \( \sigma_1 \) and \( \sigma_2 \), respectively. If \( \mu_1 < \mu_2 \), and \( \sigma_1 < \sigma_2 \), sketch both distributions on the same diagram.
3 Consider the normal probability density function:

\[ f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - 3}{3} \right)^2} \quad x \in \mathbb{R} \]

a Use your calculator to find:

\[ \int_{-\infty}^{\infty} f(x) \, dx \]

b i Express \( E(X) \) as an integral.

ii Use your calculator to evaluate the integral found in i.

c i Write down an expression for \( E(X^2) \).

ii What is the value of \( E(X^2) \)?

iii What is the value of \( \sigma \)?

4 Consider the normal probability density function:

\[ f(x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x + 4}{5} \right)^2} \quad x \in \mathbb{R} \]

a Use your calculator to find:

\[ \int_{-\infty}^{\infty} f(x) \, dx \]

b i Express \( E(X) \) as an integral.

ii Use your calculator to evaluate the integral found in i.

c i Write down an expression for \( E(X^2) \).

ii What is the value of \( E(X^2) \)?

iii What is the value of \( \sigma \)?

5 The probability density function of a normal random variable \( X \) is:

\[ f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - 3}{10} \right)^2} \]

a Write down the mean and the standard deviation of \( X \).

b Sketch the graph of \( f(x) \).

6 The probability density function of a normal random variable \( X \) is:

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( x + 3 \right)^2} \]

a Write down the mean and the standard deviation of \( X \).

b Sketch the graph of \( f(x) \).

7 The probability density function of a normal random variable \( X \) is:

\[ f(x) = \frac{1}{\sqrt{18\pi}} e^{-\frac{1}{2} \left( \frac{x}{3} \right)^2} \]

a Write down the mean and the standard deviation of \( X \).

b Sketch the graph of \( f(x) \).
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8 Describe the sequence of transformations which takes the graph of the probability density function of the standard normal distribution to the graph of the probability density function of the normal distribution with:
   a \( \mu = 3 \) and \( \sigma = 2 \)
   b \( \mu = 3 \) and \( \sigma = \frac{1}{2} \)
   c \( \mu = -3 \) and \( \sigma = 2 \)

9 Describe the sequence of transformations which takes the graphs of each of the probability density functions of the normal distribution with the given mean and standard to the graph of the probability density function of the standard normal distribution.
   a \( \mu = 3 \) and \( \sigma = 2 \)
   b \( \mu = 3 \) and \( \sigma = \frac{1}{2} \)
   c \( \mu = -3 \) and \( \sigma = 2 \)

18.2 Standardisation and the 68–95–99.7% rule

For any normal distribution it can be shown that approximately 68% of the values will lie within one standard deviation of the mean, approximately 95% of the values lie within two standard deviations of the mean and almost all (99.7%) within three standard deviations. This gives rise to what has become known as the 68–95–99.7% rule.

The 68–95–99.7% rule

For any normally distributed random variable approximately

- 68% of the values lie within one standard deviation of the mean
- 95% of the values lie within two standard deviations of the mean
- 99.7% of the values lie within three standard deviations of the mean.
If we know that a random variable is approximately normally distributed, and we know its mean and standard deviation, then we can use the 68–95–99.7% rule to quickly make some important statements about the way in which the data values are distributed.

**Example 2**

Experience has shown that the scores obtained on a commonly used IQ test can be assumed to be normally distributed with a mean $\mu = 100$ with a standard deviation $\sigma = 15$.

Approximately what percentage of the distribution lies within one, two or three standard deviations of the mean?

**Solution**

Knowing that the distribution of scores is normally distributed with $\mu = 100$ and $\sigma = 15$, the 68–95–99.7% rule means that approximately:
- 68% of the scores will lie between 85 and 115
- 95% of the scores will lie between 70 and 130
- 99.7% will lie between 55 and 145.

Statements can also be made about the percentage of scores that lie in the tails of the distribution by using the symmetry of the distribution and noting that the total area under curve is 100%.

**Example 3**

From Example 2, 95% of the scores in the IQ distribution lie between 70 and 130 (that is, within two standard deviations of the mean). What percentage of the distribution is more than two standard deviations above or below the mean (in this instance less than 70 or greater than 130)?

**Solution**

If we focus our attention on the tails of the distribution, we see that 5% of the IQ scores lie outside this region. Using the symmetry of the distribution we can say that 2.5% of the scores are below 70 and 2.5% are above 130. That as, if you obtained a score greater than 130 on this test you would have been in the top 2.5% of the group.

Clearly the standard deviation is a natural measuring stick for normally distributed data. For example, a person who obtained a score of 112 on an IQ test with a mean of $\mu = 100$ and a standard deviation $\sigma = 15$ is less than one standard deviation from the mean. Their score is typical of the group as a whole as it lies well within the middle 68% of scores. In contrast, a
person who scores 133 has done exceptionally well; their score is more than two standard deviations from the mean and this puts them in the top 2.5%. Because of the additional insight provided, it is usual to convert normally distributed data to a new set of units which shows the number of standard deviations each data value lies from the mean of the distribution. These new values are called **standardised values** or **z-values**. To standardise a data value \( x \) we first subtract the mean \( \mu \) of the normal random variable from each score and then divide the result by the standard deviation \( \sigma \).

\[
\text{standardised value} = \frac{\text{data value} - \text{mean of the normal curves}}{\text{standard deviation of the normal curve}}
\]

or symbolically

\[
z = \frac{x - \mu}{\sigma}
\]

Standardised values can be positive or negative:
- a **positive** \( z \)-value indicates that the data value it represents lies above the mean
- a **negative** \( z \)-value indicates that the data value lies below the mean.

For example, an IQ score of 90 lies below the mean and has a standardised value of:

\[
z = \frac{x - \mu}{\sigma} = \frac{90 - 100}{15} = \frac{-10}{15} \approx -0.67
\]

Clearly, not all data values in a normal distribution lie exactly one, two or three standard deviations from the mean. In these circumstances specially prepared tables or a calculator are used to give the required probability. There are as many different normal curves as there are values of \( \mu \) and \( \sigma \), but, if the measurement scale is changed to ‘standard deviations from the mean’ or \( z \)-values, all normal curves reduce to the same normal curve with a mean \( \mu = 0 \) and a standard deviation \( \sigma = 1 \). This special normal curve is known as the **standard normal**.

The figures on the right show how standardising IQ scores transforms a normal distribution with mean \( \mu = 100 \) and standard deviation \( \sigma = 15 \) into the standard normal distribution with mean \( \mu = 0 \) and standard deviation \( \sigma = 1 \).
Exercise 18B

1. What are the values of the mean and standard deviation of the normal distribution shown?

2. What are the values of the mean and standard deviation of the normal distribution shown?

3. Experience has shown that the scores obtained on a commonly used IQ test can be assumed to be normally distributed with a mean $\mu = 100$ and a standard deviation $\sigma = 15$. What percentage of scores lies above 115?

4. The heights of young women are normally distributed with a mean $\mu = 160$ cm and a standard deviation $\sigma = 8$ cm. What percentage of the women would you expect to have heights:
   a. between 152 and 168 cm?
   b. greater than 168 cm?
   c. less than 136 cm?

5. Fill in the blanks in the following paragraph:
   The age at marriage of males in the US in the 1980s was approximately normally distributed with a mean $\mu = 27.3$ years and a standard deviation $\sigma = 3.1$ years. From this data we can conclude that, in the 1980s about 95% of males married between the ages of . . . and . . .

6. Fill in the blanks in the following statement of the 68–95–99.7% rule:
   For any normal distribution, about
   $\blacksquare$ 68% of the values lie within . . . standard deviation of the mean.
   $\blacksquare$ . . .% of the values lie within two standard deviations of the mean.
   $\blacksquare$ . . .% of the values lie within . . . standard deviations of the mean.

7. If you are told that in Australian adults, nostril width is approximately normally distributed with a mean $\mu = 2.3$ cm and a standard deviation $\sigma = 0.3$ cm, find the percentage of people with nostril widths less than 1.7 cm.
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8 The distribution of IQ scores for the inmates of a certain prison is approximately normal with a mean $\mu = 85$ and a standard deviation $\sigma = 15$.

a What percentage of the prison population have an IQ of 100 or higher?
b If someone with an IQ of 70 or less can be classified as having special needs, what percentage of the prison population could be classified as having special needs?

9 The distribution of heights of navy officers was found to be normal with a mean $\mu = 175$ cm and a standard deviation $\sigma = 5$ cm. Determine:

a the percentage of navy officers with heights between 170 cm and 180 cm
b the percentage of officers with heights greater than 180 cm
c the approximate percentage of navy officers with heights greater than 185 cm

10 The distribution of blood pressures (systolic) among women of similar ages is normal with a mean of 120 (mm of mercury) and a standard deviation of 10 (mm of mercury). Determine the percentage of women with a systolic blood pressure:

a between 100 and 140
b greater than 130
c greater than 120
d between 90 and 150

11 The heights of women are normally distributed with a mean $\mu = 160$ cm and a standard deviation $\sigma = 8$ cm. What is the standardised value for the height of a woman who is:

a 160 cm tall?
b 150 cm tall?
c 172 cm tall?

12 The length of pregnancy for a human is approximately normally distributed with a mean $\mu = 270$ days and a standard deviation $\sigma = 10$ days. How many standard deviations away from the mean is a pregnancy of length:

a 256 days?
b 281 days?
c 305 days?

13 Michael scores 85 on the mathematics section of a scholastic aptitude test, the results of which are known to be normally distributed with a mean of 78 and a standard deviation of 5. Cheryl sits for a mathematics ability test and scores 27. These test scores are normally distributed with a mean of 18 and a standard deviation of 6. Assuming that both tests measure the same kind of ability, who has the better score?

14 The results obtained by a student in two subjects, Biology and History, showing mark ($x$), subject mean ($\mu$) and standard deviation ($\sigma$) are shown in the table below:

<table>
<thead>
<tr>
<th>Subject</th>
<th>Mark ($x$)</th>
<th>Mean ($\mu$)</th>
<th>Standard deviation ($\sigma$)</th>
<th>Standardised mark ($z$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biology</td>
<td>77</td>
<td>68.5</td>
<td>4.9</td>
<td></td>
</tr>
<tr>
<td>History</td>
<td>79</td>
<td>75.3</td>
<td>4.1</td>
<td></td>
</tr>
</tbody>
</table>

Complete the table by calculating the student’s standardised mark for each subject and use this information to determine in which subject the student did best relative to her peers.
Three students scored as follows on three different tests in French, English and Mathematics:

<table>
<thead>
<tr>
<th>Subject</th>
<th>Test</th>
<th>Mark (x)</th>
<th>Mean (μ)</th>
<th>Standard deviation (σ)</th>
<th>Standardised mark (z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mary</td>
<td>French</td>
<td>19</td>
<td>15</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>English</td>
<td>42</td>
<td>35</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mathematics</td>
<td>20</td>
<td>20</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Steve</td>
<td>French</td>
<td>21</td>
<td>23</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>English</td>
<td>39</td>
<td>42</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mathematics</td>
<td>23</td>
<td>18</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Sue</td>
<td>French</td>
<td>15</td>
<td>15</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>English</td>
<td>42</td>
<td>35</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mathematics</td>
<td>19</td>
<td>20</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

a  Determine the standardised mark for each student on each test.
b  Who is the best student in:
   i  French?
   ii English?
   iii Mathematics?
c  Who is the best student overall? Give reasons for your answer.

18.3 Determining normal probabilities

The CAS calculator can be used to determine areas under the standard normal curve for many different z-values, allowing us to find probabilities for values other than one, two or three standard deviations from the mean.

For the TI-Nspire, probabilities associated with the normal distribution are found in a Calculator application by accessing the Distributions menu in the Probability menu.

There are two commands which refer to the normal distribution:

1  The Normal Pdf command gives the value of the probability density function \( f(x) \) for a particular value of \( X \).

2  The Normal Cdf command gives the value of the cumulative density function \( F(x) \) — that is, the area under the probability density function, for an interval.

To determine probabilities (areas under the probability density function), select Normal Cdf. This pastes the command onto the screen. To complete the command, enter the upper and lower limits for the probability to be calculated. If the distribution is the standard normal, then no more is required.
Example 4

Suppose that $Z$ is a standard normal random variable (that is, it has a mean $\mu = 0$ and a standard deviation $\sigma = 1$). Find:

- $a$ $\Pr(-1 < Z < 2.5)$
- $b$ $\Pr(Z > 1)$

Solution

Using the TI-Nspire

- $a$ Use the $\text{Normal Cdf}$ command from the $\text{Probability}$ menu and complete as shown.
- $b$ Use the $\text{Normal Cdf}$ command and complete as shown.

The answer is $\Pr(-1 < Z < 2.5) = 0.8351$
The answer is \( \Pr(Z > 1) = 0.1587 \).

To find probabilities from other than the standard normal distribution, there are two options. The standardisation formula discussed in the last section could be used to transform any variable to standard normal. The calculator will determine the probability associated with any normal distribution directly, by entering the appropriate values for \( \mu \) and \( \sigma \).

Some problems are solved by finding the value of the random variable which corresponds to a specified area under the normal curve. For the TI-Nspire, this can be done through the command \textbf{Inverse Normal} selected from the \textbf{Distributions} menu in the \textbf{Probability} menu.

The calculator can be used to determine percentiles of any normal distribution.

### Using the Casio ClassPad

In \( \nabla \) tap in the entry line and select \textbf{Calc > distribution}.

\begin{itemize}
  \item \textbf{a} In the field below Type, tap the \( \nabla \) and scroll to select \textbf{Normal CD}, then tap \( \nabla \).
  \end{itemize}

Enter the values shown: Lower = −1, Upper = 2.5, \( \sigma = 1 \), \( \mu = 0 \).

The answer is \( \Pr(-1 \leq Z \leq 2) = 0.8351 \).

\begin{itemize}
  \item \textbf{b} For \( \Pr(Z > 1) \) enter Lower = 1, Upper = \( \infty \). The answer is \( \Pr(Z > 1) = 0.1587 \).
\end{itemize}
Example 5

Suppose $X$ is normally distributed with a mean $\mu = 100$ and a standard deviation $\sigma = 6$. Find:

a. $k$ such that $P(X \leq k) = 0.95$

b. $c_1$ and $c_2$, such that $P(c_1 < X < c_2) = 0.95$

Solution

Using the TI-Nspire

a. Use the Inverse Normal command from the Probability menu (and complete as shown.

The value of $k$ is 109.869.

b. Examination of the normal curve shows that there are (infinitely) many intervals which enclose an area of 0.95. By convention, the interval which leaves equal areas in each tail is chosen.

To find $c_1$ enter the value of 0.025 in the argument of the calculator. Enter 0.975 to find the value of $c_2$. It is found that $c_1 = 88.240$ and $c_2 = 111.760$. 

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Using the ClassPad

a In tap Calc > Distribution and scroll to Inverse Normal CD. Enter the values given for μ, σ and the area. Tail setting is set to Left to indicate we seek the value for which 95% of the area lies to the left.

b For Pr(Z > 1), enter Lower = 1, Upper = ∞.
The answer is Pr(Z > 1) = 0.1587.

Exercise 18C

1 Suppose Z is a standard normal random variable (that is, it has a mean μ = 0 and a standard deviation σ = 1). Find the following probabilities, drawing an appropriate diagram in each case:
   a Pr(Z < 2)     b Pr(Z < 2.5)     c Pr(Z ≤ 2.5)     d Pr(Z < 2.53)
   e Pr(Z ≥ 2)     f Pr(Z > 1.5)     g Pr(Z ≥ 0.34)     h Pr(Z > 1.01)

2 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:
   a Pr(Z > −2)    b Pr(Z > −0.5)    c Pr(Z > −2.5)    d Pr(Z ≥ −1.283)
   e Pr(Z < −2)    f Pr(Z < −2.33)   g Pr(Z ≤ −1.8)   h Pr(Z ≤ −0.95)

3 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:
   a Pr(−1 < Z < 1)    b Pr(−2 < Z < 2)    c Pr(−3 < Z < 3)
How do these results compare with the ‘68%–95%–99.7% rule’ discussed in Section 18.2?

4 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:
   a Pr(2 < Z < 3)    b Pr(−1.5 < Z < 2.5)
   c Pr(−2 < Z < −1.5)    d Pr(−1.4 < Z < −0.8)

5 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that Pr(Z ≤ c) = 0.9

6 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that Pr(Z ≤ c) = 0.75
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7 Suppose $Z$ is a standard normal random variable. Draw an appropriate diagram and then find the value $c$ such that $\Pr(Z \leq c) = 0.975$.

8 Suppose $Z$ is a standard normal random variable. Draw an appropriate diagram and then find the value $c$ such that $\Pr(Z \geq c) = 0.95$.

9 Suppose $Z$ is a standard normal random variable. Draw an appropriate diagram and then find the value $c$ such that $\Pr(Z \geq c) = 0.8$.

10 Suppose $Z$ is a standard normal random variable. Draw an appropriate diagram and then find the value $c$ such that $\Pr(Z \leq c) = 0.10$.

11 Suppose $Z$ is a standard normal random variable. Draw an appropriate diagram and then find the value $c$ such that $\Pr(Z \leq c) = 0.025$.

12 Let $X$ be a normal random variable with mean $\mu = 100$ and a standard deviation $\sigma = 6$. Find:
   a $\Pr(X < 110)$
   b $\Pr(X < 105)$
   c $\Pr(X > 110)$
   d $\Pr(105 < X < 110)$

13 Let $X$ be a normal random variable with mean $\mu = 40$ and a standard deviation $\sigma = 5$. Find:
   a $\Pr(X < 48)$
   b $\Pr(X < 36)$
   c $\Pr(X > 32)$
   d $\Pr(32 < X < 36)$

14 Let $X$ be a normal random variable with mean $\mu = 6$ and a standard deviation $\sigma = 2$. Find:
   a $c$ such that $\Pr(X < c) = 0.95$
   b $k$ such that $\Pr(X < k) = 0.90$

15 Let $X$ be a normal random variable with mean $\mu = 10$ and a standard deviation $\sigma = 3$. Find:
   a $c$ such that $\Pr(X < c) = 0.50$
   b $k$ such that $\Pr(X < k) = 0.975$

16 Give that $X$ is a normally distributed random variable with mean equal to 22 and standard deviation of 7, find:
   a $\Pr(X < 26)$
   b $\Pr(25 < X < 27)$
   c $\Pr(X < 26 \mid 25 < X < 27)$
   d $c$ such that $\Pr(X < c) = 0.95$
   e $k$ such that $\Pr(X > k) = 0.9$
   f $c_1$ and $c_2$ such that $\Pr(c_1 < X < c_2) = 0.95$

17 Let $X$ be a normal random variable with mean $\mu = 10$ and a standard deviation $\sigma = 0.5$. Find:
   a $\Pr(X < 11)$
   b $\Pr(X < 11 \mid X < 13)$
   c $c$ such that $\Pr(X < c) = 0.95$
   d $k$ such that $\Pr(X < k) = 0.2$
   e $c_1$ and $c_2$ such that $\Pr(c_1 < X < c_2) = 0.95$
18.8 Solving problems using the normal distribution

The normal distribution can be used to solve many practical problems, as shown in the following examples.

**Example 6**

The time taken to complete a logical reasoning task is normally distributed with a mean of 55 seconds and a standard deviation of 8 seconds.

a Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task.

b Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task, if it is known that that person took less than 60 seconds to complete the task.

**Solution**

**Using the TI-Nspire**

a Use the Normal Cdf command from the Probability menu (and complete as shown.

The answer is that

\[ \Pr(X < 50) = 0.2660 \]

b \[ \Pr(X < 50|X < 60) = \Pr \left( \frac{X < 50 \cap X < 60}{X < 60} \right) \]

by definition

\[ = \Pr \left( \frac{X < 50}{X < 60} \right) \]

\[ \therefore \Pr(X < 50|X < 60) = \frac{0.2660}{0.7340} = 0.3624 \]
Chapter 18 — The normal distribution

Using the Casio ClassPad

In tap Calc > Distribution and scroll to Normal CD in the Type area. Enter the values as shown (∞ is found by pressing Keyboard).

Tap Next to obtain the answer and the upper and lower bounds converted to z-scores.

The answer is \( \Pr(X < 50) = 0.2660 \)

Tap \( \text{Calc} \) to obtain a graph of the probability density function.

When the mean and standard deviation of a normal distribution are unknown it is sometimes necessary to transform to the standard normal. This is demonstrated in the following example.

Example 7

Limits of acceptability imposed on the lengths of a certain batch of metal rods are 1.925 cm and 2.075 cm. It is observed that, on average, 5% are rejected as undersized and 5% are rejected as oversized.

Assuming that the lengths are normally distributed find the mean and standard deviation of the distribution.

Solution

It is given that \( \Pr(X > 2.075) = 0.05 \) and \( \Pr(X < 1.925) = 0.05 \)

Symmetry tells us that the mean is equal to \( (2.075 + 1.925) / 2 = 2 \)

Transforming to the standard normal gives:

\[
\Pr\left( Z > \frac{2.075 - \mu}{\sigma} \right) = 0.05 \quad \text{and} \quad \Pr\left( Z < \frac{1.925 - \mu}{\sigma} \right) = 0.05
\]

Also \( \Pr\left( Z < \frac{2.075 - \mu}{\sigma} \right) = 0.95 \)

Using the inverse normal facility of the calculator gives:

\[
\frac{2.075 - \mu}{\sigma} = 1.6448\ldots \quad \text{and} \quad \frac{1.925 - \mu}{\sigma} = -1.6448\ldots
\]

These solve to confirm \( \mu = 2 \) and substituting in the first of the equations gives:

\[
\frac{2.075 - 2}{\sigma} = 1.6448\ldots
\]

Solving for \( \sigma \) gives:

\[
\sigma = 0.045596\ldots
\]

and to four decimal places \( \sigma = 0.0456 \)
Exercise 18D

1 Suppose that IQ scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$
   a What is the probability that a person chosen at random has an IQ:
      i greater than 110?
      ii less than 75?
      iii greater than 130 given that they have an IQ greater than 110?
   b To join an elite club one has to have an IQ in the top 5% of the population. What IQ score would be necessary to join this club?

2 The heights of women are normally distributed with a mean $\mu = 160$ cm and a standard deviation $\sigma = 8$ cm.
   a What is the probability that a woman chosen at random would be:
      i taller than 155 cm?
      ii shorter than 170 cm?
      iii taller than 170 cm given that she is between 168 and 174 cm tall?
   b What height would put a woman among the tallest 10% of the population?
   c What height would put a woman among the shortest 20% of the population?

3 The results of a mathematics exam are normally distributed with mean $\mu = 50$ and standard deviation $\sigma = 7$.
   a What is the probability that a student chosen at random has an exam score:
      i greater than 60?
      ii less than 75?
      iii greater than 60 given that they passed? (Assume a pass mark of 50%.)
   b The top 15% of the class are to be awarded a distinction. What score would be required to gain a distinction in this exam?

4 The lengths of a species of fish are normally distributed with a mean length of 40 cm and a standard deviation of 4 cm. Find the percentage of these fish having lengths:
   a greater than 45 cm
   b between 35.5 cm and 45.5 cm

5 The weight of cats is normally distributed. It is known that 10% of cats weigh more than 1.8 kg, and 15% of cats weigh less than 1.35 kg. Find the mean and the standard deviation of this distribution.

6 The marks of a large number of students in a statistics examination are normally distributed with a mean of 48 marks and a standard deviation of 15 marks.
   a If the pass mark is 53, find the percentage of students who passed the examination.
   b If 8% of students gained an A on the examination by scoring a mark of at least $c$, find the value of $c$.

7 The height of a certain population of adult males is normally distributed with mean 176 cm and standard deviation 7 cm.
   a Find the probability that the height of a randomly selected individual will exceed 190 cm.
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b If two individuals are selected at random, find the probability that both of their heights will exceed 190 cm.

c Suppose 10 individuals are selected at random. Find the probability that at least two will have heights that exceed 190 cm.

8 The volume of soft drink in a 1 litre bottle is normally distributed. The soft drink company needs to calibrate its filling machine. They don’t want to put too much soft drink into each bottle, as it adds to their expense. However, they know they will be fined if more than 2\% of bottles are more than 2 millilitres under volume. The standard deviation of the volume dispensed by the filling machine is 2.5 millilitres. What should they choose as the target volume (i.e. the mean of the distribution)? Give your answer to the nearest millilitre.

9 The weights of pumpkins sold to a greengrocer are normally distributed with a mean of 1.2 kg and a standard deviation of 0.4 kg. The pumpkins are sold in three sizes:

- small under 0.8 kg
- medium from 0.8 kg to 1.8 kg
- large over 1.8 kg

a Find the proportions of pumpkins in each of the three sizes.

b The prices of the pumpkins are $2.80 for a small, $3.50 for a medium and $5.00 for a large. Find the expected cost for 100 pumpkins chosen at random from the greengrocer's supply.

10 Potatoes are delivered to a chip factory in semitrailer loads. A sample of 1 kg of the potatoes is chosen from each load and tested for starch content. From past experience it is known that the starch content is normally distributed with a standard deviation 2.1.

a For a semi-trailer with potatoes with mean starch content of 22.0:

i What is the probability that the reading is 19.5 or less?

ii What reading will be exceeded with probability of 0.98?

b If the starch content is greater than 22.0 the potatoes cannot be used for chips, so the semitrailer load is rejected. What is the probability that a load with mean starch content of 18.0 will be rejected?

11 The amount of a certain chemical in a type A cell is normally distributed with mean of 10 and a standard deviation of 1, while the amount in a type B cell is normally distributed with a mean of 14 and a standard deviation of 2. To determine whether a cell is a type A or a type B, the amount of chemical in the cell is measured and the cell is classified as a type A if the amount is less than a specified value \( c \), and as being of type B otherwise.

a If \( c = 12 \) calculate the probability that a type A cell will be misclassified, and the probability that a type B cell will be misclassified.

b Find the value of \( c \) for which the two probabilities of misclassification are equal.
Chapter summary

A special continuous random variable is described by the normal probability density function

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} \]

where \( \mu \) and \( \sigma \) are the mean and standard deviation of the normal random variable \( X \).

The centre of the normal curve is determined by the mean, \( \mu \), and its width is determined by the standard deviation, \( \sigma \).

The 68–95–99.7% rule states that, for any normal distribution:

1. Approximately 68% of the values lie within one standard deviation of the mean.
2. Approximately 95% of the values lie within two standard deviations of the mean.
3. Approximately 99.7% of the values lie within three standard deviations of the mean.

If \( X \) is a normally distributed random variable with mean \( \mu \) and standard deviation \( \sigma \) then to standardise we first subtract the mean \( \mu \) from \( X \) and then divide the result by the standard deviation \( \sigma \). The standardised value is denoted \( z \) and has a mean \( \mu = 0 \) and standard deviation \( \sigma = 1 \).

\[ z = \frac{x - \mu}{\sigma} \]

A calculator gives the values for the cumulative distribution function of the normal distribution—i.e., areas under the standard normal curve for many different values of the random variable.

A calculator gives the inverse normal values for a random variable—that is, values corresponding to specified areas under the normal curve.

Multiple-choice questions

1. The diagram shows the graph of a normal distribution with means \( \mu \) and standard deviations \( \sigma \). Which of the following statements is true?
   - A. \( \mu = 4 \) and \( \sigma = 3 \)
   - B. \( \mu = 3 \) and \( \sigma = 4 \)
   - C. \( \mu = 4 \) and \( \sigma = 2 \)
   - D. \( \mu = 3 \) and \( \sigma = 2 \)
   - E. \( \mu = 4 \) and \( \sigma = 4 \)

2. \( \Pr(Z > 1.45) \), where \( Z \) is a standard normal random variable, is:
   - A. 0.1394
   - B. 0.8606
   - C. 0.0735
   - D. 0.9625
   - E. 0.0925

3. If \( Z \) is a standard normal random variable, and \( \Pr(Z < c) = 0.25 \), then the value of \( c \) is closest to:
   - A. 0.6745
   - B. -0.6745
   - C. 0.3867
   - D. 0.5987
   - E. -0.5987
4 The random variable $X$ has a normal distribution with mean 12 and variance of 9. If $Z$ is a standard normal random variable then the probability that $X$ is more than 15 is equal to:
A $\Pr(Z < 1)$  
B $\Pr(Z > 1)$  
C $\Pr\left(\frac{Z}{3} > \frac{1}{3}\right)$  
D $1 - \Pr\left(\frac{Z}{3} > \frac{1}{3}\right)$  
E $1 - \Pr(Z > 1)$

5 If the actual length of an AFL game is normally distributed with a mean of 102 minutes and a standard deviation of 3 minutes, then the percentage of games that last more than 110 minutes is approximately:
A 96.2%  
B 81.3%  
C 2.7%  
D 18.7%  
E 0.38%

6 If the number of goals Collingwood scores a week is a normally distributed random variable with a mean of 16 and a standard deviation of 2, then in what percentage of their matches (approximately) do they score from 10 to 22 goals?
A 5%  
B 16%  
C 68%  
D 95%  
E 99.7%

7 The amount of water Steve uses to water the garden is normally distributed with a mean of 100 litres and a standard deviation of 14 litres. On 20% of occasions it takes him more than the $k$ litres to water the garden. What is the value of $k$?
A 88.2  
B 110.7  
C 120.0  
D 111.8  
E 114.0

8 If $X$ is a normally distributed random variable with mean $\mu = 6$ and standard deviation $\sigma = 3$, then the transformation which maps the curve of the density function of $X$, $f(x)$ to the curve of the standard normal distribution is:
A $(x, y) \rightarrow \left(\frac{x - 3}{6}, 6y\right)$  
B $(x, y) \rightarrow \left(\frac{x - 6}{3}, \frac{y}{3}\right)$  
C $(x, y) \rightarrow \left(\frac{x - 6}{3}, 3y\right)$  
D $(x, y) \rightarrow \left(3(x + 6), \frac{y}{3}\right)$  
E $(x, y) \rightarrow \left(3(x + 6), \frac{y}{3}\right)$

9 The marks achieved by Angie in Politics, Mathematics and Indonesian, together with the mean and the standard deviation for each subject, are given in the following table:

<table>
<thead>
<tr>
<th>Subject</th>
<th>Mark</th>
<th>Mean ($\mu$)</th>
<th>Standard deviation ($\sigma$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics</td>
<td>72</td>
<td>72</td>
<td>5</td>
</tr>
<tr>
<td>Indonesian</td>
<td>57</td>
<td>59</td>
<td>2</td>
</tr>
<tr>
<td>Politics</td>
<td>68</td>
<td>64</td>
<td>4</td>
</tr>
</tbody>
</table>

Which of the following statements is correct?
A Angie’s best subject was Politics, followed by Mathematics and then Indonesian.  
B Angie’s best subject was Mathematics, followed by Politics and then Indonesian.  
C Angie’s best subject was Politics, followed by Indonesian and then Mathematics.  
D Angie’s best subject was Mathematics, followed by Indonesian and then Politics.  
E Angie’s best subject was Indonesian, followed by Mathematics and then Indonesian.

10 Suppose that $X$ is normally distributed with a mean of 11.3 and a standard deviation of 2.9. The values of $c_1$ and $c_2$, such that $\Pr(c_1 < X < c_2) = 0.90$ are closest to:
A 5.5, 17.1  
B 6.08, 16.52  
C 15.02, 7.58  
D 6.53, 16.07  
E 5.62, 16.98
Short-answer questions (technology-free)

1. For the standard normal random variable \( Z \), \( \Pr(Z \leq a) = p \).
   Find in terms of \( p \):
   a. \( \Pr(Z > a) \)
   b. \( \Pr(Z < -a) \)
   c. \( \Pr(-a \leq Z \leq a) \)

2. \( X \) is the normal random variable with mean 4 and standard deviation 1. \( Z \) is the standard normal random variable.
   a. If \( \Pr(X < 3) = \Pr(Z < a) \), \( a = ? \)
   b. If \( \Pr(X > 5) = \Pr(Z > b) \), \( b = ? \)
   c. \( \Pr(X > 4) = ? \)

3. A normal distribution has mean 8 and standard deviation 3. Give the rule that maps the curve of the density function to the density function of the standard normal.

4. \( X \) is a normal random variable with mean \( \mu \) and standard deviation \( \sigma \). If \( \mu < a < b \), \( \Pr(X < b) = p \) and \( \Pr(X < a) = q \), find:
   a. \( \Pr(X < a | X < b) \)
   b. \( \Pr(X < 2\mu - a) \)
   c. \( \Pr(X > b | X > a) \)

5. \( X \) is a normal random variable with mean 4 and standard deviation 2. Write each of the following probabilities in terms of \( Z \).
   a. \( \Pr(X < 5) \)
   b. \( \Pr(X < 3) \)
   c. \( \Pr(X > 5) \)
   d. \( \Pr(3 < X < 5) \)
   e. \( \Pr(3 < X < 6) \)

In questions 6 to 8 you will use the following:
\( \Pr(Z < 1) = 0.84 \), \( \Pr(Z < 2) = 0.98 \), \( \Pr(Z < 0.5) = 0.69 \)

6. A machine produces metal rods with mean diameter 2.5 mm and standard deviation 0.05 mm. Let \( X \) be the random variable of the normal distribution. Find:
   a. \( \Pr(X < 2.55) \)
   b. \( \Pr(X < 2.5) \)
   c. \( \Pr(X < 2.45) \)
   d. \( \Pr(2.45 < X < 2.55) \)

7. Nuts are packed in tins such that the mean weight of the tins is 500 g and the standard deviation is 5 g. The weights are normally distributed with random variable \( W \). Find:
   a. \( \Pr(W > 505) \)
   b. \( \Pr(500 < W < 505) \)
   c. \( \Pr(W > 505 | W > 500) \)
   d. \( \Pr(W > 510) \)

8. A random variable \( X \) has a normal distribution with mean 6 and standard deviation 1. Find:
   a. \( \Pr(X < 6.5) \)
   b. \( \Pr(6 < X < 6.5) \)
   c. \( \Pr(6.5 < X < 7) \)
   d. \( \Pr(5 < X < 7) \)

9. Suppose that three tests were given in your Mathematics course. The class means and standard deviations, together with your scores, are listed in the table:

<table>
<thead>
<tr>
<th>Test</th>
<th>( \mu )</th>
<th>( \sigma )</th>
<th>Your score</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>50</td>
<td>11</td>
<td>62</td>
</tr>
<tr>
<td>B</td>
<td>47</td>
<td>17</td>
<td>64</td>
</tr>
<tr>
<td>C</td>
<td>63</td>
<td>8</td>
<td>73</td>
</tr>
</tbody>
</table>

On which test did you do best and on which did you do worst?
Extended-response questions

1 A test devised to measure mathematical aptitude gives scores that are normally distributed with a mean of 50 and a standard deviation of 10. If we wish to categorise the results so that the highest 10% of scores are designated as high aptitude, the next 20% as moderate aptitude, the middle 40% as average, the next 20% as little aptitude and the lowest 10% as no aptitude, what ranges of scores will be covered by each of these five categories?

2 If \( X \) is normally distributed with \( \mu = 10 \) and \( \sigma = 2 \), find the value of \( k \) such that
\[
\Pr(\mu - k \leq X \leq \mu + k) = 0.95
\]

3 Records kept by a manufacturer of car tyres suggest that the distribution of the mileage from their tyres is normal, with a mean of 60 000 kilometres and a standard deviation of 5000 kilometres.
   a What proportion of the company’s tyres last:
      i less that 55 000 kilometres?
      ii more than 50 000 kilometres but less than 74 000 kilometres?
      iii more than 72 000 kilometres, given that they have already lasted more than 60 000 kilometres?
   b The company’s advertising manager wishes to claim that ‘90% of our tyres last longer than \( c \) kilometres’. What should \( c \) be?
   c What is the probability that a customer buys five tyres at the same time and finds that they all last longer than 72 000 kilometres?

4 The owner of a new van complained to the dealer that he was using, on average, 18 litres of petrol to drive 100 km. The dealer pointed out that the 15 litres/100 km referred to in an advertisement was ‘just a guide and that actual consumption will vary’. Suppose that the distribution of fuel consumption for this make of van is normal with a mean of 15 litres/100 km and standard deviation 0.75 litres/100 km.
   a How probable is a van that uses at least 18 litres/100 km?
   b What does your answer to a suggest about the manufacturer’s claim?
   c Find \( c_1 \) and \( c_2 \) such that the van’s fuel consumption is more than \( c_1 \) but less than \( c_2 \) with a probability of 0.95.

5 Suppose that \( L \), the useful life (in hours) of a fluorescent tube used for indoor gardening, is normally distributed with a mean of 600 and a standard deviation of 4. The fluorescent tubes are sold in boxes of 10. Find the probability that at least three of the tubes in a randomly selected box last longer than 605 hours.
The amount of anaesthetic required to cause surgical anaesthesia in patients is normally distributed, with a mean of 50 mg and a standard deviation of 10 mg. The lethal dose is also normally distributed with a mean of 110 mg and a standard deviation of 20 mg. If a dosage that brings 90% of patients to surgical anaesthesia were used, what percentage of patients would be killed by this dose?

The hardness of a metal may be determined by impressing a hardened point into the surface of the metal and then measuring the depth of penetration of the point. Suppose that the hardness of a particular alloy is normally distributed with mean 70 and standard deviation 3.

a If a specimen is acceptable only if its hardness is between 65 and 75, what is the probability that a randomly chosen specimen has an acceptable hardness?

b If the acceptable range of hardness was \((70 - c, 70 + c)\), for what value of \(c\) would 95% of all specimens have acceptable hardness?

c If the acceptable range is the same as in a, and the hardness of each of 10 randomly selected specimens is independently determined, what is the expected number of acceptable specimens among the 10?

d What is the probability that at most eight out of 10 independently selected specimens have a hardness less than 73.84?

e The profit on an acceptable specimen is $20.00, while unacceptable specimens result in a loss of $5.00. If \(P\) dollars is the profit on a randomly selected specimen, find the mean and variance of \(P\).

The weekly error (in seconds) of a brand of watch is known to be normally distributed. Only those watches with an error of less than 5 seconds are acceptable.

a Find the mean and standard deviation of the distribution of error if 3% of watches are rejected for losing time and 3% are rejected for gaining time.

b Determine the probability that fewer than two watches are rejected in a batch of 10 such watches.

In a given manufacturing process, components are rejected if they have a particular dimension greater than 60.4 mm or less than 59.7 mm. It is found that 3% are rejected as being too large and 5% are rejected for being too small. Assume that the dimension is normally distributed.

a Find the mean and standard deviation of the distribution of the dimension, correct to one decimal place.

b Use the result of a to find the percentage of rejects if the limits for acceptance are changed to 60.3 mm and 59.6 mm.
A brand of detergent is sold in bottles of two sizes — standard and large. For each size, the content (in litres) of a randomly chosen bottle is normally distributed with mean and standard deviation as given in the table:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard bottle</td>
<td>0.760</td>
<td>0.008</td>
</tr>
<tr>
<td>Large bottle</td>
<td>1.010</td>
<td>0.009</td>
</tr>
</tbody>
</table>

a Find the probability that a randomly chosen standard bottle contains less than 0.75 litres.

b Find the probability that a box of 10 randomly chosen standard bottles contains at least three bottles whose contents are each less than 0.75 litres.

c Using the result $E(aX - bY) = aE(X) - bE(Y)$ and $\text{Var}(aX - bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$ find the probability that there is more detergent in four randomly chosen standard bottles than in three randomly chosen large bottles. (Assume $aX - bY$ is normally distributed.)