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CHAPTER

9

Differentiation of polynomials, power functions and rational functions

Objectives

- To understand the concept of **limit**.
- To understand the **definition of differentiation**.
- To understand and use the notation for the **derivative** of a polynomial function.
- To find the **gradient** of a curve of a polynomial function by calculating its derivative.
- To differentiate functions having **negative integer powers**.
- To understand and use the **chain rule**.
- To differentiate **rational powers**.
- To understand and use the **product rule**.
- To understand and use the **quotient rule**.
- To deduce the **graph of the gradient function** from the graph of a function.

In this chapter, we review some of the important ideas and results that have been introduced in earlier studies in calculus. The chain rule, product rule and quotient rule are introduced in this chapter.

9.1 The gradient of a curve at a point

First, we develop a technique for calculating the rate of change in polynomial functions. To illustrate this, we consider this introductory example.

On Planet X, an object falls a distance of y metres in t seconds where $y = 0.8t^2$

(**Note:** On Earth the commonly used model is $y = 4.9t^2$)

Can a general expression for the speed of such an object after t seconds be found?

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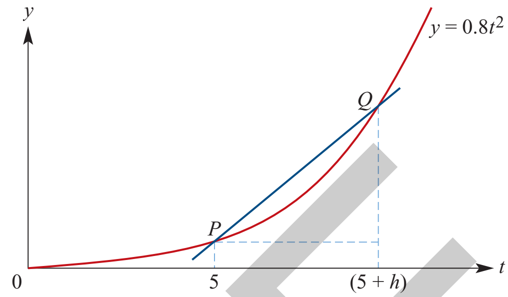
The gradient of chord PQ approximates the speed of the object. The 'shorter' this chord is made, the better the approximation.

Let P be the point on the curve where $t = 5$.

Let Q be the point on the curve corresponding to h seconds after $t = 5$, i.e. Q is the point on the curve where $t = 5 + h$



$$\text{The gradient of chord } PQ = \frac{0.8[(5+h)^2 - 5^2]}{(5+h)-5}$$
$$= \frac{0.8[(5+h)^2 - 5^2]}{h}$$
$$= 0.8(10+h)$$

The table gives the gradient for different values of h. Use your calculator to check these.

h	Gradient of PQ
0.7	8.56
0.6	8.48
0.5	8.40
0.4	8.32
0.3	8.24
0.2	8.16
0.1	8.08

If the values of h are taken to be of smaller and smaller magnitude, it is found that the gradient of chord PQ gets closer and closer to 8. The gradient at the point where $t = 5$ is 8.

Thus the speed of the object at the moment $t = 5$ is 8 m/s.

The speed of the object at the moment $t = 5$ is the limiting value of the gradient of PQ as Q approaches P.

Next, a formula for the speed of the object at any time t is needed. Let P be the point $(t, 0.8t^2)$ on the curve and Q be the point $(t+h, 0.8(t+h)^2)$.

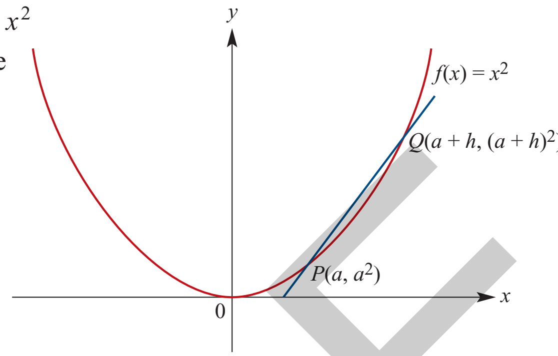
$$\text{The gradient of chord } PQ = \frac{0.8[(t+h)^2 - t^2]}{(t+h)-t}$$
$$= 0.8(2t+h)$$

From this an expression for the speed can be found. Consider the limit as h approaches 0; that is, the value of $0.8(2t+h)$ as h becomes arbitrarily small.

The speed at time t is $1.6t$ metres per second. (The gradient of the curve at the point corresponding to time t is $1.6t$.) Now that a result that gives the speed of an object at any time t has been found, the gradient of similar functions can be investigated.

Consider the function $f: R \to R, f(x) = x^2$
The gradient of the chord PQ in the figure

$$= \frac{(a+h)^2 - a^2}{a+h-a}$$
$$= \frac{a^2 + 2ah + h^2 - a^2}{a+h-a}$$
$$= 2a + h$$



and the gradient at P can be seen to be $2a$. The limit as h approaches 0 of $(2a + h)$ is $2a$. It can be seen that there is nothing special about a. So if x is a real number a similar formula holds.

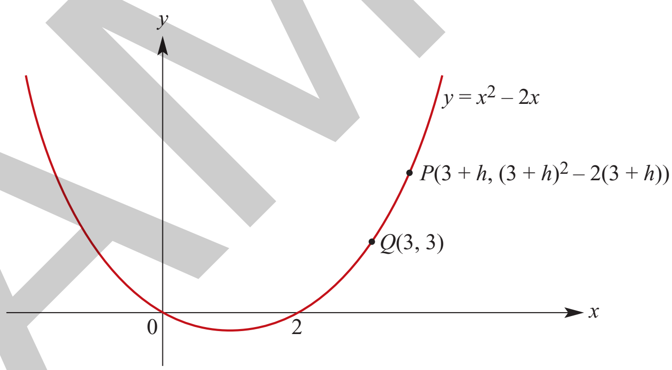
It is said that $2x$ is the **derivative** of x^2 **with respect to** x or, more briefly, the **derivative** of x^2 is $2x$.

The straight line that passes through P and that has gradient $2a$ is said to be the **tangent** to the curve at P.

From the discussion at the beginning of the chapter it was found that the derivative of $0.8t^2$ is $1.6t$.

Example 1

Find the gradient of $y = x^2 - 2x$ at the point Q with coordinates (3, 3).



Solution

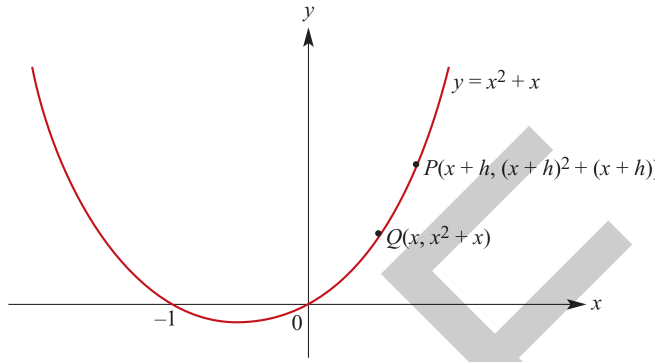
Consider chord PQ.

$$\text{Gradient of chord } PQ = \frac{(3+h)^2 - 2(3+h) - 3}{3+h-3}$$
$$= \frac{9 + 6h + h^2 - 6 - 2h - 3}{3+h-3}$$
$$= \frac{4h + h^2}{h}$$
$$= 4 + h$$

From this it can be deduced that the gradient at the point (3, 3) is 4.

Example 2

Find the derivative of $x^2 + x$ with respect to x.



Solution

$$\begin{aligned}\text{The gradient of chord } PQ &= \frac{(x+h)^2 + (x+h) - (x^2+x)}{x+h-x} \\ &= \frac{x^2 + 2xh + h^2 + x + h - (x^2+x)}{h} \\ &= \frac{2xh + h^2 + h}{h} \\ &= 2x + h + 1\end{aligned}$$

From this it can be seen that the derivative of $x^2 + x$ is $2x + 1$

The notation for limit as h approaches 0 of $2x + h + 1$ is $\lim\limits_{h \to 0} 2x + h + 1$

Example 3

Find $\lim\limits_{h \to 0} \dfrac{(x+h)^3 - x^3}{(x+h) - x}$

Solution

$$\begin{aligned}\lim_{h \to 0} \frac{(x+h)^3 - x^3}{(x+h) - x} &= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3h^2x + h^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3h^2x + h^3}{h} \\ &= \lim_{h \to 0} 3x^2 + 3hx + h^2 \\ &= 3x^2\end{aligned}$$

Note: It has been found that the derivative of x^3 is $3x^2$.

Example 4

Find:

a $\lim\limits_{h \to 0} 22x^2 + 20xh + h$ **b** $\lim\limits_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$ **c** $\lim\limits_{h \to 0} 3x$ **d** $\lim\limits_{h \to 0} 4$

Solution

a $\lim\limits_{h \to 0} 22x^2 + 20xh + h = 22x^2$

b $\lim\limits_{h \to 0} \dfrac{3x^2h + 2h^2}{h} = \lim\limits_{h \to 0} 3x^2 + 2h = 3x^2$

c $\lim\limits_{h \to 0} 3x = 3x$

d $\lim\limits_{h \to 0} 4 = 4$

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Exercise 9A

1 For a curve with equation $y = x^2 + 5x$:

a Find the gradient of the chord PQ where P is the point (2, 14) and Q is the point $(2 + h, (2 + h)^2 + 5(2 + h))$.

b Find the gradient of PQ where $h = 0.1$.

c From the result of **a** find the gradient of the curve at the point (2, 14).

2 Find:

a $\lim\limits_{h \to 0} \dfrac{4x^2h^2 + xh + h}{h}$

b $\lim\limits_{h \to 0} \dfrac{2x^3h - 2xh^2 + h}{h}$

c $\lim\limits_{h \to 0} 40 - 50h$

d $\lim\limits_{h \to 0} 5h$

e $\lim\limits_{h \to 0} 5$

f $\lim\limits_{h \to 0} \dfrac{30h^2x^2 + 20h^2x + h}{h}$

g $\lim\limits_{h \to 0} \dfrac{3h^2x^3 + 2hx + h}{h}$

h $\lim\limits_{h \to 0} 3x$

i $\lim\limits_{h \to 0} \dfrac{3x^3h - 5x^2h^2 + hx}{h}$

j $\lim\limits_{h \to 0} 6x - 7h$

3 For the curve with equation $y = x^3 - x$:

a Find the gradient of the chord PQ where P is the point (1, 0) and Q is the point $((1 + h), (1 + h)^3 - (1 + h))$.

b From the result of **a** find the gradient of the curve at the point (1, 0).

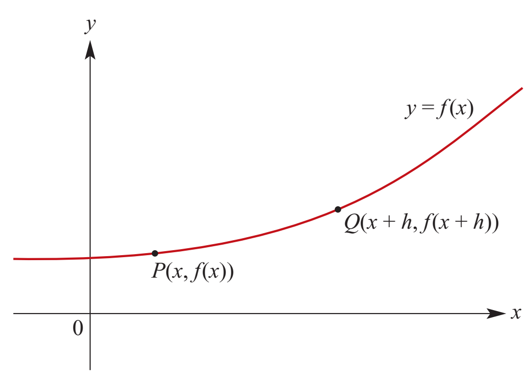
4 If $f(x) = x^2 - 2$, simplify $\dfrac{f(x + h) - f(x)}{h}$. Hence find the derivative of $x^2 - 2$.

5 Let P and Q be points on the curve $y = x^2 + 2x + 5$ at which $x = 2$ and $x = 2 + h$ respectively. Express the gradient of the line PQ in terms of h and hence find the gradient of the curve $y = x^2 + 2x + 5$ at $x = 2$.

9.2 The derived function

In this section, we develop a notation for the derivative of a function.

Consider the graph $y = f(x)$ of the function $f: R \to R$.



The gradient of the chord PQ

$$= \frac{f(x + h) - f(x)}{x + h - x}$$

$$= \frac{f(x + h) - f(x)}{h}$$

Therefore, the gradient of the graph at P is given by:

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

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This limit gives a rule for a new function called the **derived function** (or **derivative function**) and is denoted by f' where:

$$f' \colon R \to R \text{ and } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The function f is said to be **differentiable** at a point $(a, f(a))$ if $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

Example 5

Find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for each of the following:

a $f(x) = 3x^2 + 2x + 2$ **b** $f(x) = x^3 + 2$

Solution

a
$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 + 2(x+h) + 2 - (3x^2 + 2x + 2)}{h} \\
&= \frac{3(x^2 + 2xh + h^2) + 2(x+h) + 2 - (3x^2 + 2x + 2)}{h} \\
&= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 2 - 3x^2 - 2x - 2}{h} \\
&= \frac{6xh + 3h^2 + 2h}{h} = 6x + 3h + 2
\end{aligned}$$

$$\text{Therefore, } \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 6x + 3h + 2 = 6x + 2$$

b
$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^3 + 2 - (x^3 + 2)}{h} \\
&= \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2 - x^3 - 2}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2
\end{aligned}$$

$$\text{Therefore, } \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

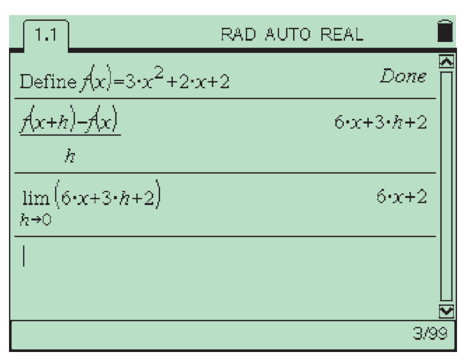
Using the TI-Nspire

Find the derivative of $f(x) = 3x^2 + 2x + 2$ by first principles.

Firstly, **Define** $f(x) = 3x^2 + 2x + 2$

Then calculate the gradient of the chord, $\dfrac{f(x+h) - f(x)}{h}$.

Finally, select **Limit** from the **Calculus** menu (menu 4 3) and complete as shown.



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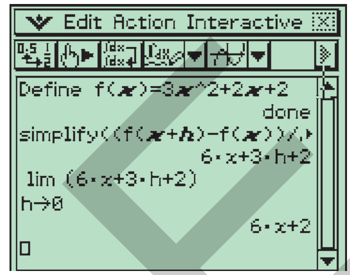
Using the Casio ClassPad

Find the derivative of
$f(x) = 3x^2 + 2x + 2$ by first principles.



Define $f(x)$ using **Interactive** > **Define** and then enter and highlight $(f(x+h) - f(x))/h$ and tap **Interactive** > **Transformation** > **simplify** to find the expression. Copy and paste the answer to a new line, highlight it, and then tap **Interactive** > **Calculation** > **lim**.

Finding $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for a given function is often referred to as finding the derivative of $f(x)$ with respect to x by **first principles**.

In this chapter only polynomial and rational functions are considered. For polynomial functions the derived function always exists and is defined for every number in the domain of f.

In Section 9.10, we discuss the existence of the derivative.

The following can be deduced from the work of the previous section:

- For $f(x) = x^2$, $f'(x) = 2x$
- For $f(x) = x^3$, $f'(x) = 3x^2$
- For $f(x) = x^4$, $f'(x) = 4x^3$
- For $f(x) = 1$, $f'(x) = 0$

This gives the following general result:

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, $n = 1, 2, 3, \ldots$ For $f(x) = 1$, $f'(x) = 0$

From the previous section it can be seen that for k, a constant:

- If $f(x) = kx^n$, then $f'(x) = knx^{n-1}$

If $g(x) = kf(x)$, where k is a constant, then $g'(x) = kf'(x)$

That is, **the derivative of a number multiple is the multiple of the derivative.**

For example, for $g(x) = 5x^2$, the derived function $g'(x) = 5(2x) = 10x$

Another important rule for differentiation is:

If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$

That is, **the derivative of the sum is the sum of the derivatives.**

For example, for $f(x) = x^2 + 2x$ the derived function $f'(x) = 2x + 2$

Example 6

Find the derivative of $x^5 - 2x^3 + 2$

Solution

If $f(x) = x^5 - 2x^3 + 2$ then:

$$\begin{aligned} f'(x) &= 5x^4 - 2(3x^2) + 2(0) \\ &= 5x^4 - 6x^2 \end{aligned}$$

Example 7

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and $f'(1)$

Solution

$$\begin{aligned} f'(x) &= 3(3x^2) - 6(2x) + 1(0) \\ &= 9x^2 - 12x \\ f'(1) &= 9 - 12 \\ &= -3 \end{aligned}$$

Example 8

Find the gradient of the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$ at the point $(1, -2)$.

Solution

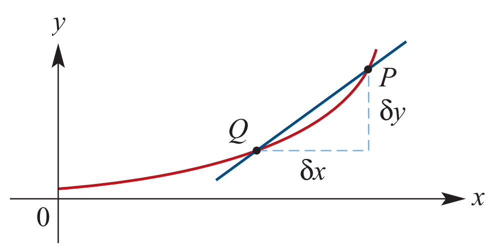
Now $f'(x) = 9x^2 - 12x$ and $f'(1) = 9 - 12 = -3$

The gradient of the curve is -3 at the point $(1, -2)$.

An alternative notation for the derivative is the following:

- If $y = x^3$, then the derivative can be denoted by $\dfrac{dy}{dx}$, so that $\dfrac{dy}{dx} = 3x^2$

In general, if y is a function of x, the derivative of y with respect to x is denoted $\dfrac{dy}{dx}$ and, with the use of a different symbol z, where z is a function of t, the derivative of z with respect to t is $\dfrac{dz}{dt}$.



In this notation d is not a factor and cannot be cancelled. This came about because in the eighteenth century the standard diagram for finding the limiting gradient was labelled as in the figure shown. ('δ' is the lower case Greek letter for 'd', and is pronounced *delta*.)

'δx' means a difference in x.

'δy' means a difference in y.

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For example:

- If $x = t^3 + t$, then $\frac{dx}{dt} = 3t^2 + 1$
- If $y = t^2$, then $\frac{dy}{dt} = 2t$
- If $y = \frac{1}{3}x^3 + x^2$, then $\frac{dy}{dx} = x^2 + 2x$

Example 9

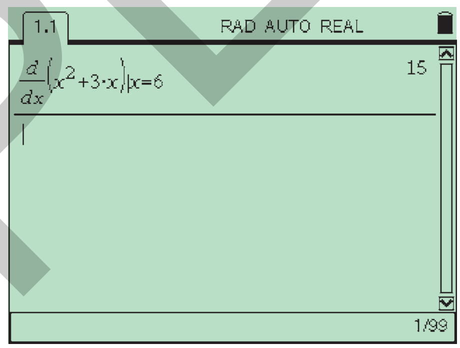
Use a CAS calculator to find the gradient of $f(x) = x^2 + 3x$ at the point on the graph where $x = 6$.

Solution

Using the TI-Nspire

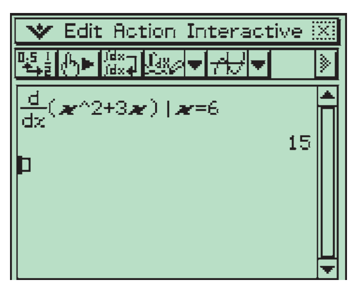
Find the derivative and evaluate for $x = 6$.

Select **Derivative** from the **Calculus** menu (menu 4 1) and complete as shown.



Using the Casio ClassPad

Press Keyboard and select the 2D CALC menu screen. Tap and 2D. Enter the derivative with respect to x 2D as shown. The | symbol is found in mth OPTN menu screen.



Example 10

The planned path for a flying saucer leaving a planet is defined by the equation:

$$y = \frac{1}{4}x^4 + \frac{2}{3}x^3 \text{ for } x > 0$$

The units are kilometres. (The x-axis is horizontal, and the y-axis vertical.)

a What will be the direction of motion when the x-value is:
i 2? **ii** 3?

b Find a point on the path where the flying saucer's path is inclined at 45° to the positive x-axis.

c Are there any other points on the path that satisfy the situation described in **b**?

Solution

a $\frac{dy}{dx} = x^3 + 2x^2$

i When $x = 2$, $\frac{dy}{dx} = 8 + 8$
$= 16$
$(\tan^{-1} 16)^\circ = 86.42^\circ$ (to the x-axis)

ii When $x = 3$, $\frac{dy}{dx} = 27 + 18$
$= 45$
$(\tan^{-1} 45)^\circ = 88.73^\circ$ (to the x-axis)

b, c When the flying saucer is flying at 45° to the direction of the x-axis, the gradient of the curve of its path is given by tan 45°.

Thus to find the point at which this happens we consider the equation:

$$\frac{dy}{dx} = \tan 45^\circ$$
$$\therefore x^3 + 2x^2 = 1$$
$$\therefore x^3 + 2x^2 - 1 = 0$$
$$\therefore (x+1)(x^2 + x - 1) = 0$$
$$\therefore x = -1 \text{ or } x = \frac{-1 \pm \sqrt{5}}{2}$$

The only acceptable solution is $x = \frac{-1+\sqrt{5}}{2}$ ($x = 0.62$), as the other two possibilities give negative values for x and we are only considering positive values for x.

Example 11

a For $y = (x+3)^2$, find $\frac{dy}{dx}$.

b For $z = (2t-1)^2(t+2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2 + 3x}{x}$, find $\frac{dy}{dx}$.

Solution

a It is first necessary to write $y = (x + 3)^2$ in expanded form.

$$\therefore y = x^2 + 6x + 9$$
$$\text{and } \frac{dy}{dx} = 2x + 6$$

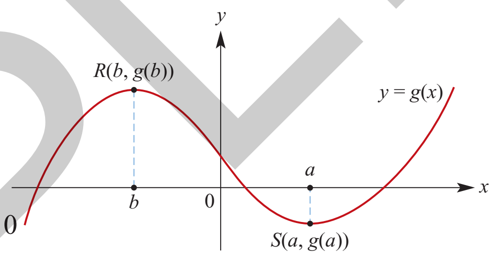
b First expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t + 2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \\ \text{and } \frac{dz}{dt} &= 12t^2 + 8t - 7 \end{aligned}$$

c First dividing by x:

$$y = x + 3$$
$$\therefore \frac{dy}{dx} = 1$$

Gradients can, of course, be negative or zero. They are not always positive. At a point $(a, g(a))$ of the graph $y = g(x)$ the gradient is $g'(a)$.



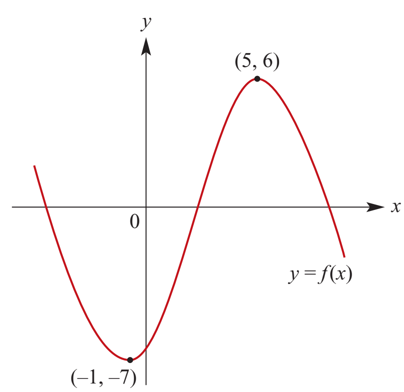
Some features of the graph are:

- For $x < b$ the gradient is positive, i.e. $g'(x) > 0$
- For $x = b$ the gradient is zero, i.e. $g'(b) = 0$
- For $b < x < a$ the gradient is negative, i.e. $g'(x) < 0$
- For $x = a$ the gradient is zero, i.e. $g'(a) = 0$
- For $x > a$ the gradient is positive, i.e. $g'(x) > 0$

Example 12

For the graph of $f: R \to R$ find:

a $\{x: f'(x) > 0\}$
b $\{x: f'(x) < 0\}$
c $\{x: f'(x) = 0\}$



Solution

a $\{x: f'(x) > 0\} = \{x: -1 < x < 5\} = (-1, 5)$
b $\{x: f'(x) < 0\} = \{x: x < -1\} \cup \{x: x > 5\} = (-\infty, -1) \cup (5, \infty)$
c $\{x: f'(x) = 0\} = \{-1, 5\}$

Exercise 9B

Skillsheet TEACHER

1 In each of the following, find $f'(x)$ by finding $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$:

a $f(x) = 5x^2$ **b** $f(x) = 3x + 2$ **c** $f(x) = 5$
d $f(x) = 3x^2 + 4x + 3$ **e** $f(x) = 5x^3 - 5$ **f** $f(x) = 5x^2 - 6x$

2 For each of the following, find $f'(x)$:

a $f(x) = x^5$ **b** $f(x) = 4x^7$

c $f(x) = 6x$ **d** $f(x) = 5x^2 - 4x + 3$

e $f(x) = 4x^3 + 6x^2 + 2x - 4$ **f** $f(x) = 5x^4 + 3x^3$

g $f(x) = -2x^2 + 4x + 6$ **h** $f(x) = 6x^3 - 2x^2 + 4x - 6$

3 For each of the following, find $\dfrac{dy}{dx}$:

a $y = -2x$ **b** $y = 7$

c $y = 5x^3 - 3x^2 + 2x + 1$ **d** $y = \dfrac{2}{5}(x^3 - 4x + 6)$

e $y = (2x + 1)(x - 3)$ **f** $y = 3x(2x - 4)$

g $y = \dfrac{10x^7 + 2x^2}{x^2}, x \neq 0$ **h** $y = \dfrac{9x^4 + 3x^2}{x}, x \neq 0$

4 **a** Differentiate $y = (2x - 1)^2$ with respect to x.

b For $y = \dfrac{x^3 + 2x^2}{x}, x \neq 0$, find $\dfrac{dy}{dx}$.

c Given that $y = 2x^3 - 6x^2 + 18x$, find $\dfrac{dy}{dx}$.

Hence show that $\dfrac{dy}{dx} > 0$ for all x.

d Given that $y = \dfrac{x^3}{3} - x^2 + x$, find $\dfrac{dy}{dx}$. Hence show that $\dfrac{dy}{dx} \geq 0$ for all x.

5 At the points on the following curves corresponding to given values of x, find the y-coordinate and the gradient:

a $y = x^2 + 2x + 1, x = 3$ **b** $y = x^2 - x - 1, x = 0$

c $y = 2x^2 - 4x, x = -1$ **d** $y = (2x + 1)(3x - 1)(x + 2), x = 4$

e $y = (2x + 5)(3 - 5x)(x + 1), x = 1$ **f** $y = (2x - 5)^2, x = 2\frac{1}{2}$

6 For the function, $f(x) = 3(x - 1)^2$, find the value(s) of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$

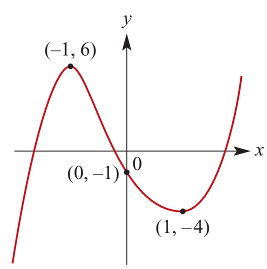
d $f'(x) < 0$ **e** $f'(x) = 10$ **f** $f(x) = 27$

7 For the graph of $y = h(x)$ illustrated, find:

a $\{x: h'(x) > 0\}$

b $\{x: h'(x) < 0\}$

c $\{x: h'(x) = 0\}$

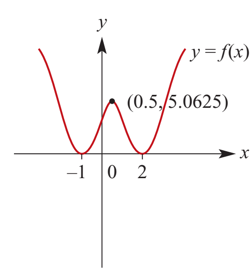


8 For the graph of $y = f(x)$ shown, find:

a $\{x: f'(x) > 0\}$

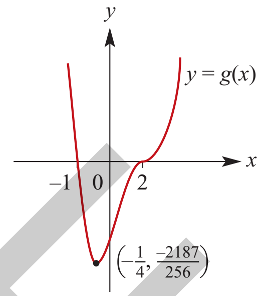
b $\{x: f'(x) < 0\}$

c $\{x: f'(x) = 0\}$



9 For the graph of $y = g(x)$ shown, find:

a $\{x: g'(x) > 0\}$
b $\{x: g'(x) < 0\}$
c $\{x: g'(x) = 0\}$



10 Find the coordinates of the parabola $y = x^2 - 2x - 8$ at which:

a the gradient is zero
b the tangent is parallel to $y = 2x + 6$
c the tangent is parallel to $3x + 2y = 8$

11 Find the coordinates of the points on the curves given by the following equations at which the gradient has the given values:

a $y = 2x^2 - 4x + 1$; gradient $= -6$
b $y = 4x^3$; gradient $= 48$
c $y = x(5 - x)$; gradient $= 1$
d $y = x^3 - 3x^2$; gradient $= 0$

9.3 Differentiating x^n where n is a negative integer

In this section, we add new functions to the family of functions for which we can find the derived functions. In particular, we consider functions that involve linear combinations of powers of x where a power may be a negative integer, for example:

- $f: R\backslash\{0\} \to R,\ f(x) = x^{-1}$
- $f: R\backslash\{0\} \to R,\ f(x) = 2x + x^{-1}$
- $f: R\backslash\{0\} \to R,\ f(x) = x + 3 + x^{-2}$

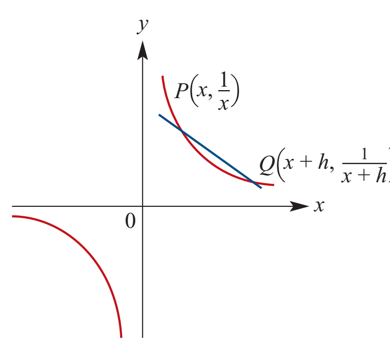
Note that we are reintroducing the function notation. This notation emphasises the need for consideration of domain.

Example 13

Let $f: R\backslash\{0\} \to R,\ f(x) = \frac{1}{x}$. Find $f'(x)$ using the definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Solution

$$\text{Gradient of chord } PQ = \frac{f(x+h) - f(x)}{x + h - x}$$
$$= \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
$$= \frac{x - (x+h)}{(x+h)x} \times \frac{1}{h}$$
$$= \frac{-h}{(x+h)x} \times \frac{1}{h}$$
$$= \frac{-1}{(x+h)x}$$

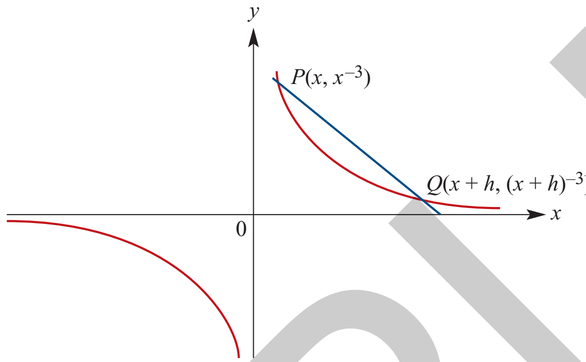


The gradient of the curve at $P = \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2} = -x^{-2}$ and $f'(x) = -x^{-2}$

Example 14

Let $f\colon R\backslash\{0\} \to R$, $f(x) = x^{-3}$. Find $f'(x)$ using the definition
$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

Solution



$$\begin{aligned}
\text{Gradient of chord } PQ &= \frac{(x+h)^{-3} - x^{-3}}{h} \\
&= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h} \\
&= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h} \\
&= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h} \\
&= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}
\end{aligned}$$

$$\begin{aligned}
\text{The gradient of the curve at } P &= \lim_{h\to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} \\
&= \frac{-3x^2}{x^6} \\
&= \frac{-3}{x^4} \\
&= -3x^{-4} \text{ and } f'(x) = -3x^{-4}
\end{aligned}$$

The generalisation of the result found in Section 9.2 can now be stated:

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, n a non-zero integer.
For $f(x) = 1$, $f'(x) = 0$
Note that for $n \leq -1$, the domain of f can be taken to be $R\backslash\{0\}$ and for $n \geq 1$ we take the domain of f to be R.

Example 15

Find the derivative of $x^4 - 2x^{-3} + x^{-1} + 2$, $x \neq 0$

Solution

If $f(x) = x^4 - 2x^{-3} + x^{-1} + 2, x \neq 0$

$$f'(x) = 4x^3 - 2(-3x^{-4}) + (-x^2) + 2(0)$$
$$= 4x^3 + 6x^{-4} - x^{-2}, x \neq 0$$

Example 16

Find the derivative f' of $f: R\backslash\{0\} \to R,\ f(x) = 3x^2 - 6x^{-2} + 1$

Solution

$$f': R\backslash\{0\} \to R,\ f'(x) = 3(2x) - 6(-2x^{-3}) + 1(0)$$
$$= 6x + 12x^{-3}$$

Example 17

Find the gradient of the curve determined by the function $f: R\backslash\{0\},\ f(x) = x^2 + \dfrac{1}{x}$ at the point (1, 2).

Solution

$$f': R\backslash\{0\} \to R,\ f'(x) = 2x + (-x^{-2})$$
$$= 2x - x^{-2}$$
$$f'(1) = 2 - 1$$
$$= 1$$

The gradient of the curve is 1 at the point (1, 2).

Exercise 9C

1 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ b $\dfrac{5}{x^3} + 6x^2$ c $\dfrac{-5}{x^3} + \dfrac{4}{x^2} + 1$

d $6x^{-3} + 3x^{-2}$ e $\dfrac{4x^2 + 2x}{x^2}$

2 Find the derivative of each of the following:

a $\dfrac{2z^2 - 4z}{z^2}, z \neq 0$ b $\dfrac{6 + z}{z^3}, z \neq 0$ c $16 - z^{-3}, z \neq 0$

d $\dfrac{4z + z^3 - z^4}{z^2}, z \neq 0$ e $\dfrac{6z^2 - 2z}{z^4}, z \neq 0$ f $\dfrac{6}{x} - 3x^2, x \neq 0$

3 Find the x-coordinates of the points on the curve $y = \dfrac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

4 Given that the curve $y = ax^2 + \dfrac{b}{x}$ has a gradient of –5 at the point (2, −2), find the value of a and b.

5 Find the gradient of the curve $y = \dfrac{2x-4}{x^2}$ at the point where the curve crosses the x-axis.

6 The gradient of the curve $y = \dfrac{a}{x} + bx^2$ at the point (3, 6) is 7. Calculate the values of a and b.

7 For the curve with equation $y = \dfrac{5}{3}x + kx^2 - \dfrac{8}{9}x^3$, calculate the possible values of k such that the tangents at the points with x-coordinates 1 and $-\dfrac{1}{2}$ respectively are perpendicular.

8 Find the gradient of each of the following curves at the stated point:

a $y = x^{-2} + x^3, x \neq 0$ at the point $\left(2,\ 8\dfrac{1}{4}\right)$

b $y = x^{-2} - \dfrac{1}{x}, x \neq 0$ at the point $\left(4, \dfrac{1}{2}\right)$

c $y = x^{-2} - \dfrac{1}{x}, x \neq 0$, at the point (1, 0)

d $y = x(x^{-1} + x^2 - x^{-3}), x \neq 0$ at the point (1, 1)

9 **a** Sketch the graph of $f\colon R\backslash\{0\} \to R,\ f(x) = \dfrac{2}{x^2}$

b Let P be the point (1, 2) and Q the point $(1+h,\ f(1+h))$. Find the gradient of chord PQ.

c Hence find the gradient of the curve $f(x) = \dfrac{2}{x^2}$ at (1, 2).

9.4 The chain rule

An expression such as $q(x) = (x^3 + 1)^2$ may be differentiated by expanding and then differentiating each term separately. This method is a great deal more tiresome for an expression such as $q(x) = (x^3 + 1)^{30}$

We express $q(x) = (x^3 + 1)^2$ as the composition of two simpler functions defined by:

$$g(x) = x^3 + 1\ (= u) \text{ and } f(u) = u^2 (= y)$$

which are ‘chained’ together as:

$$\boldsymbol{x} \xrightarrow{g} \boldsymbol{u} \xrightarrow{f} \boldsymbol{y}$$

That is, $q(x) = (x^3 + 1)^2$ is expressed as the composition of two functions f and g.

Thus $q = f \circ g$ and $q(x) = f(g(x))$

The chain rule gives a method of differentiating such functions. The chain rule states:

If $q(x) = f(g(x))$, and g is differentiable at x, and f is differentiable at $g(x)$, then the derivative of q exists and

$$q'(x) = f'(g(x))g'(x)$$

Or in the notation of Liebniz, where as above $u = g(x) = x^3 + 1$ and $y = u^2$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Proof of the chain rule

For the derivative of $f \circ g$ where $x = a$, consider the gradient of the secant:

$$\frac{f \circ g(a+h) - f \circ g(a)}{h} = \frac{f(g(a+h)) - f(g(a))}{h} \times \frac{g(a+h) - g(a)}{g(a+h) - g(a)}, \text{ provided } g(a+h) - g(a) \neq 0$$

Now write $b = g(a)$ and $b + k = g(a+h)$ so that $k = g(a+h) - g(a)$

The expression becomes:

$$\frac{f(b+k) - f(b)}{k} \times \frac{g(a+h) - g(a)}{h}$$

Since g is differentiable at a, g is also continuous at a and so:

$$\lim_{h \to 0} k = \lim_{h \to 0} [g(a+h) - g(a)] = 0$$

Thus as h tends to 0 so does k.

And thus $q'(x) = f'(g(x))g'(x)$.

Note that this proof does not hold for a function g such that $g(a+h) - g(a) = 0$ for arbitrarily chosen small h.

Example 18

Differentiate $y = (4x^3 - 5x)^{-2}$

Solution

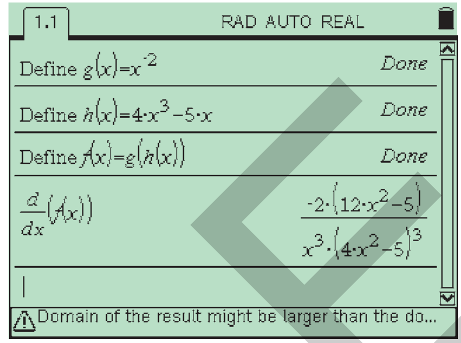
The differentiation is undertaken using both notations.

$$\begin{aligned}
\text{Let } u &= 4x^3 - 5x \\
\text{Then } y &= u^{-2} \\
\text{We have } \frac{dy}{du} &= -2u^{-3} \\
\text{and } \frac{du}{dx} &= 12x^2 - 5 \\
\therefore \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= -2u^{-3} \cdot (12x^2 - 5) \\
&= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}
\end{aligned}$$

$$\begin{aligned}
\text{Let } f(x) &= g(h(x)) \\
\text{where } h(x) &= 4x^3 - 5x \\
\text{and } g(x) &= x^{-2} \\
\text{Then } f'(x) &= g'(h(x))h'(x),\ g'(x) = -2x^{-3} \\
\text{and } h'(x) &= 12x^2 - 5 \\
\text{Then } g'(h(x)) &= -2(h(x))^{-3} = -2(4x^3 - 5x)^{-3} \\
\text{Therefore } f'(x) &= -2(4x^3 - 5x)^{-3} \times (12x^2 - 5) \\
&= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}
\end{aligned}$$

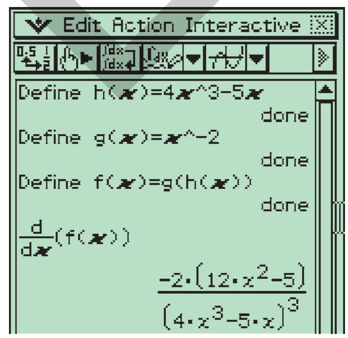
Using the TI-Nspire

Define $g(x)$ and $h(x)$
Define $f(x) = g(h(x))$
Select **Derivative** from the **Calculus** menu (menu 4 1) and complete as shown.



Using the Casio ClassPad

Define $g(x)$ and $h(x)$, then define $f(x) = g(h(x))$ and take the derivative of $f(x)$.



Example 19

Find the gradient of the curve with equation $y = \dfrac{16}{3x^2 + 1}$ at the point (1, 4).

Solution

Let $u = 3x^2 + 1$. Then $y = 16u^{-1}$

So $\dfrac{du}{dx} = 6x$ and $\dfrac{dy}{dx} = -16u^{-2}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -16u^{-2} \cdot 6x \\ &= \frac{-96x}{(3x^2 + 1)^2}\end{aligned}$$

$\therefore$ at $x = 1$ the gradient is $\dfrac{-96}{16} = -6$

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Example 20

Find the derivative of $y = (3x + 4)^{20}$

Solution

Let $u = 3x + 4$. Then $y = u^{20}$

So $\dfrac{du}{dx} = 3$ and $\dfrac{dy}{du} = 20u^{19}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 20u^{19} \cdot 3 \\ &= 60(3x + 4)^{19}\end{aligned}$$

Example 21

Use the chain rule to prove $\dfrac{dy}{dx} = nx^{n-1}$ for $y = x^n$ where n is a negative integer.
(Assume the result for n a positive integer and $x > 0$.)

Solution

Let n be a negative integer and $y = x^n$. Then $y = \dfrac{1}{x^{-n}}$ and $-n$ is a positive integer.

Let $u = x^{-n}$. Then $y = \dfrac{1}{u} = u^{-1}$

Thus $\dfrac{dy}{du} = -u^{-2}$ and $\dfrac{du}{dx} = -nx^{-n-1}$ ($-n$ is a positive integer)

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -u^{-2} \cdot (-nx^{-n-1}) \\ &= nx^{-n-1}(x^{-n})^{-2} \\ &= nx^{-n-1}(x^{2n}) \\ &= nx^{n-1}\end{aligned}$$

Example 22

Given that $f(x) = (x^2 + 1)^3$, find $f'(x)$.

Solution

Now $f = k \circ g$ where $k(x) = x^3$ and $g(x) = x^2 + 1$

It follows that $k'(x) = 3x^2$ and $g'(x) = 2x$

Since by the chain rule $f'(x) = k'(g(x))g'(x)$ we have in this case

$$f'(x) = 3(g(x))^2\, 2x$$

which yields $f'(x) = 6x(x^2 + 1)^2$

Exercise 9D

1 Differentiate each of the following with respect to x:

a $(x^2+1)^4$ **b** $(2x^2-3)^5$ **c** $(6x+1)^4$ **d** $(ax+b)^n$

e $(ax^2+b)^n$ **f** $(1-x^2)^{-3}$ **g** $\left(x^2-\dfrac{1}{x^2}\right)^{-3}$ **h** $(1-x)^{-1}$

2 Differentiate each of the following with respect to x:

a $(x^2+2x+1)^3$ **b** $(x^3+2x^2+x)^4$ **c** $\left(6x^3+\dfrac{2}{x}\right)^4$ **d** $(x^2+2x+1)^{-2}$

3 Differentiate each of the following with respect to x, giving the answer in terms of $f(x)$ and $f'(x)$:

a $[f(x)]^n$, where n is a positive integer **b** $\dfrac{1}{f(x)}$ where $f(x)\neq 0$

9.5 Differentiating rational powers $\left(x^{\frac{p}{q}}\right)$

Using the chain rule in the form $\dfrac{dy}{du}=\dfrac{dy}{dx}\cdot\dfrac{dx}{du}$

with $y=u$ we have $1=\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}$

and thus

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}},\ \text{for } \frac{dx}{dy}\neq 0$$

Let $y=x^{\frac{1}{n}}$ where $n\in Z\backslash\{0\}$ and $x>0$

Then $y^n=x$ and $\dfrac{dx}{dy}=ny^{n-1}$

From the above results we have

$$\begin{aligned}\frac{dy}{dx}&=\frac{1}{ny^{n-1}}\\&=\frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}}\\&=\frac{1}{n}x^{\frac{1}{n}-1}\end{aligned}$$

$$\text{For } y=x^{\frac{1}{n}},\ \frac{dy}{dx}=\frac{1}{n}x^{\frac{1}{n}-1};\ n\in Z\backslash\{0\} \text{ and } x>0$$

If n is odd then the result is also true for $x<0$, but for both odd and even n the result does not hold when $x=0$.

This result may now be extended to any rational power.

Let $y = x^{\frac{p}{q}}$, where $p, q \in Z\backslash\{0\}$.

We write $y = \left(x^{\frac{1}{q}}\right)^p$. Let $u = x^{\frac{1}{q}}$. Then $y = u^p$

The chain rule yields
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= pu^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1}\\ &= p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1}\\ &= \frac{p}{q}x^{\frac{p}{q}-\frac{1}{q}}x^{\frac{1}{q}-1}\\ &= \frac{p}{q}x^{\frac{p}{q}-1}\end{aligned}$$

If q is odd then the result is also true for $x < 0$, but for both odd and even q the result does not hold when $x = 0$.

Thus we have the result for any non-zero rational power and, in fact, it is true for any non-zero real power:

for $f(x) = x^a$, $f'(x) = ax^{a-1}$, for $x > 0$ and $a \in R$

Example 23

Find the derivative of each of the following with respect to x:

a $2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}$ **b** $\sqrt[3]{x^2 + 2x}$

Solution

a
$$\begin{aligned}\frac{d\left(2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}\right)}{dx} &= 2 \times \frac{-1}{5}x^{-\frac{6}{5}} + 3 \times \frac{2}{7}x^{-\frac{5}{7}}\\ &= -\frac{2}{5}x^{-\frac{6}{5}} + \frac{6}{7}x^{-\frac{5}{7}}\end{aligned}$$

b
$$\begin{aligned}\frac{d(\sqrt[3]{x^2 + 2x})}{dx} &= \frac{d\left((x^2 + 2x)^{\frac{1}{3}}\right)}{dx}\\ &= (2x + 2) \times \frac{1}{3}(x^2 + 2x)^{-\frac{2}{3}} \qquad \text{(chain rule)}\\ &= \frac{2x + 2}{3\sqrt[3]{(x^2 + 2x)^2}}\end{aligned}$$

Exercise 9E

1 Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{5}}$ **b** $x^{\frac{5}{2}}$ **c** $x^{\frac{5}{2}} - x^{\frac{3}{2}}, x > 0$

d $3x^{\frac{1}{2}} - 4x^{\frac{5}{3}}$ **e** $x^{-\frac{6}{7}}$ **f** $x^{-\frac{1}{4}} + 4x^{\frac{1}{2}}$

2 Find the gradient of each of the following at the stated value for x:

a $f(x) = x^{\frac{1}{3}}$ where $x = 27$ **b** $f(x) = x^{\frac{1}{3}}$ where $x = -8$

c $f(x) = x^{\frac{2}{3}}$ where $x = 27$ **d** $f(x) = x^{\frac{5}{4}}$ when $x = 16$

3 Find the derivative of each of the following with respect to x:

a $\sqrt{2x+1}$ **b** $\sqrt{4-3x}$ **c** $\sqrt{x^2+2}$

d $\sqrt[3]{4-3x}$ **e** $\dfrac{x^2+2}{\sqrt{x}}$ **f** $3\sqrt{x}(x^2+2x)$

4 **a** Show that $\dfrac{d}{dx}(\sqrt{x^2 \pm a^2}) = \dfrac{x}{\sqrt{x^2 \pm a^2}}$

b Show that $\dfrac{d}{dx}(\sqrt{a^2 - x^2}) = \dfrac{-x}{\sqrt{a^2 - x^2}}$

5 If $y = (x + \sqrt{x^2+1})^2$, show $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{x^2+1}}$

6 Find the derivative with respect to x of each of the following:

a $\sqrt{x^2+2}$ **b** $\sqrt[3]{x^3-5x}$ **c** $\sqrt[5]{x^2+2x}$

9.6 Product rule

In the next two sections, we introduce two more rules for differentiation. The first of these is the **product rule**.

Let $F(x) = f(x) \cdot g(x)$

If $f'(x)$ and $g'(x)$ exist, then $F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

For example, if $F(x) = (x^2 + 3x)(4x + 5)$, F can be considered as the product of two functions f and g where $f(x) = x^2 + 3x$ and $g(x) = 4x + 5$

The product rule gives:

$$\begin{aligned}F'(x) &= f(x) \cdot g'(x) + g(x) \cdot f'(x)\\ &= (x^2+3x) \cdot 4 + (4x+5) \cdot (2x+3)\\ &= 4x^2 + 12x + 8x^2 + 22x + 15\\ &= 12x^2 + 34x + 15\end{aligned}$$

This could also have been achieved by first multiplying $x^2 + 3x$ by $4x + 5$ and then differentiating.

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Proof of the product rule

By the definition of the derivative of F, we have:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

Adding and subtracting $f(x+h) \cdot g(x)$:

$$F'(x) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x) + [f(x+h) \cdot g(x) - f(x+h) \cdot g(x)]}{h}$$

Regrouping:

$$F'(x) = \lim_{h \to 0} \frac{[f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)]}{h}$$
$$F'(x) = \lim_{h \to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$
$$= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h}\right] + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right]$$

and, since f and g are differentiable:

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

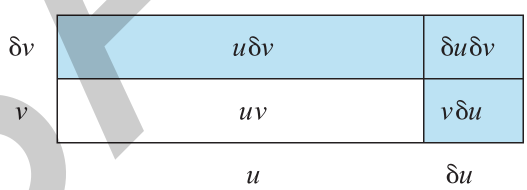
The product rule is restated:

> If $F(x) = f(x) \cdot g(x)$, then
> $F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

The product rule may also be stated in Leibniz notation and a geometric interpretation is given.

If $y = uv$ where u and v are functions of x

> $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$



In the figure above, $y = uv$, δy is the shaded area, as explained below.

$$\therefore \delta y = (u + \delta u)(v + \delta v) - uv$$
$$= uv + v\delta u + u\delta v + \delta u\delta v - uv$$
$$= v\delta u + u\delta v + \delta u\delta v$$
$$\therefore \frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x$$

When $\delta x \to 0$,

$$\frac{\delta v}{\delta x} = \frac{dv}{dx}, \frac{\delta u}{\delta x} = \frac{du}{dx} \text{ and } \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example 24

Differentiate each of the following with respect to x:

a $(2x^2 + 1)(5x^3 + 16)$ **b** $x^3(3x - 5)^4$

Solution

a Let $y = (2x^2 + 1)(5x^3 + 16)$

Let $u = 2x^2 + 1$ and $v = 5x^3 + 16$

Then $\frac{du}{dx} = 4x$ and $\frac{dv}{dx} = 15x^2$

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (2x^2 + 1) \cdot 15x^2 + (5x^3 + 16) \cdot 4x\\ &= 30x^4 + 15x^2 + 20x^4 + 64x\\ &= 50x^4 + 15x^2 + 64x\end{aligned}$$

b Let $y = x^3(3x - 5)^4$

Let $u = x^3$ and $v = (3x - 5)^4$

Then $\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = 12(3x - 5)^3$ (using the chain rule).

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= 12x^3(3x - 5)^3 + (3x - 5)^4 \cdot 3x^2\\ &= (3x - 5)^3[12x^3 + 3x^2(3x - 5)]\\ &= (3x - 5)^3[12x^3 + 9x^3 - 15x^2]\\ &= (3x - 5)^3(21x^3 - 15x^2)\\ &= 3x^2(7x - 5)(3x - 5)^3\end{aligned}$$

Example 25

For $F\colon R\backslash\{0\} \to R$, $F(x) = x^{-3}(10x^2 - 5)^3$, find $F'(x)$.

Solution

Let $f(x) = x^{-3}$ and $g(x) = (10x^2 - 5)^3$

Then $f'(x) = -3x^{-4}$ and $g'(x) = 60x(10x^2 - 5)^2$ (chain rule)

$$\begin{aligned}\therefore F'(x) &= x^{-3} \times 60x(10x^2 - 5)^2 + (10x^2 - 5)^3 \cdot -3x^{-4} \\ &= (10x^2 - 5)^2[60x^{-2} + (10x^2 - 5) \cdot -3x^{-4}] \\ &= (10x^2 - 5)^2\left[\frac{60x^2 - 30x^2 + 15}{x^4}\right] \\ &= \frac{(10x^2 - 5)^2(30x^2 + 15)}{x^4}\end{aligned}$$

Exercise 9F

Find the derivative of each of the following with respect to x, using the product rule:

1 $(2x^2 + 6)(2x^3 + 1)$ **2** $3x^{\frac{1}{2}}(2x + 1)$ **3** $3x(2x - 1)^3$

4 $4x^2(2x^2 + 1)^2$ **5** $(3x + 1)^{\frac{3}{2}}(2x + 4)$ **6** $(x^2 + 1)\sqrt{2x - 4}$

7 $x^3(3x^2 + 2x + 1)^{-1}$ **8** $x^4\sqrt{2x^2 - 1}$ **9** $x^2\sqrt[3]{x^2 + 2x}$

9.7 Quotient rule

Let $F(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist then:

$$F'(x) = \frac{g(x) . f'(x) - f(x) . g'(x)}{[g(x)]^2}$$

For example, if $F(x) = \dfrac{x^3 + 2x}{x^5 + 2}$, we see that F can be considered as a quotient of two functions f and g where $f(x) = x^3 + 2x$ and $g(x) = x^5 + 2$

The quotient rule gives:

$$\begin{aligned}F'(x) &= \frac{(x^5 + 2)(3x^2 + 2) - (x^3 + 2x)5x^4}{(x^5 + 2)^2} \\ &= \frac{3x^7 + 6x^2 + 2x^5 + 4 - 5x^7 - 10x^5}{(x^5 + 2)^2} \\ &= \frac{-2x^7 - 8x^5 + 6x^2 + 4}{(x^5 + 2)^2}\end{aligned}$$

Proof of the quotient rule

By the definition of derivative of F, we have:

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right] \times \frac{1}{h} \\
&= \lim_{h\to 0} \frac{f(x+h).g(x) - f(x).g(x+h)}{g(x+h)g(x)} \times \frac{1}{h}
\end{aligned}
$$

Adding and subtracting $f(x)\cdot g(x)$ in the numerator:

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x) - f(x)\cdot g(x)}{g(x+h)\cdot g(x)} \times \frac{1}{h} \\
&= \lim_{h\to 0} \left(\frac{g(x)[f(x+h)-f(x)]}{h} - \frac{f(x)[g(x+h)-g(x)]}{h}\right) \times \frac{1}{g(x+h)g(x)} \\
&= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}
\end{aligned}
$$

The quotient rule may be also stated in Leibniz notation:

> If $y = \dfrac{u}{v}$, $v \neq 0$ where u and v are functions of x
>
> $$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Example 26

Find the derivative of $\dfrac{x-2}{x^2+4x+1}$ with respect to x.

Solution

Let $y = \dfrac{x-2}{x^2+4x+1}$

The quotient rule gives:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{x^2+4x+1-(2x+4)(x-2)}{(x^2+4x+1)^2} \\
&= \frac{x^2+4x+1-(2x^2-8)}{(x^2+4x+1)^2} \\
&= \frac{-x^2+4x+9}{(x^2+4x+1)^2}
\end{aligned}
$$

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Exercise 9G

1 Find the derivative of each of the following with respect to x:

a $\dfrac{x}{x+4}$ b $\dfrac{x^2-1}{x^2+1}$ c $\dfrac{x^{\frac{1}{2}}}{1+x}$ d $\dfrac{(x+2)^3}{x^2+1}$

e $\dfrac{x-1}{x^2+2}$ f $\dfrac{x^2+1}{x^2-1}$ g $\dfrac{3x^2+2x+1}{x^2+x+1}$ h $\dfrac{2x+1}{2x^3+2x}$

2 Find the y-coordinate and the gradient at points on each of the following curves corresponding to the given value of x:

a $y = (2x+1)^4x^2;\ x = 1$ b $y = x^2\sqrt{x+1};\ x = 0$

c $y = x^2(2x+1)^{\frac{1}{2}};\ x = 0$ d $y = \dfrac{x}{x^2+1};\ x = 1$

e $y = \dfrac{2x+1}{x^2+1};\ x = 1$

3 For each of the following, find $f'(x)$:

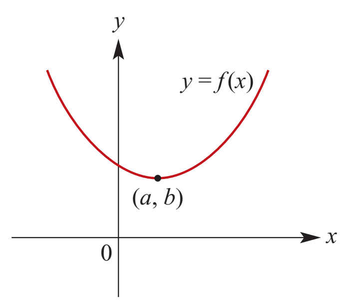
a $f(x) = (x+1)\sqrt{x^2+1}$ b $f(x) = (x^2+1)\sqrt{x^3+1},\ x > -1$

c $f(x) = \dfrac{2x+1}{x+3}$

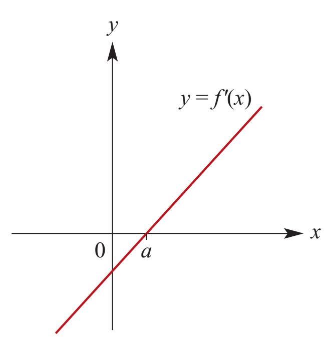
9.8 The graph of the gradient function

The graph of the derivative function is also considered in Section 9.10 where differentiability is considered. Consider the quadratic function with rule $y = f(x)$. The vertex is at the point with coordinates (a, b).

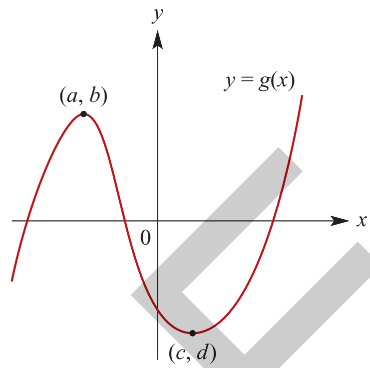
For $x < a,\ f'(x) < 0$
For $x = a,\ f'(x) = 0$
For $x > a,\ f'(x) > 0$



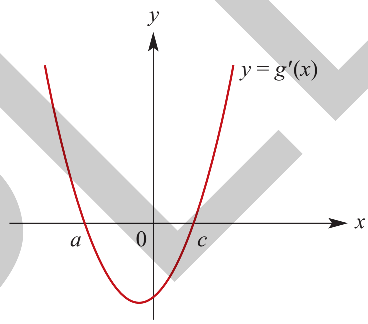
The graph of the derivative function with rule $y = f'(x)$ is therefore as shown to the right. The derivative function f' is linear as f is quadratic.



Consider the cubic function with rule $y = g(x)$.
For $x < a$, $f'(x) > 0$
For $x = a$, $f'(x) = 0$
For $a < x < c$, $f'(x) < 0$
For $x = c$, $f'(x) = 0$
For $x > c$, $f'(x) > 0$



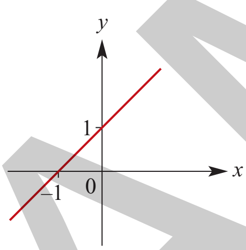
The graph of the derivative function with rule $y = g'(x)$ is therefore as shown to the right. The derivative is known to be quadratic as g is cubic.



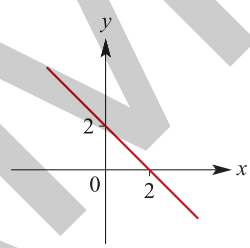
Example 27

Sketch the graph of the derivative function for each of the functions of the graphs shown:

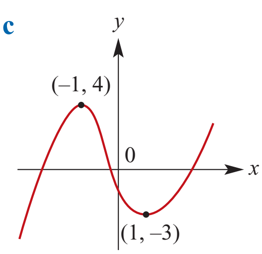
a



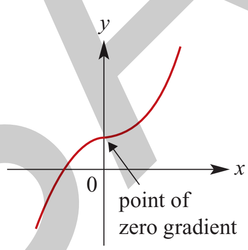
b



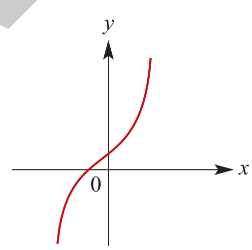
c



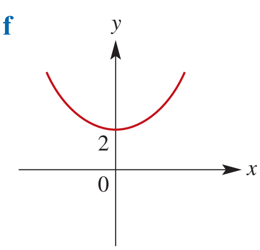
d



e



f



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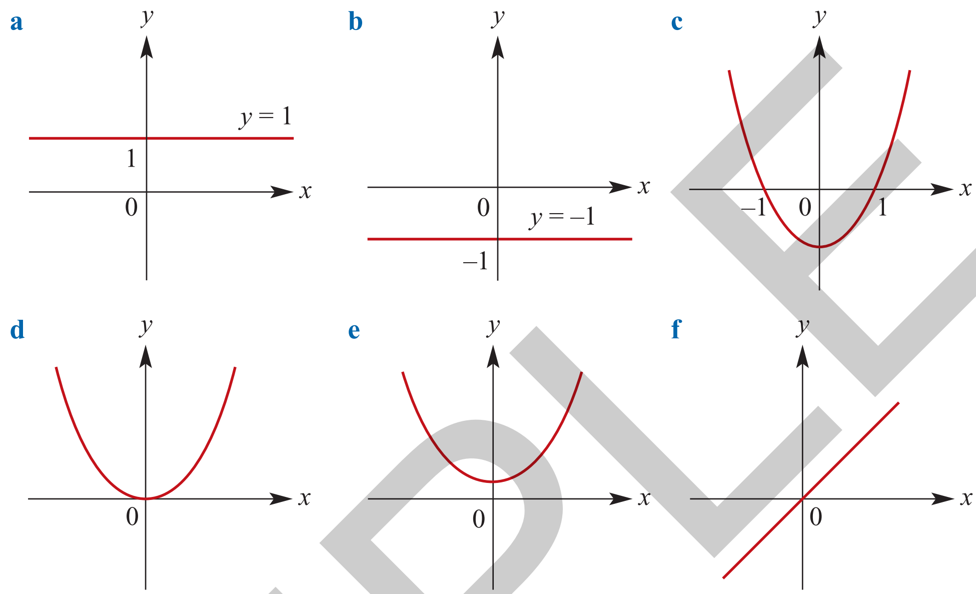
Solution

Note: Not all features of the graphs are known.



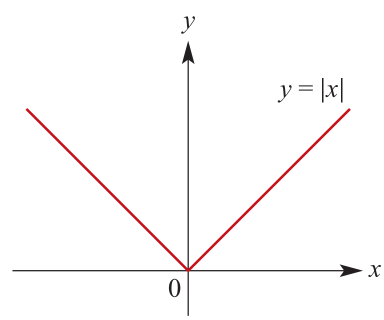
In Section 9.10, differentiability of a function is considered formally. Consider the absolute value function:

$$f: R \to R, f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Consider the gradient of a chord connecting the points (0, 0) and $(h, f(h))$ on the graph of $f(x) = |x|$

$$\frac{f(0+h) - f(0)}{h} = \begin{cases} \dfrac{h}{h} & h > 0 \\ \dfrac{-h}{h} & h < 0 \end{cases}$$

$$= \begin{cases} 1 & h > 0 \\ -1 & h < 0 \end{cases}$$



As the value for the gradient is not unique, we say $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist, i.e., f is not differentiable at $x = 0$.

Note: The gradient to the left of 0 is –1 and to the right of 0 the gradient is 1. The idea of left and right limits is further explored in Section 9.9.

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Example 28

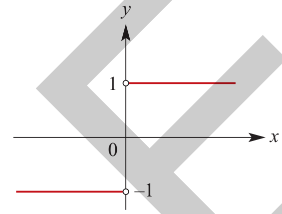
Let $f: R \to R$, $f(x) = |x|$

Sketch the graph of the derivative for a suitable domain.

Solution

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$f'(x)$ is not defined at $x = 0$.



Using the TI-Nspire

The derivative of $|x|$ yields the function **sing** (x).

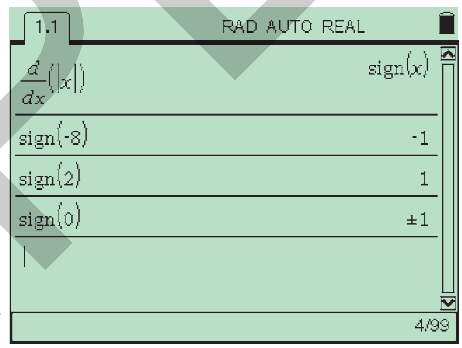
This gives the result of –1 if $x < 0$ and 1 if $x > 0$. (Somewhat confusingly, it returns ± 1 if $x = 0$).

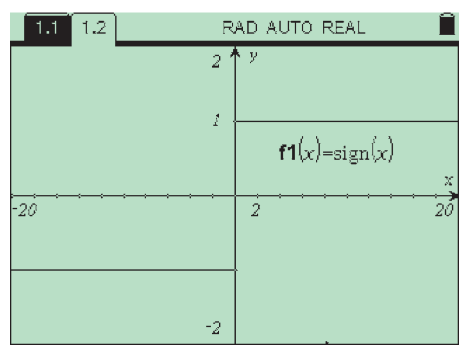
Note that the function **sign( )** can be found in the **Number Tools** menu (menu 2 7 4), but it may also be typed directly using the letters on the keyboard.

Open a **Graphs & Geometry** application (home 2) and let $f1(x) =$ **sing** (x).

Select an appropriate **window setting** (menu 4 1).

The graph of **sing (*x*)** is shown.





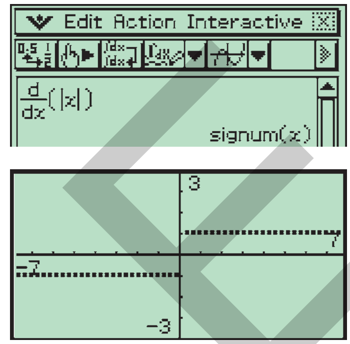
Using the Casio ClassPad

The derivative of $|x|$ yields the function **signum**(x).

This gives a result of –1 if $x < 0$ and 1 if $x > 0$. (Somewhat confusingly, it returns ± 1 if $x = 0$.)

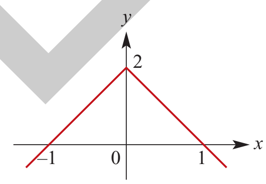
It will be used in the following example.

The graph of **signum**(x) is given with the graph mode set to [·····].



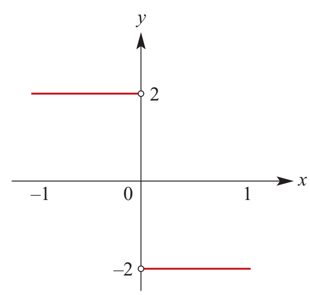
Example 29

Draw a sketch graph of f' where the graph of f is as illustrated. Indicate where f' is not defined.



Solution

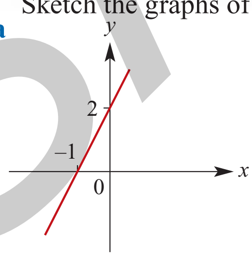
The derivative does not exist at $x = 0$, i.e. the function is not differentiable at $x = 0$.



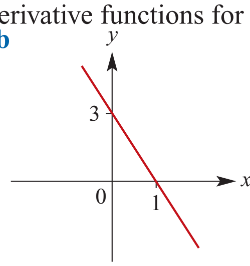
Exercise 9H

1 Sketch the graphs of the derivative functions for each of the functions with graphs shown:

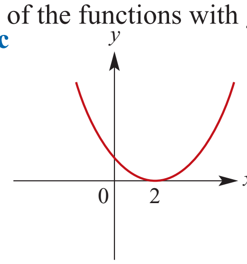
a



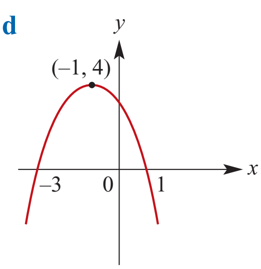
b



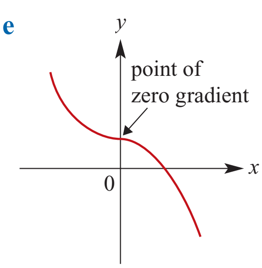
c



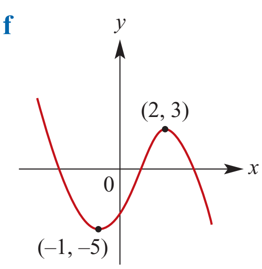
d



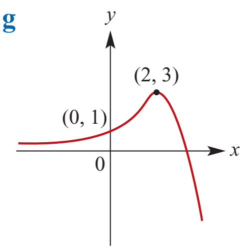
e



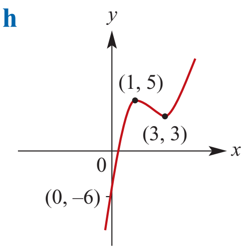
f



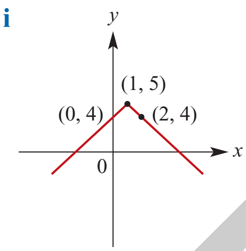
g



h



i



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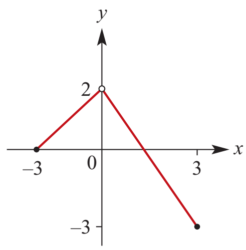

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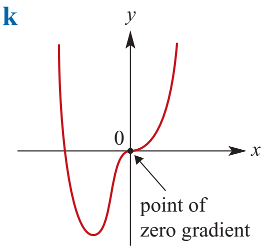

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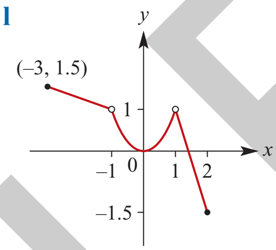

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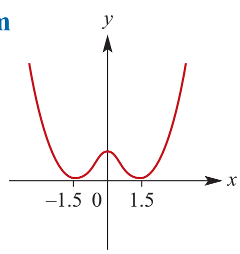

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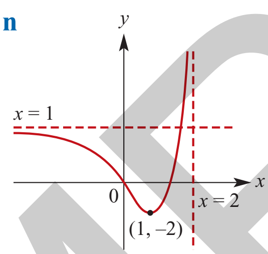

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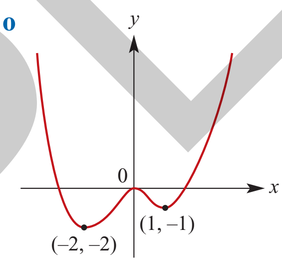

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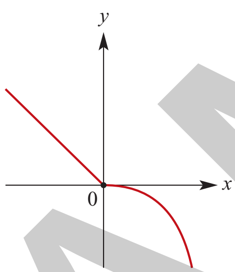

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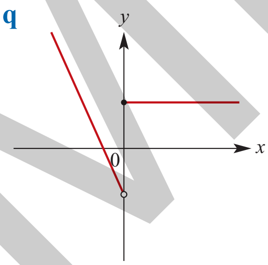
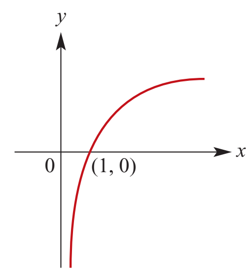


r



2 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = (x^2 - 2x)^2$

b Using the same screen plot the graph of $y = f'(x)$. (Do not attempt to determine the rule for $f'(x)$ first.)

c Use a calculator to determine $f'(x)$ for:

i $x = 0$ **ii** $x = 2$ **iii** $x = 1$ **iv** $x = 4$

d For $0 \le x \le 1$ find the value of x for which:

i $f(x)$ is a maximum **ii** $f'(x)$ is a maximum

3 On the same screen plot the graphs of $y = f(x)$ and $y = f'(x)$ where $f(x) = \dfrac{x^3}{3} - x^2 + x + 1$. Comment.

4 On the same screen plot the graphs of $y = g(x)$ and $y = g'(x)$ where $g(x) = x^3 + 2x + 1$. Comment.

5 a On the same screen plot the graphs of $y = h(x)$ and $y = h'(x)$ where $h(x) = x^4 + 2x + 1$

b Find the value(s) of x for which:

i $h(x) = 3$ ii $h'(x) = 3$

6 Plot the graph of $y = f(x)$ and $y = f'(x)$ on the one set of axes where $f(x) = 3x^3 + 4x$

a Find the minimum gradient of the graph of $y = f(x)$

b Find the values of x for which $f'(x) = 6$

7 Consider the function $g(x) = \dfrac{f(2 + x) - f(2)}{x}$ where $-0.5 < x < 0.5$ and $x \neq 0$

a Plot the graph of $g(x)$ when:

i $f(x) = x^3$ ii $f(x) = 4x^2$ iii $f(x) = x^4$

b Consider the table of values with the increment below. Start at $x = -0.5$

i 0.1 ii 0.05

9.9 Review of limits and continuity*

Limits

The limit of a function with rule $f(x)$ is said to be the value that $f(x)$ approaches as x approaches a given value. $\lim\limits_{x \to a} f(x) = p$ means that as x approaches a, $f(x)$ approaches p. An important idea is that it is possible to get as close as desired to p as x approaches a.

Note that $f(x)$ may or may not be defined at $x = a$.

With many functions $f(a)$ is defined, so to evaluate the limit we simply substitute the value a into the rule for the function.

Example 30

If $f(x) = -4x^2$, find $\lim\limits_{x \to 2}(-4x^2)$

Solution

Since $f(x) = -4x^2$ is defined at $x = 2$:

$$\begin{aligned}\lim_{x \to 2}(-4x^2) &= -4(2)^2 \\ &= -16\end{aligned}$$

If the function is not defined at the value where the limit is to be found, the procedure is different.

* This topic is included in *Essential Mathematical Methods 1 & 2 CAS*.

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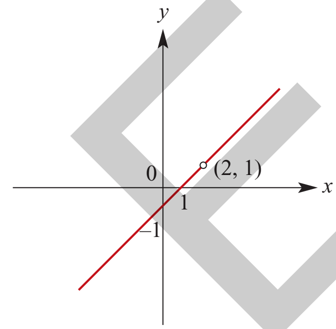
Example 31

For $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$, $x \neq 2$, find $\lim\limits_{x \to 2} f(x)$.

Solution

Observe that $f(x)$ is defined for $x \in R \backslash \{2\}$.
It is apparent that as x takes values closer and closer to 2, regardless of whether x approaches 2 from the left or from the right, the values of $f(x)$ become closer and closer to 1.

That is, $\lim\limits_{x \to 2} f(x) = 1$



This can be seen by observing:

$$\begin{aligned} f(x) &= \frac{(x-1)(x-2)}{x-2} \\ &= x - 1, \quad x \neq 2 \end{aligned}$$

The graph of $f: R \backslash \{2\} \to R$, $f(x) = x - 1$ is shown.

The following are important results that are useful for the evaluation of limits:

- $\lim\limits_{x \to c} (f(x) + g(x)) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$
  i.e., the limit of the sum is the sum of the limits.
- $\lim\limits_{x \to c} (kf(x)) = k \lim\limits_{x \to c} f(x)$, k being a given number (non-zero)
- $\lim\limits_{x \to c} (f(x)g(x)) = \lim\limits_{x \to c} f(x) \lim\limits_{x \to c} g(x)$
  i.e., the limit of the product is the product of the limits.
- $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$, provided $\lim\limits_{x \to c} g(x) \neq 0$
  i.e., the limit of the quotient is the quotient of the limits.

Example 32

Find:

a $\lim\limits_{x \to 0} x^2 + 2$ **b** $\lim\limits_{x \to 3} \dfrac{x^2 - 3x}{x - 3}$ **c** $\lim\limits_{x \to 2} \dfrac{(x^2 - x - 2)}{x - 2}$

d $\lim\limits_{x \to 3} (2x + 1)(3x - 2)$ **e** $\lim\limits_{x \to 3} \dfrac{x^2 - 7x + 10}{x^2 - 25}$

Solution

a $\lim\limits_{x\to 0}(x^2+2) = \lim\limits_{x\to 0} x^2 + \lim\limits_{x\to 0} 2 = 0 + 2 = 2$

b $\lim\limits_{x\to 3} \dfrac{x^2-3x}{x-3} = \lim\limits_{x\to 3} \dfrac{x(x-3)}{x-3} = \lim\limits_{x\to 3} x = 3$

c $\lim\limits_{x\to 2} \dfrac{(x^2-x-2)}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+1)}{x-2} = \lim\limits_{x\to 2}(x+1) = 3$

d $\lim\limits_{x\to 3}(2x+1)(3x-2) = \lim\limits_{x\to 3}(2x+1)\lim\limits_{x\to 3}(3x-2) = 7 \times 7 = 49$

e $\lim\limits_{x\to 3} \dfrac{x^2-7x+10}{x^2-25} = \lim\limits_{x\to 3} \dfrac{(x-2)(x-5)}{(x+5)(x-5)} = \lim\limits_{x\to 3} \dfrac{(x-2)}{(x+5)} = \dfrac{1}{8}$

Limit notation

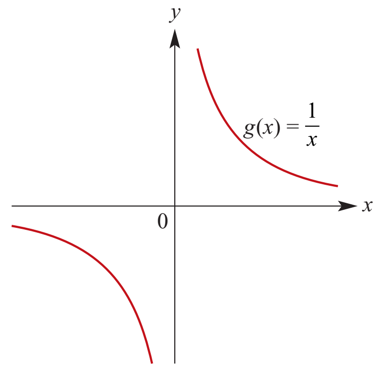
The notation of limits is used to describe the behaviour of graphs.

Consider $f: R\backslash\{0\} \to R,\ f(x) = \dfrac{1}{x^2}$. Observe that as $x \to 0$, both from the left and the right, $f(x)$ increases without bound. The limit notation for this is $\lim\limits_{x\to 0} f(x) = \infty$

For $g: R\backslash\{0\} \to R,\ g(x) = \dfrac{1}{x}$, the behaviour of $g(x)$ as x approaches 0 from the left is different from the behaviour as x approaches 0 from the right.

With limit notation this is written as:

$$\lim_{x\to 0^-} g(x) = -\infty \text{ and } \lim_{x\to 0^+} g(x) = \infty$$



Now examine this function as the magnitude of x becomes very large. It can be seen that as x increases without bound through positive values, the corresponding values of $g(x)$ approach zero. Likewise as x decreases without bound through negative values, the corresponding values of $f(x)$ also approach zero.

Symbolically this is written as:

$$\lim_{x\to\infty} g(x) = 0^+ \text{ and } \lim_{x\to-\infty} g(x) = 0^-$$

Many functions approach a limiting value, or limit, as x approaches $\pm\infty$.

Left and right limits

An idea that is useful in the following discussion is the existence of limits from the left and right.

If the value of $f(x)$ approaches the number p as x approaches a from the right-hand side, then it is written as $\lim\limits_{x\to a^+} f(x) = p$ and, if the value of $f(x)$ approaches the number p as x approaches a from the left-hand side, it is written as $\lim\limits_{x\to a^-} f(x) = p$

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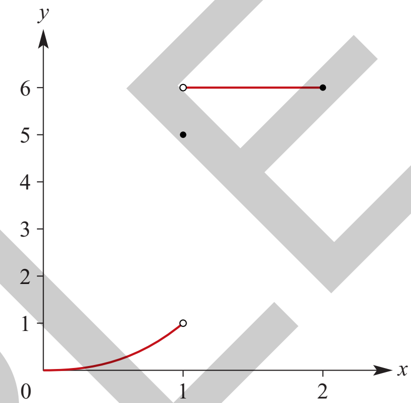
The limit as x approaches a exists only if the limit from the left and right both exist and are equal. Then:

$$\lim_{x \to a} f(x) = p$$

The following is an example of the limit not existing for a particular value.

$$\text{Let } f(x) = \begin{cases} x^3 & \text{for } 0 \leq x < 1 \\ 5 & \text{for } x = 1 \\ 6 & \text{for } 1 < x \leq 2 \end{cases}$$



It is clear from the graph of f that $\lim_{x \to 1} f(x)$ does not exist. However, if x is allowed to approach 1 from the left, then $f(x)$ approaches 1. On the other hand, if x is allowed to approach 1 from the right, then $f(x)$ approaches 6.

Also $f(1) = 5$.

Continuity at a point — informal definition

A function with rule $f(x)$ is said to be continuous when $x = a$ if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break. Otherwise there is said to be a discontinuity at $x = a$.

A more formal definition of continuity is:

> A function f is continuous at a point a if $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ all exist and are equal.

Or, equivalently:

> A function f is continuous at the point $x = a$ if the following three conditions are met:

1 $f(x)$ is defined at $x = a$ **2** $\lim_{x \to a} f(x)$ exists **3** $\lim_{x \to a} f(x) = f(a)$

A function is said to be discontinuous at a point if it is not continuous at that point. We say that a function is continuous everywhere if it is continuous for all real numbers.

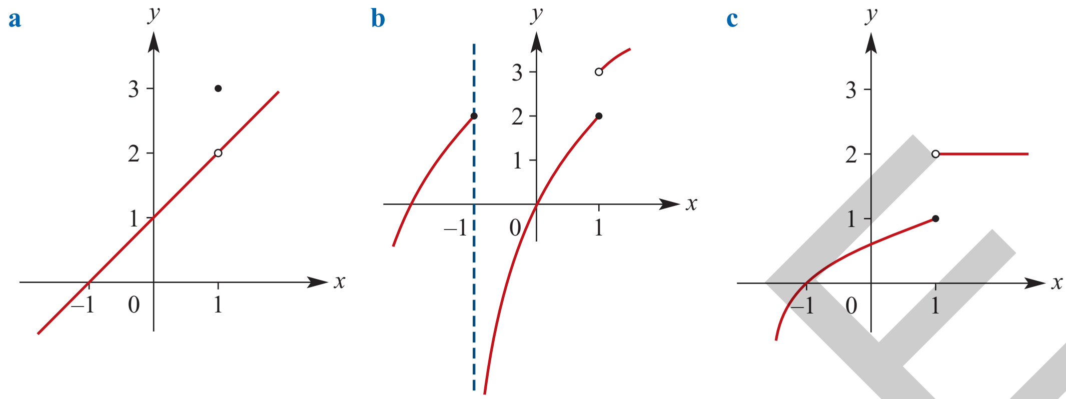
The polynomial functions are all continuous for R. Most of the functions considered in this course are continuous for their domains.

The function with rule $f(x) = \dfrac{1}{x}$ does have a discontinuity, where $x = 0$, as $f(0)$ is not defined. It is continuous everywhere in its domain.

Hybrid functions, as introduced in Chapter 5, provide examples of functions that have points of discontinuity where the function is defined.

Example 33

State the values for x for which each of the functions whose graphs are shown below have a discontinuity:



Solution

a Discontinuity at $x = 1$ as $f(1) = 3$ but $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2$

b Discontinuity at $x = -1$, as $f(-1) = 2$ and $\lim_{x \to -1^-} f(x) = 2$ but $\lim_{x \to -1^+} f(x) = -\infty$ and a discontinuity at $x = 1$, as $f(1) = 2$ and $\lim_{x \to 1^-} f(x) = 2$ but $\lim_{x \to 1^+} f(x) = 3$

c Discontinuity at $x = 1$, as $f(1) = 1$ and $\lim_{x \to 1^-} f(x) = 1$ but $\lim_{x \to 1^+} f(x) = 2$

Example 34

For each of the functions following, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why each is a discontinuity:

a $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

c $f(x) = \begin{cases} x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -2x + 1 & \text{if } x \geq 0 \end{cases}$

d $f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

e $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$

Solution

a $f(0) = 0$ but $\lim_{x \to 0^-} f(x) = 1$. Therefore a discontinuity at $x = 0$.

b $f(0) = 0$ but $\lim_{x \to 0^-} f(x) = 1$. Therefore a discontinuity at $x = 0$.

c $f(-1) = -1$ but $\lim_{x \to -1^+} f(x) = 1$. Therefore a discontinuity at $x = 1$.
$f(0) = 1$ but $\lim_{x \to 0^-} f(x) = 0$. Therefore a discontinuity at $x = 0$.

d No discontinuity **e** No discontinuity

Exercise 9I

1 Find the following limits:

a $\lim_{x \to 2} 17$ **b** $\lim_{x \to 6}(x - 3)$ **c** $\lim_{x \to \frac{1}{2}}(2x - 5)$

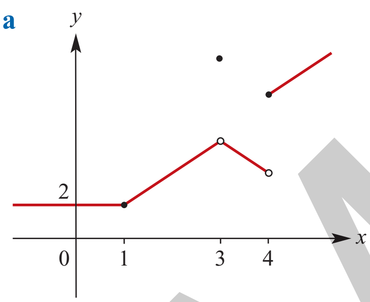
d $\lim_{t \to -3} \frac{(t + 2)}{(t - 5)}$ **e** $\lim_{t \to 2} \frac{t^2 + 2t + 1}{t + 1}$ **f** $\lim_{x \to 0} \frac{(x + 2)^2 - 4}{x}$

g $\lim_{t \to 1} \frac{t^2 - 1}{t - 1}$ **h** $\lim_{x \to 9} \sqrt{x + 3}$ **i** $\lim_{x \to 0} \frac{x^2 - 2x}{x}$

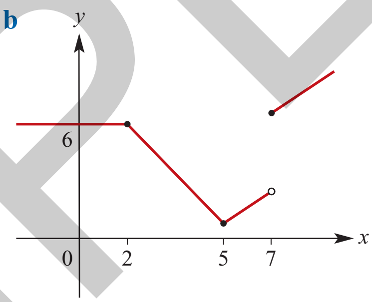
j $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$ **k** $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 + 5x - 14}$ **l** $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 6x + 5}$

2 For each of the following graphs, give the values of x for which a discontinuity occurs. Give reasons.

a



b



3 For each of the functions following, state the values of x for which there is a discontinuity and use the definition of continuity in terms of $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why each stated value of x corresponds to a discontinuity:

a $f(x) = \begin{cases} 3x & \text{if } x \geq 0 \\ -2x + 2 & \text{if } x < 0 \end{cases}$ **b** $f(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 1 \\ -2x + 1 & \text{if } x < 1 \end{cases}$

c $f(x) = \begin{cases} -x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -3x + 1 & \text{if } x \geq 0 \end{cases}$

4 The rule of a particular function is given below. For what values of x is the graph of this function continuous?

$$y = \begin{cases} 2, & x < 1 \\ (x - 4)^2 - 9, & 1 \leq x < 7 \\ x - 7, & x \geq 7 \end{cases}$$

9.10 Differentiability

If a function is differentiable at x, then it is also continuous at x. The converse, however, is not true. A function that is continuous at x is not necessarily differentiable at x. Consider, for example, the function $f(x) = |x|$ where it was found, in Section 9.8, that:

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$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The function $f(x) = |x|$ is not differentiable at the point $x = 0$, although it is continuous at that point.

Example 35

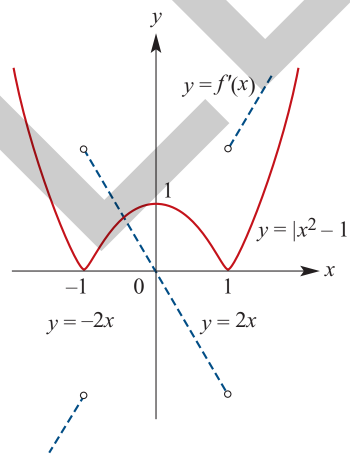
Find the derivative of $f(x) = |x^2 - 1|$ and sketch the graphs of $y = f(x)$ and $y = f'(x)$

Solution

$f(x) = g(h(x))$ where $g(x) = |x|$ and $h(x) = x^2 - 1$

The chain rule gives:

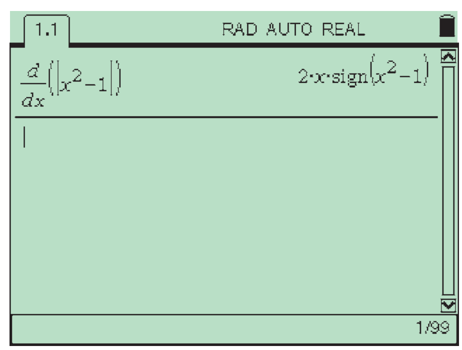
$$\begin{aligned} f'(x) &= g'(h(x))\,h'(x) \\ &= 2x \times \begin{cases} 1 & \text{if } h(x) > 0 \\ -1 & \text{if } h(x) < 0 \end{cases} \\ &= \begin{cases} 2x & \text{if } h(x) > 0 \\ -2x & \text{if } h(x) < 0 \end{cases} \\ &= \begin{cases} 2x & \text{if } x^2 - 1 > 0 \\ -2x & \text{if } x^2 - 1 < 0 \end{cases} \\ &= \begin{cases} 2x & \text{if } x > 1 \text{ or } x < -1 \\ -2x & \text{if } -1 < x < 1 \end{cases} \end{aligned}$$



It is clear that the function is not differentiable at $x = 1$ or $x = -1$.

Using the TI-Nspire

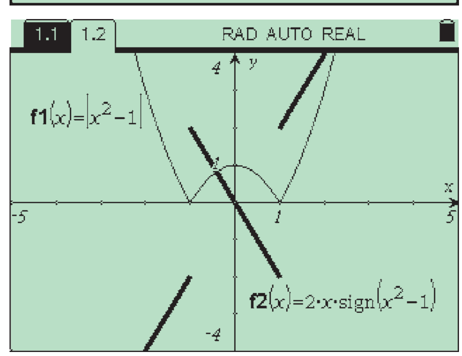
Use the **Derivative** template from the **Calculus** menu (menu 4 1) to find the derivative of $|x^2 - 1|$.



Open a **Graphs & Geometry** application (⌂ 2) and let $f1(x) =$ **abs**$(x^2 - 1)$ and $f2(x) = 2x$ **sign** $(x^2 - 1)$.

Select an appropriate **window setting** (menu 4 1).

The two graphs are shown. The graph of the derivative has **medium line weight**, available from the **Attributes** menu (menu 1 4).



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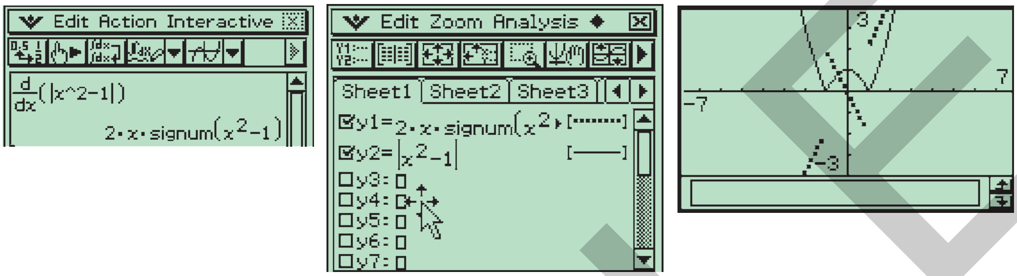
Using the Casio ClassPad

The derivative of $|x^2 - 1|$ is found with the result graphed in dot mode together with the graph of the function $f(x)$ as a solid line.



Example 36

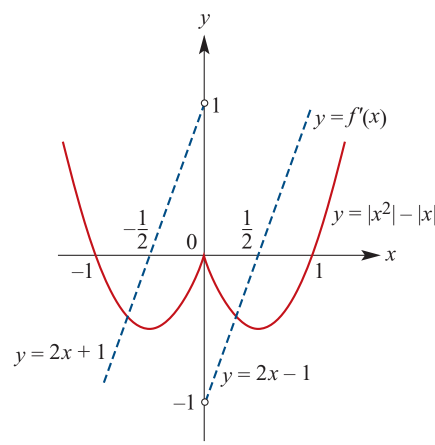
Find the derivative of $f(x) = |x|^2 - |x|$ and sketch the graph of $y = f(x)$ and $y = f'(x)$

Solution

$f(x) = g(h(x))$ where $g(x) = x^2 - x$ and
$h(x) = |x|$

The chain rule gives:

$$
\begin{aligned}
f'(x) &= g'(h(x))h'(x) \\
&= (2h(x) - 1)h'(x) \\
&= (2|x| - 1) \times \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \\
&= \begin{cases} (2|x| - 1) & \text{if } x > 0 \\ -(2|x| - 1) & \text{if } x < 0 \end{cases} \\
&= \begin{cases} (2|x| - 1) & \text{if } x > 0 \\ -2|x| + 1 & \text{if } x < 0 \end{cases} \\
&= \begin{cases} 2x - 1 & \text{if } x > 0 \\ 2x + 1 & \text{if } x < 0 \end{cases}
\end{aligned}
$$



This may also be obtained by first writing:

$$
\begin{aligned}
f(x) &= |x|^2 - |x| \\
&= \begin{cases} x^2 - x & \text{if } x > 0 \\ x^2 + x & \text{if } x < 0 \end{cases}
\end{aligned}
$$

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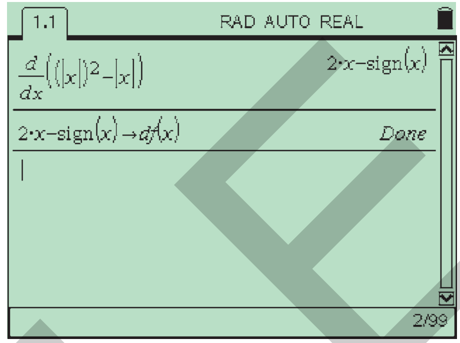
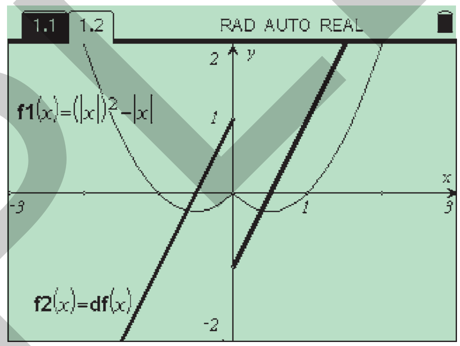
Using the TI-Nspire

Use the **Derivative** template from the **Calculus** menu (menu 4 1) to find the derivative of $|x|^2 - |x|$.
Store (ctrl sto→ var) this derivative as $df(x)$ as shown.

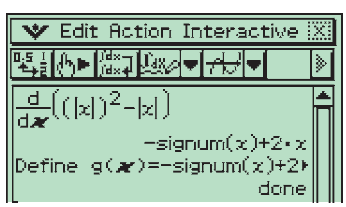


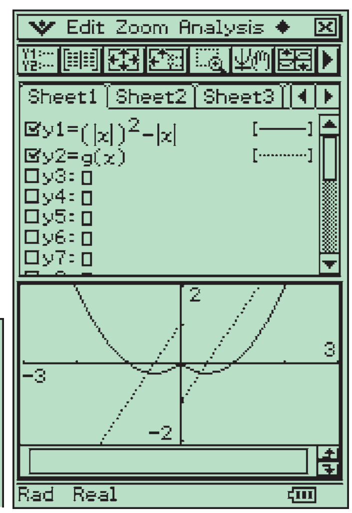
Open a **Graphs & Geometry** application (⌂ 2) and let $f1(x) = \text{abs}(x)^2 - \text{abs}(x)$ and $f2(x) = df(x)$.
Select an appropriate **window setting** (menu 4 1).
The two graphs are shown. The graph of the derivative has **medium line weight**, available from the **Attributes** menu (menu 1 4).



Using the Casio ClassPad

The derivative of $|x|^2 - |x|$ is found using Main. The derivative is defined as $g(x)$ using **Interactive** > **define**.
The graphs of the function and its derivative are graphed, each with a different style.
Note: **Edit**, **Cut** & **Paste** can be used to transfer the function and its derivative as required within and across program areas.





There are hybrid functions that are differentiable for R. The smoothness of the 'joins' determines if this is the case.

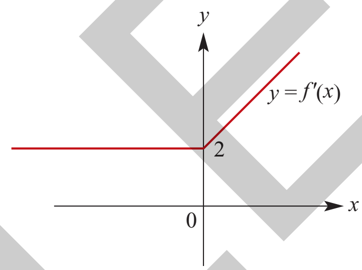
Example 37

For the function with rule $f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ 2x + 1 & \text{if } x < 0 \end{cases}$, find $f'(x)$ and sketch the graph of $y = f'(x)$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$$

In particular $f'(0)$ is defined and is equal to 2. Also, $f(0) = 1$. The two sections of the graph join smoothly at (0, 1).



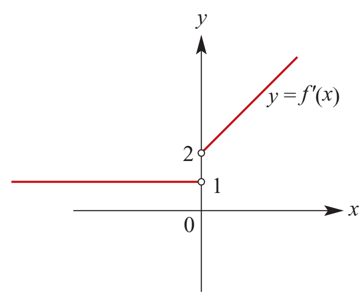
Example 38

For the function with rule $f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases}$, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$$

$f'(0)$ is not defined as the limits from the left and right are not equal. The function is differentiable for $R \setminus \{0\}$.



To test if a hybrid function is differentiable at a join at $x = a$:

1 test whether the function is continuous at $x = a$

2 test whether $\lim_{x \to a^-} f'(x)$ and $\lim_{x \to a^+} f'(x)$ exist and are equal.

Example 39

For the function with rule $f(x) = x^{\frac{1}{3}}$ state when the derivative is defined and sketch the graph of the derivative function.

Solution

By the rule $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. The derivative is not defined at $x = 0$.

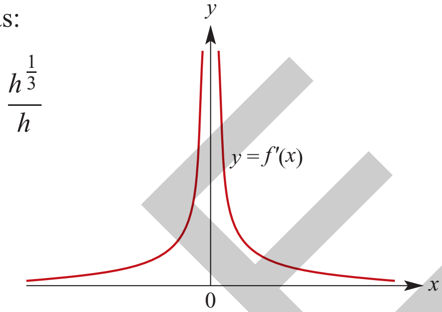
This is clear through the definition of limit as:

$$f'(0) = \lim_{h \to 0} \frac{(0+h)^{\frac{1}{3}} - 0^{\frac{1}{3}}}{(0+h) - 0} = \lim_{h \to 0} \frac{h^{\frac{1}{3}}}{h}$$

and thus the derivative is not defined.

It can be seen that $\lim_{h \to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} = \infty$

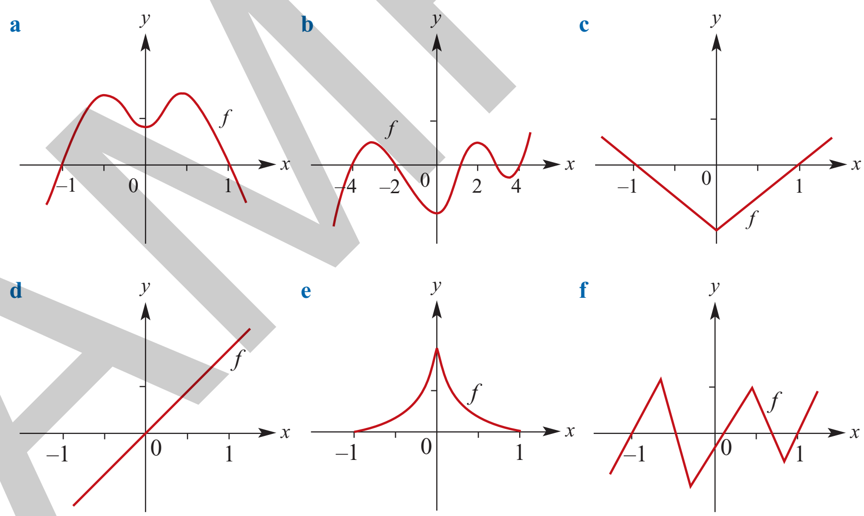
The graph of the derivative is as shown.



The function $f(x) = x^{\frac{1}{3}}$ is continuous at $x = 0$ but not differentiable.

Exercise 9J

1 In each of the figures below, a function graph f is given. Sketch the graph of f'. Your sketch of f' cannot be exact, but $f'(x)$ should be equal to zero at values of x for which the gradient of f is zero; $f'(x)$ should be less than zero where the original graph slopes downward; and so on.



2 For the function with rule $f(x) = \begin{cases} -x^2 + 3x + 1 & \text{if } x \geq 0 \\ 3x + 1 & \text{if } x < 0 \end{cases}$, find $f'(x)$ and sketch the graph of $y = f'(x)$

3 For the function with rule $f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 1 \\ -2x + 3 & \text{if } x < 1 \end{cases}$, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$

4 For the function with rule $f(x) = \begin{cases} -x^2 - 2x + 1 & \text{if } x \geq -1 \\ -2x + 3 & \text{if } x < -1 \end{cases}$, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$

5 For each of the following, give the set of values for which the derivative is defined, give the derivative and sketch the graph of the derivative function:

a $f(x) = (x-1)^{\frac{1}{3}}$ **b** $f(x) = x^{\frac{1}{5}}$ **c** $f(x) = x^{\frac{2}{3}}$ **d** $f(x) = (x+2)^{\frac{2}{5}}$

6 **a** For $f(x) = |x^2 - 4x|$ find $f'(x)$ for $x \in R \backslash \{0, 4\}$.

b For $f(x) = |x|^2 - 4|x|$ find $f'(x)$ for $x \in R \backslash \{0\}$.

9.11 Miscellaneous exercises

Exercise 9K

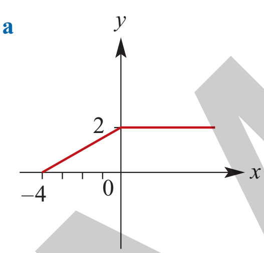
1 For $y = x^2 + 1$:

a Find the average rate of change of y with respect to x over the interval [3, 5].

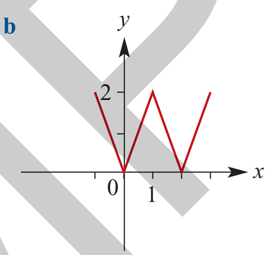
b Find the instantaneous rate of change of y with respect to x at the point $x = -4$.

2 Match the graphs of the functions shown in **a–f** with the graphs of their derivatives **A–F**:

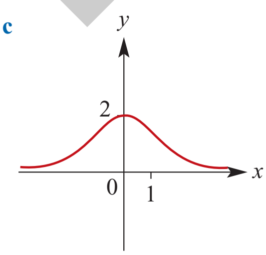
a



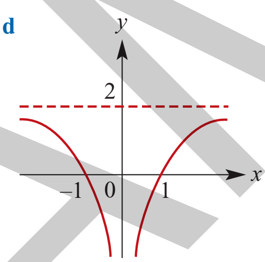
b



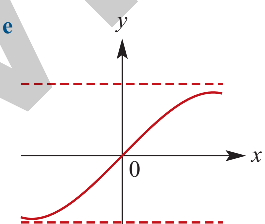
c



d



e



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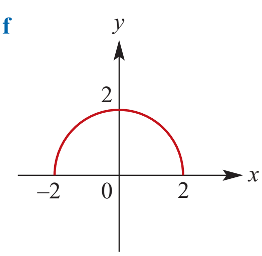

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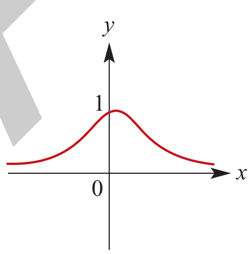

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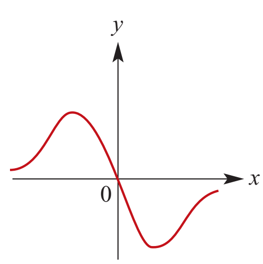

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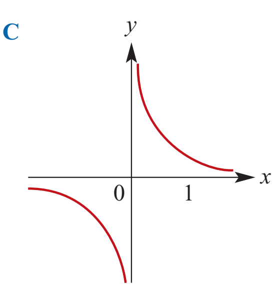

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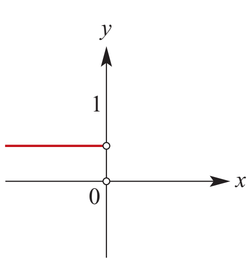

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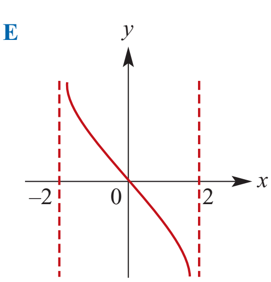

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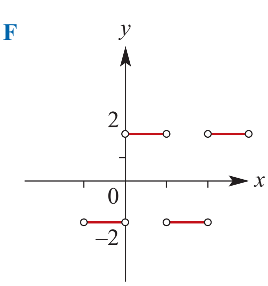

3 Let $y = \frac{x^2 - 1}{x^4 - 1}$

a Find $\frac{dy}{dx}$.   **b** Find $\left\{x: \frac{dy}{dx} = 0\right\}$

4 If $f(3) = -2$ and $f'(3) = 5$, find $g'(3)$ if:

a $g(x) = 3x^2 - 5f(x)$ **b** $g(x) = \frac{3x + 1}{f(x)}$

5 If $f(4) = 6$ and $f'(4) = 2$, find $g'(4)$ if:

a $g(x) = \sqrt{x}\, f(x)$ **b** $g(x) = \frac{f(x)}{x}$

6 Given that $f'(x) = \sqrt{3x + 4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$

7 If $f(x) = 2x^2 - 3x + 5$, find:

a $f'(x)$ **b** $f'(0)$ **c** $\{x: f'(x) = 1\}$

8 If $f(x) = \frac{1}{3x - 1}$, find $f'(2)$.

9 If $y = 1 - x^2$ prove that $x\frac{dy}{dx} + 2 = 2y$ for all values of x.

10 If $A = 4\pi r^2$, calculate $\frac{dA}{dr}$ when $r = 3$.

11 At what point on the graph of $y = 1.8x^2$ is the gradient 1?

12 If $y = 3x^2 - 4x + 7$, find the value of x such that $\frac{dy}{dx} = 0$

13 If $y = x^3$, prove that $x\frac{dy}{dx} = 3y$ for all values of x.

14 If $y = \frac{x^2 + 2}{x^2 - 2}$, find $\frac{dy}{dx}$.

15 If $z = 3y^2 + 4$ and $y = 2x - 1$, find $\frac{dz}{dx}$

16 If $y = (5 - 7x)^9$, calculate $\frac{dy}{dx}$.

17 If $y = 3x^{\frac{1}{3}}$, find $\frac{dy}{dx}$ when $x = 27$.

18 If $y = \sqrt{5 + x^2}$, find $\frac{dy}{dx}$ when $x = 2$.

19 Find $\frac{dy}{dx}$ when $x = 1$, given that $y = (x^2 + 3)(2 - 4x - 5x^2)$

20 If $y = \frac{x}{1 + x^2}$, find $\frac{dy}{dx}$ when $x = 1$.

21 If $y = \frac{2 + x}{x^2 + x + 1}$, find $\frac{dy}{dx}$ when $x = 0$.

22 Let $f(x) = \frac{1}{2x + 1}$

a Use the definition of derivative to find $f'(x)$.

b Find the gradient of the curve of f at the point $(0, 1)$.

23 Let $f(x) = x^3 + 3x^2 - 1$. Find:

a $\{x: f'(x) = 0\}$ **b** $\{x: f'(x) > 0\}$ **c** $\{x: f'(x) < 0\}$

24 Let $y = \dfrac{x}{1 - x}$

a Find $\dfrac{dy}{dx}$. **b** Write $\dfrac{dy}{dx}$ in terms of y.

25 If $y = (x^2 + 1)^{-\frac{3}{2}}$, find $\dfrac{dy}{dx}$.

26 If $P = 3s^2 - 3s$, find $\dfrac{dP}{ds}$ when $s = 1$.

27 If $y = x^4$, prove that $x\dfrac{dy}{dx} = 4y$

MC TEST D&D TEST

Chapter summary

- For points $P(x, f(x))$ and $Q(x+h, f(x+h))$ on the graph of $y = f(x)$ the gradient of the chord PQ is:

$$\frac{f(x+h) - f(x)}{h}$$

- The gradient of the graph of $y = f(x)$ at P is defined as:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- For $f: R \to$ R, the derived function is denoted by f',where $f': R \to R$ and $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, $n = 1, 2, 3, \ldots$ and for $f(x) = 1$, $f'(x) = 0$
- If $g(x) = kf(x)$, where k is a constant, then $g'(x) = kf'(x)$
- If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$
- If y is a function of x, then the derivative of y with respect to x is denoted by $\frac{dy}{dx}$
- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, n is a non-zero integer.
  For $f(x) = 1$, $f'(x) = 0$
  For $n \leq -1$, the domain of f is taken to be $R\backslash\{0\}$, and for $n \geq 1$, the domain of f is taken to be R.
- **The chain rule**

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$$

  and in function notation:

$$(f \circ g)'(x) = f'(g(x))g'(x), \text{ where } f \circ g(x) = f(g(x))$$

- For $f(x) = x^a$, $f'(x) = ax^{a-1}$, for $x > 0$ and $a \in R$.
- **The product rule**

  If $F(x) = f(x) \cdot g(x)$, then:
  $F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

- In Leibniz notation:

  If $y = uv$, where, u and v are functions of x

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

- **The quotient rule**

  If $F(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$, then:

$$F'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

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In Leibniz notation:

$$\text{If } y = \frac{u}{v} \text{ where } u \text{ and } v \text{ are functions of } x,\, v \neq 0$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

- The following are important results that useful for the evaluation of limits:
  - $\lim\limits_{x\to c}(f(x)+g(x)) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$
    i.e., the limit of the sum is the sum of the limits.
  - $\lim\limits_{x\to c}(kf(x)) = k\lim\limits_{x\to c} f(x)$, k being a given number (non-zero)
  - $\lim\limits_{x\to c}(f(x)g(x)) = \lim\limits_{x\to c} f(x)\lim\limits_{x\to c} g(x)$
    i.e., the limit of the product is the product of the limits.
  - $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$, provided $\lim\limits_{x\to c} g(x) \neq 0$
    i.e., the limit of the quotient is the quotient of the limits.
- A function f is continuous at the point $x = a$ if the following three conditions are met:
  1 $f(x)$ is defined at $x = a$ **2** $\lim\limits_{x\to a} f(x)$ exists **3** $\lim\limits_{x\to a} f(x) = f(a)$
- A function is said to be discontinuous at a point if it is not continuous at that point. We say that a function is continuous everywhere if it is continuous for all real numbers.
- A function f is said to be differentiable at x if $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ exists.

Multiple-choice questions

1 If $f(x) = \dfrac{4x^4 - 12x^2}{3x}$, then $f'(x)$ is equal to:

A $\dfrac{16x^3 - 24x}{3}$ **B** $4x^2 - 4$ **C** $\dfrac{16x^3 - 24x}{3x}$ **D** $4x^2 - 8x$ **E** $\dfrac{8x^3 - 16x}{3x}$

2 For $f\colon R\backslash\{7\} \to R$ where $f(x) = 5 + \dfrac{5}{(7-x)^2}$, $f'(x) > 0$ for:

A $R\backslash\{7\}$ **B** R **C** $x < 7$ **D** $x > 7$ **E** $x > 5$

3 Let $y = f(g(x))$ where $g(x) = 2x^4$. Then $\dfrac{dy}{dx}$ is equal to:

A $8x^3 f'(2x^4)$ **B** $8x^2 f(4x^3)$ **C** $8x^4 f(x) f'(x^3)$
D $2f(x)f'(x^3)$ **E** $8x^3$

4 Which of the following is not true for the curve of $y = f(x)$ where $f(x) = x^{\frac{1}{3}}$?

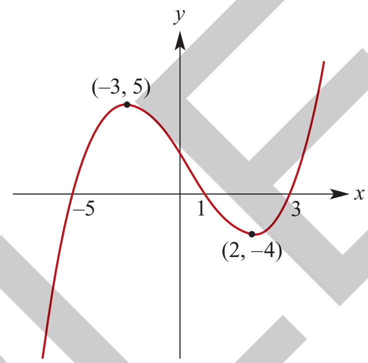
A The gradient is defined for all real numbers.
B The curve passes through the origin.
C The curve passes through the points with coordinates (1, 1) and (−1, −1).
D For $x > 0$ the gradient is positive. **E** For $x > 0$ the gradient is decreasing.

5 The graph of the function with rule $y = \dfrac{k}{2(x^3+1)}$ has gradient 1 when $x = 1$. The value of k is:

A 1 **B** $\dfrac{-8}{3}$ **C** $\dfrac{-1}{2}$ **D** -4 **E** $\dfrac{-1}{4}$

6 For the graph shown, the gradient is positive for:

A $-3 < x < 2$
B $-3 \le x \le 2$
C $x < -3$ or $x > 2$
D $x \le -3$ or $x \ge 2$
E $-3 \le x \le 3$



7 For the function $f(x) = 4x(2-3x)$, $f'(x) < 0$ for:

A $x < \dfrac{1}{3}$ **B** $0 < x < \dfrac{2}{3}$ **C** $x = \dfrac{1}{3}$

D $x > \dfrac{1}{3}$ **E** $x = 0, \dfrac{2}{3}$

8 For $y = \sqrt{3 - 2f(x)}$, $\dfrac{dy}{dx}$ is equal to:

A $\dfrac{2f'(x)}{\sqrt{3-2f(x)}}$ **B** $\dfrac{-1}{2\sqrt{3-2f(x)}}$ **C** $\dfrac{1}{2}\sqrt{3-2f'(x)}$

D $\dfrac{3}{2}\dfrac{1}{[3-2f'(x)]}$ **E** $\dfrac{-f'(x)}{\sqrt{3-2f(x)}}$

9 The point on the curve defined by the equation $y = (x+3)(x-2)$ where the gradient is -7 has coordinates:

A $(-4, 6)$ **B** $(-4, 0)$ **C** $(-3, 0)$ **D** $(-3, -5)$ **E** $(-2, 0)$

10 The function $y = ax^2 - bx$ has a zero gradient only for $x = 2$. The x-axis intercepts of the graph of this function are:

A $\dfrac{1}{2}, \dfrac{-1}{2}$ **B** $0, 4$ **C** $0, -4$ **D** $0, \dfrac{1}{2}$ **E** $0, \dfrac{-1}{2}$

Short-answer questions (technology-free)

1 Differentiate each of the following with respect to x:

a $x + \sqrt{1-x^2}$ **b** $\dfrac{4x+1}{x^2+3}$ **c** $\sqrt{1+3x}$

d $\dfrac{2+\sqrt{x}}{x}$ **e** $(x-9)\sqrt{x-3}$ **f** $x\sqrt{1+x^2}$

g $\dfrac{x^2-1}{x^2+1}$ **h** $\dfrac{x}{x^2+1}$ **i** $(2+5x^2)^{\frac{1}{3}}$

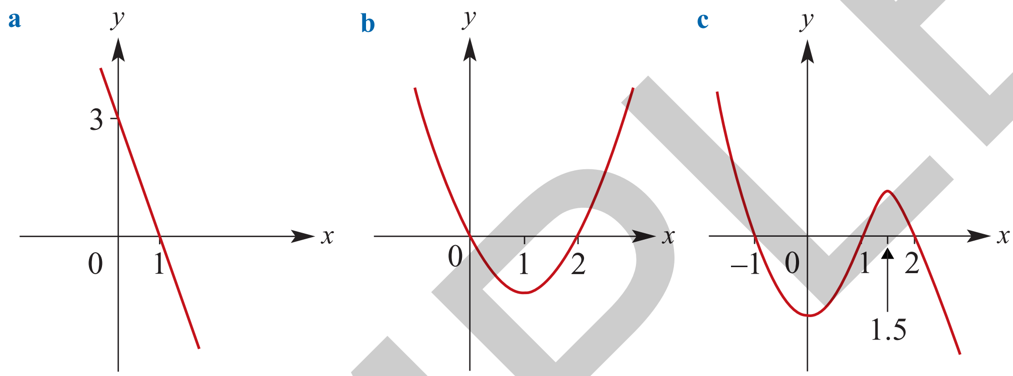
j $\dfrac{2x+1}{x^2+2}$ **k** $(3x^2+2)^{\frac{2}{3}}$

2 Find the gradient of the curve of each of the following functions at the point corresponding to the x-value given:

a $y = 3x^2 - 4;\ x = -1$ **b** $y = \dfrac{x-1}{x^2+1};\ x = 0$

c $y = (x-2)^5;\ x = 1$ **d** $y = (2x+2)^{\frac{1}{3}};\ x = 3$

3 Sketch the graphs of the derivative functions for each of the following functions from the graphs shown:



4 Find the derivative of $\left(4x + \dfrac{9}{x}\right)^2$ and values of x at which the derivative is zero.

5 **a** For $y = \dfrac{2x-3}{x^2+4}$, show that $\dfrac{dy}{dx} = \dfrac{8+6x-2x^2}{(x^2+4)^2}$

b Find the values of x for which y and $\dfrac{dy}{dx}$ are both positive.

6 Find the derivative of each of the following, given that f is a differentiable function for all real numbers.

a $xf(x)$ **b** $|f(x)|$, given that $f(x) \geq 0$ only for $x \in [0, 4]$ **c** $\dfrac{x^2}{[f(x)]^2}$

Extended-response questions

1 **a** For the functions f and g that are defined and differentiable for all real numbers, it is known that:

- $f(1) = 6,\ g(1) = -1,\ g(6) = 7$ and $f(-1) = 8$
- $f'(1) = 6,\ g'(1) = -2,\ f'(-1) = 2$ and $g'(6) = -1$

Find:

i $(f \circ g)'(1)$ **ii** $(gof)'(1)$ **iii** $(fg)'(1)$

iv $(gf)'(1)$ **v** $\left(\dfrac{f}{g}\right)'(1)$ **vi** $\left(\dfrac{g}{f}\right)'(1)$

b It is known that f is a cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

2 Let f be a function, which is differentiable for R, with graph defined by the rule $y = f(x)$. It is known that:

- $f'(x) = 0$ for $x = 1$ and $x = 5$
- $f'(x) > 0$ for $x > 5$ and $x < 1$
- $f'(x) < 0$ for $1 < x < 5$
- $f(1) = 6$ and $f(5) = 1$

a For $y = f(x + 2)$ find the values of x for which:

i $\frac{dy}{dx} = 0$ **ii** $\frac{dy}{dx} > 0$

b For $y = f(x - 2)$:

i Find the values of x for which $\frac{dy}{dx} = 0$

ii Find the coordinates of the points on the graph for which $\frac{dy}{dx} = 0$

c For $y = f(2x)$:

i Find the values of x for which $\frac{dy}{dx} = 0$

ii Find the coordinates of the points on the graph for which $\frac{dy}{dx} = 0$

d For $y = f\left(\frac{x}{2}\right)$:

i Find the values of x for which $\frac{dy}{dx} = 0$

ii Find the coordinates of the points on the graph for which $\frac{dy}{dx} = 0$

e For $y = 3f\left(\frac{x}{2}\right)$:

i Find the values of x for which $\frac{dy}{dx} = 0$

ii Find the coordinates of the points on the graph for which $\frac{dy}{dx} = 0$

3 Let $f(x) = (x - \alpha)^n(x - \beta)^m$ where m and n are positive integers with $m > n$ and $\beta > \alpha$

a Solve the equation $f(x) = 0$ for x

b Find $f'(x)$.

c Solve the equation $f'(x) = 0$ for x

d **i** If m and n are odd, find the set of values for which $f'(x) > 0$

ii If m is odd and n is even, find the set of values for which $f'(x) > 0$

4 Consider the function with rule $f(x) = \frac{x^n}{1 + x^n}$ where n is a positive even integer.

a Show that $f(x) = 1 - \frac{1}{x^n + 1}$

b Find $f'(x)$.

c Show that $0 \le f(x) < 1$ for all x.

d State the set of values for which $f'(x) = 0$

e State the set of values for which $f'(x) > 0$

f Show that f is an even function.