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CHAPTER

12

Integration

Objectives

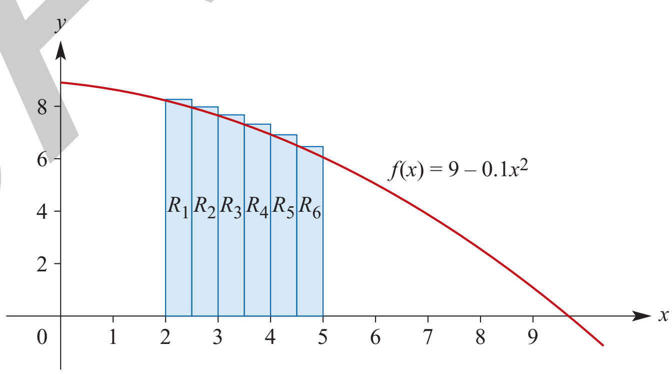
- To use numerical methods **to determine area**.
- To **integrate functions with rule of the form $f(x) = (ax + b)^n$, $n \in Q$**.
- To integrate **polynomial functions**.
- To integrate **trigonometric functions**.
- To integrate **exponential functions**.
- To determine **areas under curves** by integration.
- To use integration to **solve problems**.

12.1 Approximations leading to the definite integral

In this section we consider three methods for determining the area under a graph.

The left endpoint estimate

First, an approximation is found for the area of the shaded region by dividing the region into rectangles as illustrated. The width of each rectangle is 0.5.

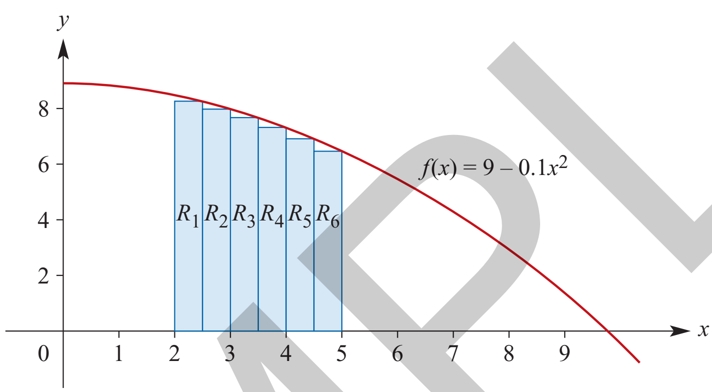


The area of rectangle $R_1 = 0.5 f(2.0) = 4.30$ square units.
The area of rectangle $R_2 = 0.5 f(2.5) = 4.1875$ square units.
The area of rectangle $R_3 = 0.5 f(3.0) = 4.05$ square units.

The area of rectangle $R_4 = 0.5f(3.5) = 3.8875$ square units.
The area of rectangle $R_5 = 0.5f(4.0) = 3.70$ square units.
The area of rectangle $R_6 = 0.5f(4.5) = 3.4875$ square units.
The sum of the areas of the rectangles $= 23.6125$ square units.
The left endpoint estimate will be:

- **larger** than the actual area for a graph which is **decreasing** over the interval.
- **smaller** than the actual area for a graph which is **increasing** over the interval.

The right endpoint estimate



Again the rectangles have width 0.5 and we find:

- The area of rectangle $R_1 = 0.5f(2.5) = 4.1875$ square units.
- The area of rectangle $R_2 = 0.5f(3.0) = 4.05$ square units.
- The area of rectangle $R_3 = 0.5f(3.5) = 3.8875$ square units.

$\downarrow$

- The area of rectangle $R_6 = 0.5f(5.0) = 3.25$ square units.
- The sum of the areas of the rectangles $= 22.5625$ square units.

This is called the right endpoint estimate for the area.

For f decreasing over $[a, b]$, left endpoint estimate $\geq$ true area $\geq$ right endpoint estimate.
For f increasing over $[a, b]$, left endpoint estimate $\leq$ true area $\leq$ right endpoint estimate.

From this it can be seen that a further estimate for the area may be achieved by the average of the two estimates:

$$\text{average} = \frac{\text{left endpoint estimate} + \text{right endpoint estimate}}{2}$$

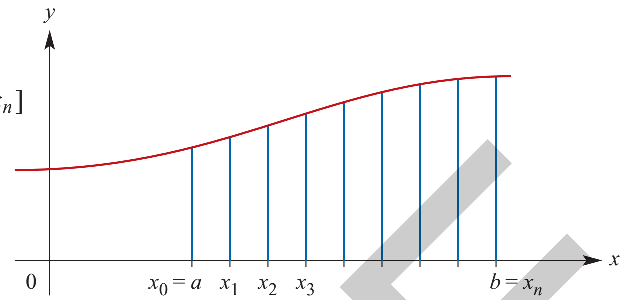
For the example discussed above, average $= 23.0875$ square units.

It is clear that if narrower strips are chosen then the estimate is closer to the true value. This is time-consuming to do by hand but a computer program or spreadsheet makes the process quite manageable.

If the interval $[a, b]$ on the x-axis is divided into n equal sub-intervals

$$[a, x_1], [x_1, x_2], [x_2, x_3], \ldots [x_{n-1}, x_n]$$

the two methods may be summarised as follows.



The left endpoint estimate:

$$L_n = \frac{b-a}{n}[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$$

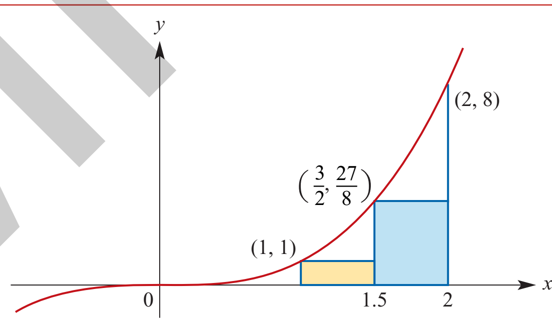
The right endpoint estimate:

$$R_n = \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)]$$

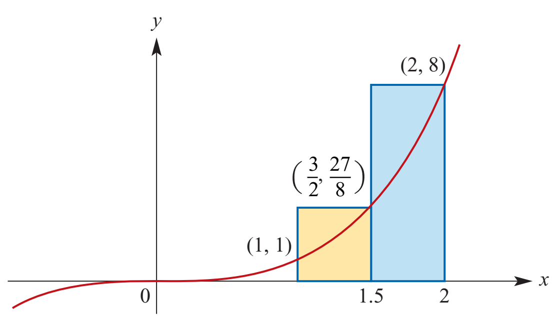
The methods are not limited to situations where the graph is either increasing or decreasing for all the interval, but may be used to determine the area under any continuous curve. These methods may be applied to any continuous function on an interval $[a, b]$ to determine an approximate value of the area.

Example 1

a Using the two rectangles shown find an approximation for the shaded area. (This is the left endpoint method.)



b Using the two rectangles shown find an approximation for the shaded area. (This is the right endpoint method.)



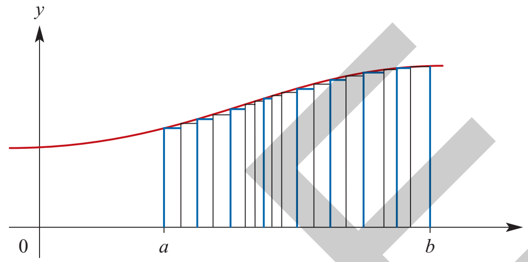
Solution

a Area $= \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{27}{8} = \frac{1}{2} + \frac{27}{16} = \frac{35}{16} = 2\frac{3}{16}$ square units

b Area $= \frac{1}{2} \times \frac{27}{8} + \frac{1}{2} \times 8 = \frac{27}{16} + 4 = 5\frac{11}{16}$ square units

It can be seen from the above that for a continuous function f such that $f(x)$ is positive for all x in the interval $[a, b]$, and if the interval $[a, b]$ is partitioned into arbitrarily small intervals, then the **area** under the curve between $x = a$ and $x = b$ can be defined by this limit process.

In the diagram on the right rectangles formed from partitions are shown. They can be of varying width, but the limiting process must take the width of all the rectangles to approach zero.



The definite integral

Suppose that f is a function that is continuous for a closed interval $[a, b]$ and $f(x)$ is positive for x a member of this interval. Then the area under the curve of $y = f(x)$ from $x = a$ to $x = b$ is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$, and is given the symbol $\int_a^b f(x)dx$. The function f is called the integrand and a and b are the upper and lower limits of the integral.

This limiting process can be expressed as

$$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\delta x_i$$

where the interval $[a, b]$ is partitioned into n subintervals, with the ith subinterval of length δx_i and containing x_i^*, and $\delta x = \max\{\delta x_i : i = 1, 2, \ldots n\}$

When a function is linear or a hybrid composed of linear constituents, then the area may be found using geometric techniques.

Example 2

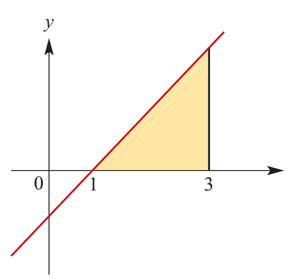
Evaluate each of the following definite integrals by using an area formula:

a $\int_1^3 (x - 1)dx$ **b** $\int_{-1}^3 |x - 1|dx$ **c** $\int_1^2 (x + 1)dx$

Solution

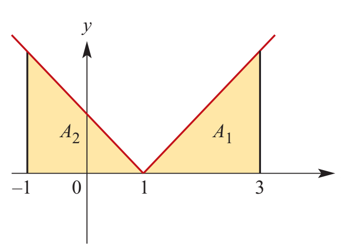
a The area of the triangle $= \frac{1}{2} \times 2 \times 2$
$= 2$ square units

Therefore $\int_1^3 (x - 1)dx = 2$



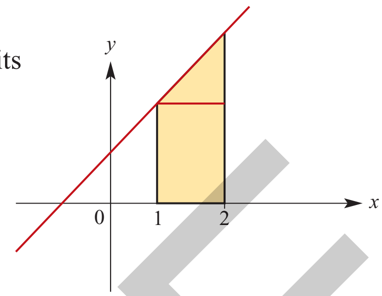
b The area $= A_1 + A_2 = 2 + \frac{1}{2} \times 2 \times 2 = 4$ square units

Therefore $\int_{-1}^3 |x - 1|dx = 4$



c The required region is a trapezium.

Hence area $= \frac{1}{2} \times 1 \times (2+3) = \frac{5}{2}$ square units

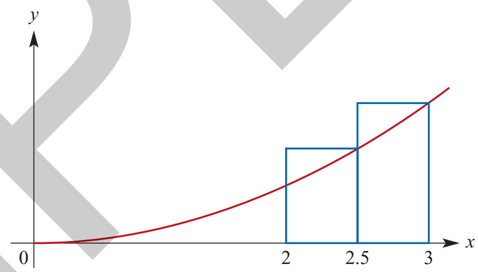


A calculus method for determining areas will be introduced in Section 12.5.

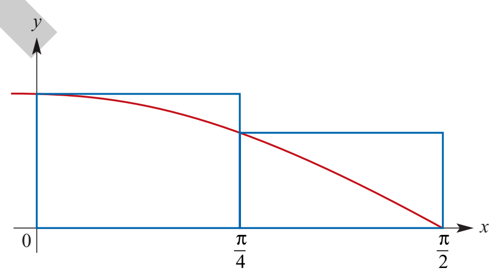
Exercise 12A

1 Use two rectangles to approximate the area contained between the curve and the x-axis. Use the method indicated and give your answer correct to two decimal places.

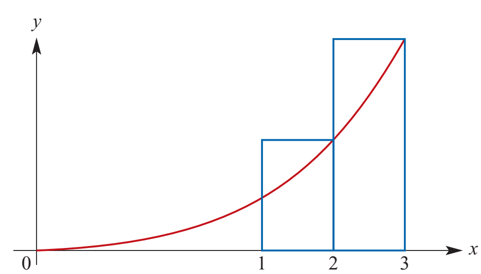
a $y = \frac{1}{2}x^2$ between $x = 2$ and $x = 3$ using the right endpoint method



b $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ using the left endpoint method



c $y = \frac{1}{2}x^3$ between $x = 1$ and $x = 3$ using the right endpoint method

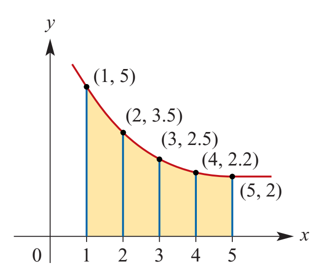


2 Calculate:

a the left endpoint estimate

b the right endpoint estimate

to find the area of the region shaded in the figure. (Use sub-intervals as shown.)

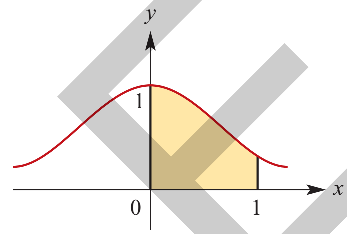


3 Calculate an approximation to an area under the graph of $y = x(4 - x)$ between $x = 0$ and $x = 4$ using:

a 4 strips of width 1.0 (Do by hand.)

b 20 strips of width 0.2 (You can use a calculator program if you wish).

4 The graph is that of $y = \dfrac{1}{1 + x^2}$. It is known that the area of the shaded region is $\dfrac{\pi}{4}$.



a Apply the right endpoint rule with strips of width 0.25 to find the area under the curve.

b Find an approximate value for π. How could you improve the approximation?

5 A table of values is given for the rule $y = f(x)$.

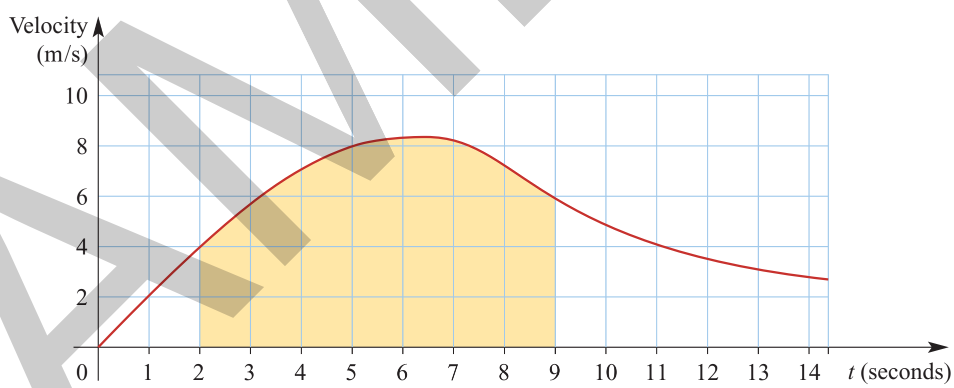
x	0	1	2	3	4	5	6	7	8	9	10
y	3	3.5	3.7	3.8	3.9	3.9	4.0	4.0	3.7	3.3	2.9

Find the area enclosed by the graph of $y = f(x)$, the lines $x = 0$ and $x = 10$, and the x-axis by:

a the left endpoint estimate

b the right endpoint estimate

6 The graph shows the velocity (in m/s) of a body at time t seconds.



a Use the left endpoint rule to find the area of the shaded region.

b What does this area represent?

7 Calculate, by using the right endpoint estimate, an approximation to the area under the graph of $y = 2^x$ between $x = 0$ and $x = 3$, using strips of width 0.5. Write your answer correct to two decimal places.

8 Using area formula evaluate each of the following definite integrals:

a $\displaystyle\int_2^5 (x - 2)dx$

b $\displaystyle\int_{-1}^5 |x - 2|dx$

c $\displaystyle\int_1^2 (2x + 1)dx$

12.2 Antidifferentiation

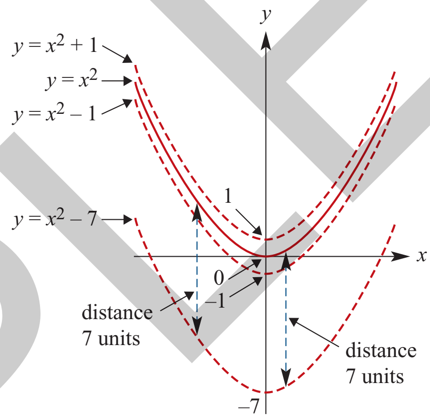
The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$ it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Consider the polynomial functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$

Then $f'(x) = 2x$ and $g'(x) = 2x$

i.e. the two different functions have the same derived function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be antiderivatives of $2x$. If two functions have the same derivative on an interval, then they differ by a constant. If two functions have the same derived function, then the graph of one function is obtained by a translation parallel to the y-axis of the other. Several antiderivatives of $2x$ are illustrated on the right.



Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.

The general antiderivative of $2x$ is $x^2 + c$ where c is an arbitrary real number. The notation of Leibniz is used to state this with symbols:

$$\int 2x\,dx = x^2 + c$$

The reason why the symbol is the same as that used for the definite integral of Section 12.1 will become evident in Section 12.5. This is read as 'the general antiderivative of $2x$ with respect to x is equal to x^2 squared plus c' or 'the *indefinite* integral of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we write:

$$\int 2x\,dx = \{f(x)\colon f'(x) = 2x\} = \{x^2 + c : c \in R\}$$

The set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. Set notation is not used in the following, but it is advisable to keep it in mind when considering further results.

In general:

> If $F'(x) = f(x)$,
> $\int f(x)dx = F(x) + c$, where c is an arbitrary real number.

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The antiderivative of x^r, $r \neq -1$

From previous work it is known that:

$$f(x) = x^3 \text{ implies } f'(x) = 3x^2$$
$$f(x) = x^8 \text{ implies } f'(x) = 8x^7$$
$$f(x) = x^{\frac{3}{2}} \text{ implies } f'(x) = \frac{3}{2}x^{\frac{1}{2}}$$
$$f(x) = x^{-4} \text{ implies } f'(x) = -4x^{-5}$$

Reversing this process gives:

$$\int 3x^2 dx = x^3 + c, \text{ where } c \text{ is an aribitrary constant}$$
$$\int 8x^7 dx = x^8 + c, \text{ where } c \text{ is an aribitrary constant}$$
$$\int \frac{3}{2}x^{\frac{1}{2}} dx = x^{\frac{3}{2}} + c, \text{ where } c \text{ is an aribitrary constant}$$
$$\int -4x^{-5} dx = x^{-4} + c, \text{ where } c \text{ is an aribitrary constant}$$

and from this:

$$\int x^2 dx = \frac{1}{3}x^3 + c$$
$$\int x^7 dx = \frac{1}{8}x^8 + c$$
$$\int x^{\frac{1}{2}} dx = \frac{2}{3}x^{\frac{3}{2}} + c$$
$$\int x^{-5} dx = -\frac{1}{4}x^{-4} + c$$

Generalising, it is seen that:

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c, r \in Q\backslash\{-1\}$$

Note: This definition can only be applied for suitable values of x for a given value of r. For example if $r = \frac{1}{2}$, $x \in R^+$ is a suitable restriction.

For $r = -2$, $x \in R\backslash\{0\}$ and if $r = 3$, $x \in R$.

The following results are recorded, following immediately from the corresponding results for differentiation in Chapter 9:

$$\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx$$

and $$\int kf(x) dx = k \int f(x) dx \text{ where } k \text{ is a real number}$$

Example 3

Find the general antiderivative (indefinite integral) of each of the following:

a $3x^5$ **b** $3x^2 + 4x^{-2} + 3$

Solution

a $\int 3x^5 dx = 3\int x^5 dx$

$$= 3 \times \frac{x^6}{6} + c$$

$$= \frac{x^6}{2} + c$$

b $\int 3x^2 + 4x^{-2} + 3dx = 3\int x^2 dx + 4\int x^{-2} dx + 3\int 1 dx$

$$= 3\left(\frac{x^3}{3}\right) + \frac{4x^{-1}}{-1} + \frac{3x}{1} + c$$

$$= x^3 - \frac{4}{x} + 3x + c$$

Example 4

Find y in terms of x if:

a $\frac{dy}{dx} = \frac{1}{x^2}$ **b** $\frac{dy}{dx} = 3\sqrt{x}$ **c** $\frac{dy}{dx} = x^{\frac{3}{4}} + x^{-\frac{3}{4}}$

Solution

a $\int \frac{1}{x^2} dx = \int x^{-2} dx$

$$= \frac{x^{-1}}{-1} + c$$

$$\therefore y = \frac{-1}{x} + c$$

b $\int 3\sqrt{x} dx = 3\int x^{\frac{1}{2}} dx$

$$= 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore y = 2x^{\frac{3}{2}} + c$$

c $\int x^{\frac{3}{4}} + x^{-\frac{3}{4}} dx$

$$\therefore y = \frac{4}{7}x^{\frac{7}{4}} + 4x^{\frac{1}{4}} + c$$

In some situations extra information that makes it possible to determine a unique antiderivative is given.

Example 5

It is known that $f'(x) = x^3 + 4x^2$ and $f(0) = 0$. Find $f(x)$.

Solution

$$\int x^3 + 4x^2 dx = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

$$\therefore f(x) = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

and as $f(0) = 0$, $c = 0$

$$\therefore f(x) = \frac{x^4}{4} + \frac{4x^3}{3}$$

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Antidifferentiation can be used to find the equation of a curve from its derivative. That is, given information about the gradient of a curve and a point on the curve the equation of the curve can be found.

Example 6

For the curve $y = f(x)$, $f'(x) = 2x$. If the curve passes through the point $(-1, 4)$, find the equation of the curve.

Solution

$$f'(x) = 2x \text{ implies } f(x) = \int 2x\,dx$$

$$\int 2x\,dx = \frac{2x^2}{2} + c = x^2 + c$$

$$\therefore f(x) = x^2 + c$$

but $f(-1) = 4$ so $4 = (-1)^2 + c$

$$\therefore c = 3 \text{ and } f(x) = x^2 + 3$$

Exercise 12B

1 Find:

a $\int \frac{1}{2}x^3\,dx$ **b** $\int 5x^3 - 2x\,dx$ **c** $\int \frac{4}{5}x^3 - 3x^2\,dx$ **d** $\int (2 - z)(3z + 1)\,dz$

2 Find:

a $\int 3x^{-2}\,dx$ **b** $\int 2x^{-4} + 6x\,dx$ **c** $\int \sqrt{x}(2 + x)\,dx$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}}\,dx$ **e** $\int \frac{3z^4 + 2z}{z^3}\,dz$ **f** $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}}\,dx$

3 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = 2x - 3$ and $y = 1$ when $x = 1$ **b** $\frac{dy}{dx} = x^3$ and $y = 6$ when $x = 0$

c $\frac{dy}{dx} = x^{\frac{1}{2}} + x$ and $y = 6$ when $x = 4$

4 A curve with equation $y = f(x)$ passes through the point $(2, 0)$ and $f'(x) = 3x^2 - \frac{1}{x^2}$. Find $f(x)$.

5 Find s in terms of t if $\frac{ds}{dt} = 3t - \frac{8}{t^2}$ and $s = 1\frac{1}{2}$ when $t = 1$.

6 The curve $f(x)$ for which $f'(x) = 16x + k$, where k is a constant, has a stationary point at $(2, 1)$. Find:

a the value of k **b** the value of $f(x)$ when $x = 7$

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12.3 Antidifferentiation of $(ax + b)^r$

Case 1: $r \geq 1$

For $f\colon R \to R,\ f(x) = (ax + b)^r,\ r \in Q, r \geq 1$

$$f'(x) = ar(ax + b)^{r-1} \quad \text{(using the chain rule)}$$

Case 2: $r < 1$

For $f\colon S \to R,\ f(x) = (ax + b)^r,\ r \in Q\backslash\{0\}, r < 1$

$$f'(x) = ar(ax + b)^{r-1}$$

for a suitable domain S.

Thus for a suitable domain the following holds:

$$\int (ax + b)^r\,dx = \frac{1}{a(r + 1)}(ax + b)^{r+1} + c,\quad r \neq -1$$

The result does not hold for $r = -1$.

Example 7

Find the general antiderivative of:

a $(3x + 1)^5$ **b** $(2x - 1)^{-2}$

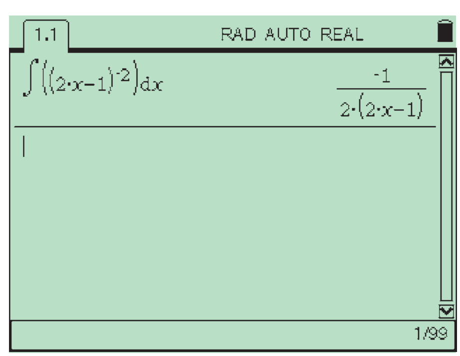
Solution

a
$$\int (3x + 1)^5 dx = \frac{1}{3(5 + 1)}(3x + 1)^6 + c$$
$$= \frac{1}{18}(3x + 1)^6 + c$$

b
$$\int (2x - 1)^{-2} dx = \frac{1}{2(-2 + 1)}(2x - 1)^{-1} + c$$
$$= -\frac{1}{2}(2x - 1)^{-1} + c$$

Using the TI-Nspire

Use the **Integral** template from the **Calculus** menu (menu 4 2) find the integral of $(2x - 1)^{-2}$.

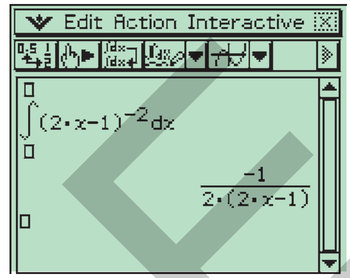


Using the Casio ClassPad

Find the integral of $(2x-1)^{-2}$.

Enter and highlight the expression, then tap **Interactive** > **Calculation** > $\int$.

Note that the two boxes on the integral allow for definite integrals to be evaluated. This is covered later in the chapter.



Case 3: $r = -1$

But what happens when $r = -1$? In other words, what is $\int \frac{1}{(ax+b)}dx$?

Remember that $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

Thus, $\int \frac{1}{x}dx = \log_e x + c$, provided of course that $x > 0$.

If $x < 0$, then $\frac{d}{dx}\log_e(-x) = -1 \times \frac{1}{-x} = x$, so that $\int \frac{1}{x}dx = \log_e(-x)$ for $x < 0$.

Since $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$

$$\int \frac{1}{x}dx = \log_e |x| + c$$

or, more generally,

$$\int \frac{1}{(ax+b)}dx = \frac{1}{a}\log_e |ax+b|$$

Example 8

a Find the general antiderivative of $\frac{2}{3x-2}$

b Given $\frac{dy}{dx} = \frac{3}{x}$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

c Given $\frac{dy}{dx} = \frac{3}{x}$ and $y = 10$ when $x = -1$, find an expression for y in terms of x.

Solution

a $\displaystyle\int \frac{2}{3x-2}\,dx = \frac{1}{3} \times 2\log_e|3x-2| + c = \frac{2}{3}\log_e|3x-2| + c$

b $\displaystyle y = \int \frac{3}{x}\,dx = 3\log_e|x| + c$

When $x = 1$, $y = 10$, and therefore

$10 = 3\log_e 1 + c = 0 + c$ or $c = 10$

Thus $y = 3\log_e|x| + 10$

c $\displaystyle y = \int \frac{3}{x}\,dx = 3\log_e|x| + c$

When $x = -1$, $y = 10$ and therefore

$10 = 3\log_e|-1| + c$

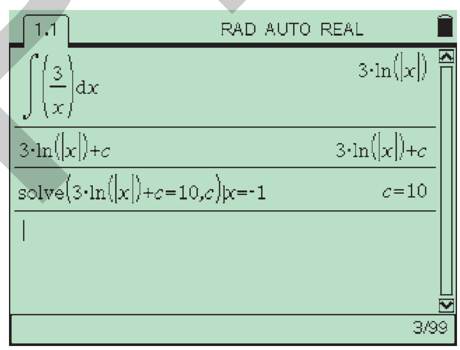
Hence $c = 10$ and therefore $y = \log_e|x| + 10$

Using the TI-Nspire

Use the **Integral** template from the **Calculus** menu (menu 4 2) find *any* antiderivative of $\frac{3}{x}$.

Add c to find the *general* antiderivative of $\frac{3}{x}$.

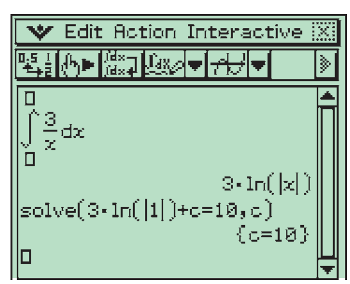
Use **solve( )** to determine the value of c as shown.



Using the Casio ClassPad

Enter and highlight the expression $\frac{3}{x}$ then tap **Interactive** > **Calculation** > $\int$.

Note that the ClassPad does not add c to the indefinite integral.



Copy and paste the answer to the next line, replace the x with 1 and add the c to complete the equation $3\ln|1| + c = 0$. Then tap **Interactive** > **Equation/inequality** > **solve** and ensure the variable is set to c.

Exercise 12C

1 Find:

a $\int (2x-1)^2 dx$ b $\int (2-t)^3 dt$ c $\int (5x-2)^3 dx$

d $\int (3x+6)^{\frac{1}{2}} dx$ e $\int (3x+6)^{-\frac{1}{2}} dx$ f $\int (2x-4)^{\frac{7}{2}} dx$

g $\int (3x+11)^{\frac{4}{3}} dx$ h $\int \sqrt{(2-3x)}dx$ i $\int (5-2x)^4 dx$

j $\int (3-4x)^{-5} dx$

2 Find an antiderivative of each of the following:

a $\dfrac{1}{2x}$ b $\dfrac{1}{3x+2}$ c $\dfrac{3}{1-4x}$

d $\dfrac{x}{x+1}$ e $\dfrac{x+1}{x}$ f $\dfrac{1}{(x+1)^2}$

g $\dfrac{(x+1)^2}{x}$ h $\dfrac{2x+1}{x+1}$ i $\dfrac{3}{(x-1)^3}$

3 Find:

a $\int \dfrac{5}{x} dx$ b $\int \dfrac{3}{x-4} dx$ c $\int \dfrac{10}{2x+1} dx$

d $\int \dfrac{6}{5-2x} dx$ e $\int 6(1-2x)^{-2} dx$ f $\int (4-3x)^{-1} dx$

4 Find y in terms of x for each of the following:

a $\dfrac{dy}{dx} = \dfrac{1}{2x}$ and $y = 2$ when $x = e^2$ b $\dfrac{dy}{dx} = \dfrac{2}{5-2x}$ and $y = 10$ when $x = 2$

5 A curve with equation $y = f(x)$ passes through the point $(5+e, 10)$ and $f'(x) = \dfrac{10}{x-5}$. Find the equation of the curve.

6 Given that $\dfrac{dy}{dx} = \dfrac{3}{x-2}$ and $y = 10$ when $x = 0$, find an expression for y in terms of x.

7 Given that $\dfrac{dy}{dx} = \dfrac{5}{2-4x}$ and $y = 10$ when $x = -2$, find an expression for y in terms of x.

12.4 The antiderivative of e^{kx}

In Chapter 11 it was found that for

$$f\colon R \to R,\ f(x) = e^{kx}$$
$$f'\colon R \to R,\ f'(x) = ke^{kx}$$

Thus $$\int e^{kx} dx = \frac{1}{k}e^{kx} + c,\ k \neq 0$$

Example 9

Find the general antiderivative of each of the following:

a e^{4x} **b** $e^{5x} + 6x$ **c** $e^{3x} + 2$ **d** $e^{-x} + e^{x}$

Solution

a $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$

b $\int e^{5x} + 6x\,dx = \frac{1}{5}e^{5x} + 3x^2 + c$

c $\int e^{3x} + 2\,dx = \frac{1}{3}e^{3x} + 2x + c$

d $\int e^{-x} + e^{x}\,dx = -e^{-x} + e^{x} + c$

Example 10

If the gradient at a point (x, y) on a curve is given by $5e^{2x}$ and the curve passes through $(0, 7.5)$, find the equation of the curve.

Solution

Let the curve have the equation $y = f(x)$

Then $f'(x) = 5e^{2x}$ and

$$\int 5e^{2x}\,dx = \frac{5}{2}e^{2x} + c$$

$$\therefore f(x) = \frac{5}{2}e^{2x} + c$$

but $f(0) = 7.5$

$$\therefore 7.5 = \frac{5}{2}e^{0} + c$$

$$= 2.5 + c$$

$$\therefore c = 5$$

so that $f(x) = \frac{5}{2}e^{2x} + 5$

Exercise 12D

1 Find:

a $\int e^{2x} - e^{\frac{x}{2}}\,dx$ **b** $\int \frac{e^{2x} + 1}{e^{x}}\,dx$ **c** $\int 2e^{3x} - e^{-x}\,dx$ **d** $\int 5e^{\frac{x}{3}} - 3e^{\frac{x}{5}}\,dx$

Back to Menu

2 Find y in terms of x for each of the following:

a $\dfrac{dy}{dx} = e^{2x} - x$ and $y = 5$ when $x = 0$ **b** $\dfrac{dy}{dx} = \dfrac{3 - e^{2x}}{e^x}$ and $y = 4$ when $x = 0$

3 Given that $\dfrac{dy}{dx} = ae^{-x} + 1$ and that when $x = 0$, $\dfrac{dy}{dx} = 3$ and $y = 5$, find the value of y when $x = 2$.

4 The curve for which $\dfrac{dy}{dx} = e^{kx}$, where k is a constant, is such that the tangent at $(1, e^2)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k **b** the equation of the curve

12.5 The fundamental theorem of calculus and the definite integral

The integrals that you have learned to evaluate by calculus techniques to date are known as **indefinite integrals** because they are only defined to within an arbitrary constant, for example $\int 3x^2dx = x^3 + c$. In general terms, we write $\int f(x)dx = G(x) + c$; that is, the integral of $f(x)$ is $G(x)$ plus a constant, where $G(x)$ is the antiderivative of $f(x)$.

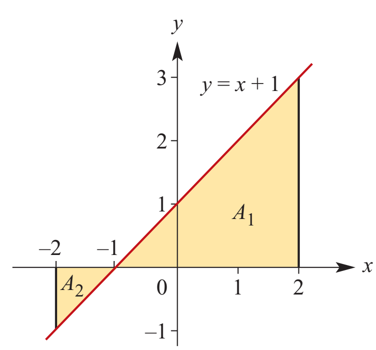
Consideration of the **definite integral** will now be resumed and the connection established. In Section 12.1 the following definition was given.

Suppose that f is a function that is continuous for a closed interval $[a, b]$ and $f(x)$ is positive for x, a member of this interval. Then the area under the curve of $y = f(x)$ from $x = a$ to $x = b$ is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$. and is given the symbol $\int_a^b f(x)dx$. The function f is called the integrand and a and b are the upper and lower bounds of the integral.

The fundamental theorem of integral calculus provides a connection between the area definition of the definite integral and the antiderivatives discussed previously. An outline of the proof is given in the final section of this chapter. First, attention must be given to the case when $f(x) < 0$.

The net signed area

For $f(x) = x + 1$



$$A_1 = \frac{1}{2} \times 3 \times 3$$
$$= 4\frac{1}{2} \quad \text{(area of a triangle)}$$
$$A_2 = \frac{1}{2} \times 1 \times 1$$
$$= \frac{1}{2}$$

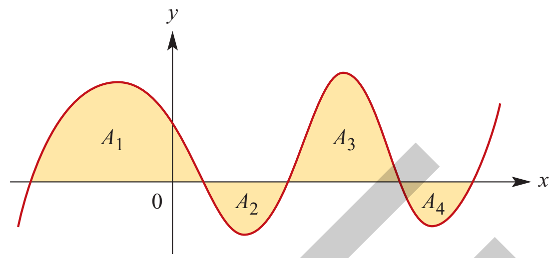
The total area $= A_1 + A_2$
$= 5$

The **net signed area** $= A_1 - A_2$
$= 4$

Regions below the x-axis have **negative signed areas**. Regions above the x-axis have **positive signed areas**.

The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$

The net signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$



$\int_a^b f(x)dx$ for any continuous function f, for the interval $[a, b]$, is the net signed area of the regions contained between the curve and the x-axis for $x \in [a, b]$.

It can be determined by a limiting process as discussed briefly in the first section of this chapter.

The fundamental theorem of integral calculus is as follows.

If f is a continuous function on the interval $[a, b]$, then:

$$\int_a^b f(x)dx = G(b) - G(a) \text{ where } G \text{ is any antiderivative of } f.$$

$\int_a^b f(x)dx$ is called the **definite integral** from $x = a$ to $x = b$. The number a is called the **lower limit of integration** and b is called the **upper limit of integration**. The function f is called the **integrand**. To facilitate setting out we sometimes write $G(b) - G(a) = [G(x)]_a^b$.

Example 11

Evaluate the definite integral: $\int_1^2 x\,dx$

Solution

$$\int x dx = \frac{1}{2}x^2 + c$$

so

$$\int_1^2 x dx = \frac{1}{2} \times 2^2 + c - \left(\frac{1}{2} \times 1^2 + c\right)$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

Note that the arbitrary constant cancels out. Because of this we ignore it when evaluating definite integrals. We also use the more compact notation $G(b) - G(a) = [G(x)]_a^b$ to help with setting out, as shown in the following example:

$$\int_1^2 x dx = \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{2^2}{2} - \frac{1^2}{2}$$

$$= \frac{3}{2}$$

Example 12

Evaluate the definite integral: $\int_0^1 2e^{-2x}\,dx$

Solution

$$\begin{aligned}\int_0^1 2e^{-2x}\,dx &= \left[\frac{2}{-2}e^{-2x}\right]_0^1\\ &= -1\left(e^{-2\times 1} - e^{-2\times 0}\right)\\ &= -1\left(e^{-2} - 1\right)\\ &= 1 - e^{-2}\end{aligned}$$

Properties of the definite integral

1 $\int_a^a f(x)\,dx = 0$

2 $\int_a^b f(x)\,dx = \int_c^b f(x)\,dx + \int_a^c f(x)\,dx,\ a < c < b$

3 $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4 $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (Note that this is a definition.)

Example 13

Evaluate each of the following definite integrals:

a $\int_2^3 x^2\,dx$ **b** $\int_3^2 x^2\,dx$ **c** $\int_0^4 e^{2x} + 1\,dx$ **d** $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

e $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx$ **f** $\int_{-1}^4 \frac{1}{5-x}\,dx$ **g** $\int_4^5 \frac{1}{5-2x}\,dx$

Solution

a $\int_2^3 x^2\,dx$

$$\begin{aligned}&= \left[\frac{x^3}{3}\right]_2^3\\ &= \frac{27}{3} - \frac{8}{3}\\ &= 9 - 2\frac{2}{3}\\ &= \frac{19}{3}\end{aligned}$$

b $\int_3^2 x^2\,dx$

$$\begin{aligned}&= \left[\frac{x^3}{3}\right]_3^2\\ &= \frac{8}{3} - \frac{27}{3}\\ &= -\frac{19}{3}\end{aligned}$$

c $\int_0^4 e^{2x} + 1\,dx$

$$\begin{aligned}&= \left[\frac{1}{2}e^{2x} + x\right]_0^4\\ &= \frac{1}{2}e^8 + 4 - \left(\frac{1}{2}e^0 + 0\right)\\ &= \frac{1}{2}(e^8 + 7)\end{aligned}$$

d $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1$

$= \frac{2}{3} + \frac{2}{5}$

$= \frac{16}{15}$

e $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx$

$= \left[\frac{4}{3}x^{\frac{3}{2}} + 2e^{\frac{x}{2}}\right]_1^4$

$= \frac{4}{3} \times 8 + 2e^2 - \left(\frac{4}{3} + 2e^{\frac{1}{2}}\right)$

$= \frac{28}{3} + 2e^2 - 2e^{\frac{1}{2}}$

$= 2\left(\frac{14}{3} + e^2 - e^{\frac{1}{2}}\right)$

f $\int_{-1}^4 \frac{1}{5-x}\,dx$

$= [-\log_e|5-x|]_{-1}^4$

$= -(\log_e 1 - \log_e 6)$

$= \log_e 6$

g $\int_4^5 \frac{1}{5-2x}\,dx$

$= -\frac{1}{2}\left[\log_e|5-2x|\right]_4^5$

$= -\frac{1}{2}(\log_e|-5| - \log_e|-3|)$

$= -\frac{1}{2}\log_e\frac{5}{3}$

$= \frac{1}{2}\log_e\frac{3}{5}$

Exercise 12E

1 Evaluate each of the following:

a $\int_1^2 x^2\,dx$ **b** $\int_{-1}^3 x^3\,dx$ **c** $\int_0^1 x^3 - x\,dx$

d $\int_{-1}^2 (x+1)^2\,dx$ **e** $\int_1^2 \frac{1}{x^2}\,dx$ **f** $\int_1^4 x^{\frac{1}{2}} + 2x^2\,dx$

g $\int_0^2 x^3 + 2x^2 + x + 2\,dx$ **h** $\int_1^4 2x^{\frac{3}{2}} + 5x^3\,dx$

2 Evaluate each of the following:

a $\int_0^1 (2x+1)^3\,dx$ **b** $\int_0^2 (4x+1)^{-\frac{1}{2}}\,dx$ **c** $\int_1^2 (1-2x)^2\,dx$

d $\int_0^1 (3-2x)^{-2}\,dx$ **e** $\int_0^2 (3+2x)^{-3}\,dx$ **f** $\int_{-1}^1 (4x+1)^3\,dx$

g $\int_0^1 \sqrt{2-x}\,dx$ **h** $\int_3^4 \frac{1}{\sqrt{2x-4}}\,dx$ **i** $\int_0^1 \frac{1}{(3+2x)^2}\,dx$

3 Evaluate each of the following:

a $\int_0^1 e^{2x}\,dx$ **b** $\int_0^1 e^{-2x} + 1\,dx$ **c** $\int_0^1 2e^{\frac{x}{3}} + 2\,dx$ **d** $\int_{-2}^2 \frac{e^x + e^{-x}}{2}\,dx$

4 Given that $\int_0^4 h(x)\,dx = 5$, evaluate:

a $\int_0^4 2h(x)dx$ **b** $\int_0^4 (h(x)+3)dx$ **c** $\int_4^0 h(x)dx$

d $\int_0^4 (h(x)+1)dx$ **e** $\int_0^4 (h(x)-x)dx$

5 Evaluate each of the following:

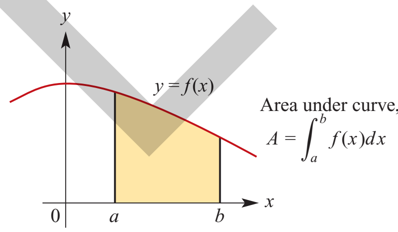
a $\int_0^4 \frac{1}{x-6}dx$ **b** $\int_{-3}^0 \frac{1}{3-2x}dx$ **c** $\int_5^6 \frac{3}{2x+7}dx$

12.6 Area under a curve

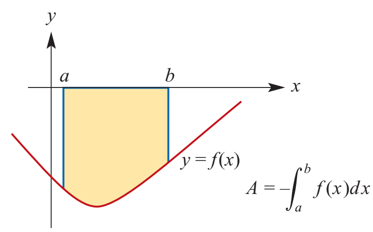
Recall that the definite integral $\int_a^b f(x)dx$ gives the net signed area 'under' the curve.

Finding the area of a region

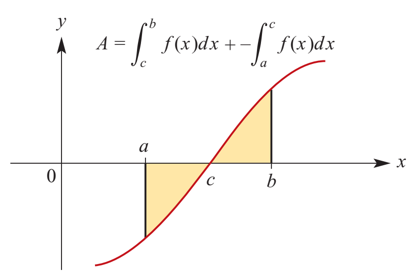
- If $f(x) \geq 0$ for all $x \in [a, b]$, the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)dx = G(b) - G(a)$ where G is an antiderivative of f.



- If $f(x) \leq 0$ for all $x \in [a, b]$, the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $A = -\int_a^b f(x)dx$



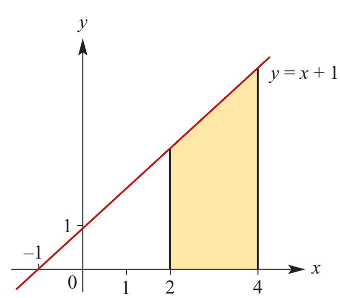
- If $c \in (a, b)$, $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area A of the shaded region is given by $A = \int_c^b f(x)dx + -\int_a^c f(x)dx$



Note: In determining the areas 'under' curves, the sign of $f(x)$ in the given interval is the critical factor.

Example 14

Find the area of the region between the x-axis, the line $y = x + 1$ and the lines $x = 2$ and $x = 4$. Check the answer by working out the area of the trapezium.



Solution

$$\text{Area} = \int_2^4 x + 1 dx$$

An antiderivative of $x + 1$ is $\frac{x^2}{2} + x$.

$$\begin{aligned}\text{We write } \int_2^4 x + 1 dx &= \left[\frac{x^2}{2} + x\right]_2^4 \\ &= \left(\frac{4^2}{2} + 4\right) - \left(\frac{2^2}{2} + 2\right) \\ &= 12 - 4 \\ &= 8\end{aligned}$$

The shaded region has area 8 square units.

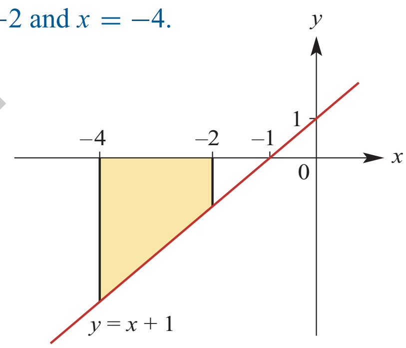
Check: $\quad$ Area of trapezium = average height × base

$$\begin{aligned} &= \frac{3+5}{2} \times 2 \\ &= 8 \text{ square units}\end{aligned}$$

Example 15

Find the area under the line $y = x + 1$ and between $x = -2$ and $x = -4$.

Note: The negative is introduced as the area is the negative of the integral from -4 to -2.



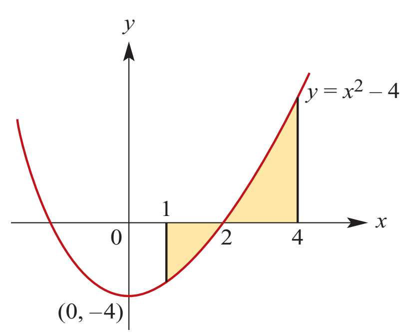
Solution

$$\begin{aligned}\text{Area} &= -\int_{-4}^{-2} x + 1 dx \\ &= -\left[\frac{x^2}{2} + x\right]_{-4}^{-2} \\ &= -(0 - 4)\end{aligned}$$

The area of the shaded region is 4 square units.

Example 16

Find the exact area of the shaded region.



Solution

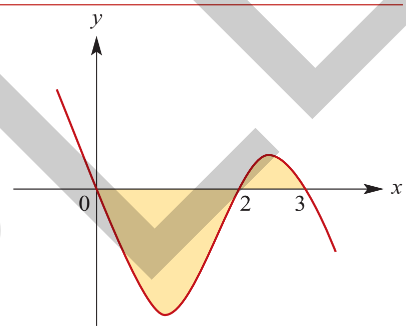
$$\begin{aligned}\text{Area} &= \int_2^4 x^2 - 4dx + -\int_1^2 x^2 - 4dx \\ &= \left[\frac{x^3}{3} - 4x\right]_2^4 - \left[\frac{x^3}{3} - 4x\right]_1^2 \\ &= \frac{64}{3} - 16 - \left(\frac{8}{3} - 8\right) - \left[\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right]\end{aligned}$$

$$= \frac{56}{3} - 8 - \left[\frac{7}{3} - 4\right]$$
$$= \frac{56}{3} - \frac{7}{3} - 4$$
$$= \frac{49}{3} - \frac{12}{3}$$
$$= \frac{37}{3}$$

The area is $\frac{37}{3}$ square units.

Example 17

Find the exact area of the regions enclosed by the graph of $y = x(2 - x)(x - 3)$ and the x-axis.



Solution

$$y = x(-x^2 + 5x - 6)$$
$$= -x^3 + 5x^2 - 6x$$

Note: There is no need to find the coordinates of the stationary points.

$$\text{Area} = \int_2^3 (-x^3 + 5x^2 - 6x)dx + -\int_0^2 (-x^3 + 5x^2 - 6x)dx$$
$$= \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_2^3 + -\left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_0^2$$
$$= \left(\frac{-81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right) - \left(-4 + \frac{40}{3} - 12\right)$$
$$= \frac{-81}{4} + 18 + 32 - \frac{80}{3}$$
$$= 50 - \frac{243 + 320}{12}$$
$$= 3\frac{1}{12}$$

Area is $3\frac{1}{12}$ square units.

Exercise 12F

1 Find the exact area of the region(s) bounded by the x-axis and the graph of each of the following:

a $y = 3x^2 + 2$ between $x = 0$ and $x = 1$ (sketch the graph first)

b $y = x^3 - 8$ between $x = 2$ and $x = 4$ (sketch the graph first)

c $y = 4 - x$ between:

  i $x = 0$ and $x = 4$ (sketch the graph first)

  ii $x = 0$ and $x = 6$ (sketch the graph first)

2 Find the exact area bounded by the x-axis and the graph of each of the following:

a $y = x^2 - 2x$ **b** $y = (4 - x)(3 - x)$ **c** $y = (x + 2)(7 - x)$

d $y = x^2 - 5x + 6$ **e** $y = 3 - x^2$ **f** $y = x^3 - 6x^2$

3 For each of the following sketch a graph to illustrate the region for which the definite integral gives the area:

a $\int_1^4 2x + 1\, dx$ **b** $\int_0^3 3 - x\, dx$ **c** $\int_0^4 x^2\, dx$

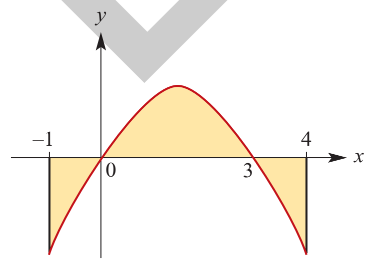
d $\int_{-1}^1 4 - 2x^2\, dx$ **e** $\int_2^4 \sqrt{x}\, dx$ **f** $\int_0^1 (1 - x)(1 + x)^2\, dx$

4 Find the exact area of the region bounded by the curve $y = 3x + 2x^{-2}$, the lines $x = 2$ and $x = 5$ and the x-axis.

5 Sketch the graph of $f(x) = 1 + x^3$ and find the exact area of the region bounded by the curve and the axes.

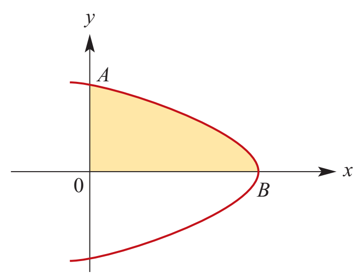
6 **a** Evaluate $\int_{-1}^4 x(3 - x)dx$.

b Find the exact area of the shaded region in the figure.



7 **a** In the figure, the graph of $y^2 = 9(1 - x)$ is shown. Find the coordinates of A and B.

b Find the exact area of the shaded region by evaluating $\int_0^b \left(1 - \frac{y^2}{9}\right) dy$ for a suitable choice of b.



8 **a** Show that $a^x = e^{x(\log_e a)}$ for $a > 0$.

b Hence find the derivative and an antiderivative of a^x.

c Hence or otherwise, show that the area under the curve $y = a^x$ $(a \geq 0)$ between the lines $x = 0$ and $x = b$ is $\frac{1}{\log_e a}(a^b - 1)$

9 Sketch the graph of $f(x) = 4e^{2x} + 3$ and find the exact area of the region enclosed between the curve, the axes and the line $x = 1$.

10 Sketch the graph of $y = x(2 - x)(x - 1)$ and find the exact area of the region enclosed between the curve and the x-axis.

11 Sketch the graph of $y = \frac{1}{2 - 3x}$ and find the exact area of the region enclosed by the curve, the x-axis and the lines with equations $x = -3$ and $x = -2$.

12 Sketch the graph of $y = 2 + \dfrac{1}{x+4}$ and find the exact area of the region enclosed by the curve, the axes and the line $x = -2$.

12.7 Integration of circular functions

Recall the following results from Chapter 11:

- If $f(x) = \sin(kx + a)$, then $f'(x) = k\cos(kx + a)$
- If $g(x) = \cos(kx + a)$, then $g'(x) = -k\sin(kx + a)$

Thus the following results hold:

$$\int \sin(kx + a)dx = -\frac{1}{k}\cos(kx + a) + c$$
$$\int \cos(kx + a)dx = \frac{1}{k}\sin(kx + a) + c$$

Example 18

Find antiderivatives of each of the following:

a $\sin\left(3x + \dfrac{\pi}{4}\right)$ **b** $\dfrac{1}{4}\sin 4x$

Solution

a $-\dfrac{1}{3}\cos\left(3x + \dfrac{\pi}{4}\right) + c$ **b** $-\dfrac{1}{16}\cos 4x + c$

Example 19

Find the exact value of each of the following definite integrals:

a $\displaystyle\int_0^{\frac{\pi}{4}} \sin 2x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{2}} 2\cos x + 1dx$

Solution

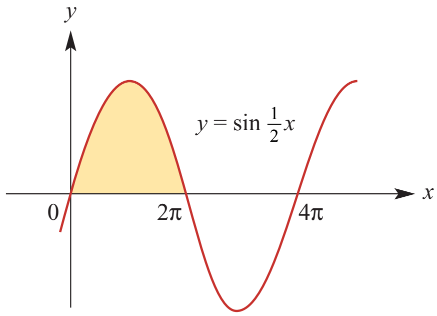
a
$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin 2x\,dx &= \left[\frac{-1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}\\ &= \frac{-1}{2}\cos\frac{\pi}{2} - \left(-\frac{1}{2}\cos 0\right)\\ &= 0 + \frac{1}{2}\\ &= \frac{1}{2}\end{aligned}$$

b
$$\begin{aligned}\int_0^{\frac{\pi}{2}} 2\cos x + 1dx &= [2\sin x + x]_0^{\frac{\pi}{2}}\\ &= 2\sin\frac{\pi}{2} + \frac{\pi}{2} - (2\sin 0 + 0)\\ &= 2 + \frac{\pi}{2}\end{aligned}$$

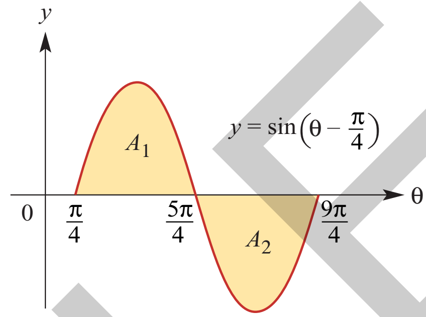
Example 20

Find the exact area of the shaded region for each of the graphs:

a



b



Solution

a Area $= \int_0^{2\pi} \sin \frac{1}{2}x \, dx$

$= \left[-2\cos\frac{1}{2}x\right]_0^{2\pi}$

$= -2\cos\pi - (-2\cos 0)$

$= 4$

$\therefore$ area of shaded region = 4 square units

b Regions A_1 and A_2 must be considered separately.

Area of $A_1 = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$

$= \left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$= -(\cos\pi - \cos 0)$

$= 2$

Area of $A_2 = -\int_{\frac{5\pi}{4}}^{\frac{9\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$

$= \left[\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{5\pi}{4}}^{\frac{9\pi}{4}}$

$= \cos 2\pi - \cos\pi$

$= 2$

$\therefore$ area of shaded region = 4 square units

Exercise 12G

1 Find an antiderivative of each of the following:

a $\cos 3x$ **b** $\sin\frac{1}{2}x$ **c** $3\cos 3x$ **d** $2\sin\frac{1}{2}x$

e $\sin\left(2x - \frac{\pi}{3}\right)$ **f** $\cos 3x + \sin 2x$ **g** $\cos 4x - \sin 4x$

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h $-\frac{1}{2}\sin 2x + \cos 3x$ **i** $-\frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right)$ **j** $\sin \pi x$

2 Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin x\,dx$ **b** $\int_0^{\frac{\pi}{4}} \cos 2x\,dx$ **c** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos\theta\,d\theta$

d $\int_0^{\frac{\pi}{2}} \sin\theta + \cos\theta\,d\theta$ **e** $\int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta$ **f** $\int_0^{\frac{\pi}{3}} (\cos 3\theta + \sin 3\theta)\,d\theta$

g $\int_0^{\frac{\pi}{3}} \left[\cos 3\theta + \sin\left(\theta - \frac{\pi}{3}\right)\right]d\theta$ **h** $\int_0^{\pi} \left(\sin\frac{x}{4} + \cos\frac{x}{4}\right)dx$

i $\int_0^{\frac{\pi}{4}} \sin\left(2x - \frac{\pi}{3}\right)dx$ **j** $\int_0^{\pi} \left(\cos 2x - \sin\frac{x}{2}\right)dx$

3 Calculate the exact area of the region bounded by the curve $y = \sin\frac{1}{2}x$, the x-axis and the line $x = \frac{\pi}{2}$.

4 Draw a graph to illustrate the area given by the definite integral and evaluate the integral for each of the following:

a $\int_0^{\frac{\pi}{4}} \cos x\,dx$ **b** $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$ **c** $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos 2x\,dx$

d $\int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta\,d\theta$ **e** $\int_0^{\frac{\pi}{2}} \sin 2\theta + 1\,d\theta$ **f** $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta\,d\theta$

5 Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{\pi}{4}\right)dx$ **b** $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{6}\right)dx$

c $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right)dx$ **d** $\int_0^{\frac{\pi}{4}} \cos(3\pi - x)\,dx$

6 Sketch the curve $y = 2 + \sin 3x$ for the interval $0 \le x \le \frac{2\pi}{3}$ and calculate the exact area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

12.8 Miscellaneous exercises

Example 21

If $f(x) = \log_e(x^2 + 1)$:

a Show that $f'(x) = \frac{2x}{x^2 + 1}$ **b** Hence evaluate $\int_0^2 \frac{x}{x^2 + 1}dx$

Solution

a Let $y = \log_e(x^2 + 1)$

and $v = x^2 + 1$

Then $\dfrac{dv}{dx} = 2x$

and $y = \log_e v$

$$\therefore \frac{dy}{dv} = \frac{1}{v}$$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

$$= \frac{1}{v} \cdot 2x$$

i.e. $f'(x) = \dfrac{2x}{x^2 + 1}$

b
$$\int_0^2 \frac{x}{x^2+1} dx = \frac{1}{2} \int_0^2 \left(\frac{2x}{x^2+1}\right) dx$$
$$= \frac{1}{2}[\log_e(x^2+1)]_0^2$$
$$= \frac{1}{2}[\log_e 5 - \log_e 1]$$
$$= \frac{1}{2}\log_e 5$$

Example 22

If $f(x) = \dfrac{\cos x}{\sin x}$:

a Show that $f'(x) = \dfrac{-1}{\sin^2 x}$

b Hence evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x} dx$

Solution

a Using the quotient rule:

$$f'(x) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$
$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$
$$= \frac{-1}{\sin^2 x}$$

b
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x} dx = -\left[\frac{\cos x}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= -\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} + \frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$$
$$= 1$$

Example 23

a If $f(x) = x\log_e(kx)$, find $f'(x)$ and hence find $\int \log_e(kx)dx$ where k is a positive real constant.

b If $f(x) = x^2\log_e(kx)$, find $f'(x)$ and hence find $\int x\log_e(kx)dx$ where k is a positive real constant.

Solution

a $f'(x) = \log_e(kx) + x \times \dfrac{1}{x} = \log_e(kx) + 1$

Antidifferentiating both sides of the equation with respect to x gives:

$$\int f'(x)dx = \int \log_e(kx)\,dx + \int 1dx$$

$$x\log_e(kx) + c_1 = \int \log_e(kx)dx + x + c_2$$

Therefore $\displaystyle\int \log_e(kx)dx = x\log_e(kx) - x + c_1 - c_2$

And $\displaystyle\int \log_e(kx)\,dx = x\log_e(kx) - x + c$

b $f'(x) = 2x \log_e(kx) + x^2 \times \dfrac{1}{x}$
$\quad = 2x \log_e(kx) + x$

Antidifferentiating both sides of the equation with respect to x gives:

$$\int f'(x)dx = \int 2x \log_e(kx)dx + \int x\,dx$$

$$x^2 \log_e(kx) + c_1 = \int 2x \log_e(kx)dx + \frac{x^2}{2} + c_2$$

$$\text{Therefore } \int x \log_e(kx)dx = \frac{1}{2}x^2 \log_e(kx) - \frac{x^2}{4} + c$$

Using the TI-Nspire

It is not possible to find rules for antiderivatives of all functions, for example e^{-x^2}. It is possible to find approximations for definite integrals that involve functions for which a rule for the antiderivative cannot be determined.

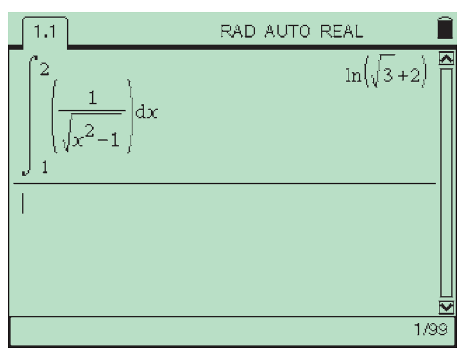
In other cases a CAS calculator can be used to find an antiderivative of a function for which the rule does exist, but students are not required to be able to find this rule by hand. The following three examples illustrate these cases.

Example 24

Find $\int_1^2 \dfrac{1}{\sqrt{x^2-1}}dx$.

Solution

Use the **Integral** template from the **Calculus** menu (menu 4 2) and complete as shown.



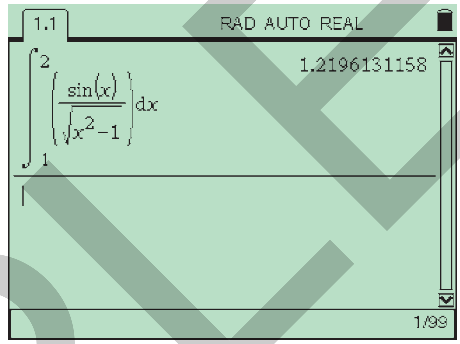
Example 25

Find $\int_1^2 \frac{\sin(x)}{\sqrt{x^2-1}}dx$, correct to two decimal places.

Solution

Use the **Integral** template from the **Calculus** menu (menu 4 2) and complete as shown.

Note that it is impossible to find an antiderivative of $\frac{\sin(x)}{\sqrt{x^2-1}}$, so the calculator returns an approximate (decimal) answer.

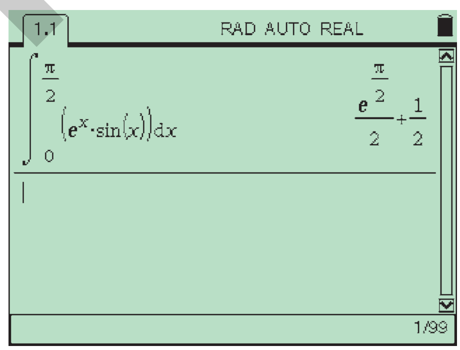


Example 26

Find $\int_1^{\frac{\pi}{2}} e^x \sin(x)\,dx$

Solution

Use the **Integral** template from the **Calculus** menu (menu 4 2) and complete as shown.



Exercise 12H

1 Find the exact value of each of the following:

a $\int_1^4 \sqrt{x}\,dx$

b $\int_{-1}^1 (1+x)^2 dx$

c $\int_0^8 \sqrt[3]{x}\,dx$

d $\int_0^{\frac{\pi}{3}} \cos 2x - \sin \frac{1}{2}x\,dx$

e $\int_1^2 e^{2x} + \frac{4}{x}dx$

f $\int_0^{\frac{\pi}{2}} (\sin 2x + \cos 3x)dx$

g $\int_0^{\pi} \sin\frac{x}{4} + \cos\frac{x}{4}dx$

h $\int_0^{\frac{\pi}{2}} 5x + \sin 2x\,dx$

i $\int_1^4 \left(2+\frac{1}{x}\right)^2 dx$

j $\int_0^1 x^2(1-x)dx$

2 Find the exact area of the region between the graph of $f(x) = \sin x$, the x-axis and the line $x = \frac{\pi}{3}$.

3 Find the value of the following definite integrals, correct to two decimal places:

a $\displaystyle\int_0^{20} 10\cos\left(\frac{\pi x}{40}\right) e^{\frac{x}{80}}\,dx$

b $\displaystyle\int_2^5 \frac{e^x}{(x-1)^2}\,dx$

c $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos x}{(x-1)^2}\,dx$

d $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{100\cos x}{x^2}\,dx$

e $\displaystyle\int_0^{\pi} e^{\left(\frac{x}{10}\right)^2} \sin x\,dx$

f $\displaystyle\int_0^{\frac{\pi}{4}} \cos^3(x)e^{-x}\,dx$

4 **a** Show that $\dfrac{2x+3}{x-1} = 2 + \dfrac{5}{x-1}$ **b** Hence evaluate $\displaystyle\int_2^4 \frac{2x+3}{x-1}\,dx$

5 **a** Show that $\dfrac{5x-14}{x-2} = 5 + \dfrac{\;}{x-2}$ **b** Hence evaluate $\displaystyle\int_3^4 \frac{5x-14}{x-2}\,dx$

6 **a** If $y = \left(1 - \frac{1}{2}x\right)^8$, find $\frac{dy}{dx}$. Hence or otherwise, find $\displaystyle\int \left(1 - \frac{1}{2}x\right)^7 dx$

b If $y = \log_e|\cos x|$, find $\frac{dy}{dx}$. Hence evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \tan x\,dx$

7 Find a function f such that $f'(x) = \sin\left(\frac{1}{2}x\right)$ and $f\left(\frac{4\pi}{3}\right) = 2$

8 For each of the following, find $f(x)$:

a $f'(x) = \cos 2x$ and $f(\pi) = 1$

b $f'(x) = \frac{3}{x}$ and $f(1) = 6$

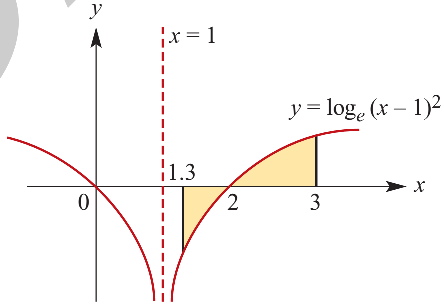
c $f'(x) = e^{\frac{x}{2}}$ and $f(0) = 1$

9 Find $\frac{d}{dx}(x\sin 3x)$, and hence evaluate $\displaystyle\int_0^{\frac{\pi}{6}} x\cos 3x\,dx$

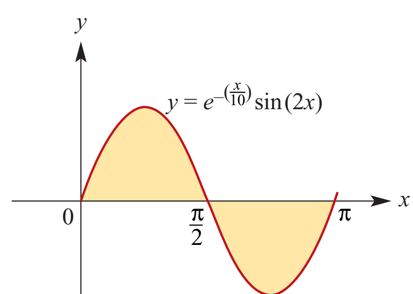
10 The curve with equation $y = a + b\sin\frac{\pi x}{2}$ passes through the points (0, 1) and (3, 3). Find a and b. Find the area of the region enclosed between this curve, the x-axis and the lines $x = 0$ and $x = 1$.

11 Find the area of the shaded region for each of the following shaded areas, correct to three decimal places:

a



b



12 Evaluate $\int_0^{\pi} e^{-\left(\frac{x}{10}\right)} \sin(2x)dx$, correct to four decimal places.

13 The gradient of a curve with equation $y = f(x)$ is given by $f'(x) = x + \sin 2x$ and $f(0) = 1$. Find $f(x)$.

14 Let $f(x) = g'(x)$ and $h(x) = k'(x)$, where $g(x) = (x^2 + 1)^3$ and $k(x) = \sin x^2$. Find:

a $\int f(x)dx$ **b** $\int h(x)dx$ **c** $\int f(x) + h(x)dx$

d $\int -f(x)dx$ **e** $\int (f(x) - 4)dx$ **f** $\int 3h(x)dx$

15 a Differentiate $\frac{\sin x}{\cos x}$ and hence find an antiderivative of $\frac{1}{\cos^2 x}$

b Differentiate $\log_e(3x^2 + 7)$ and hence evaluate $\int_0^2 \frac{x}{(3x^2 + 7)}dx$

c Find the derivatives of $x + \sqrt{1 + x^2}$ and $\log_e(x + \sqrt{1 + x^2})$

By simplifying your last result if necessary, evaluate $\int_0^1 \frac{1}{\sqrt{1 + x^2}}dx$

16 Sketch the graph of $y = \frac{2}{x - 1} + 4$ and evaluate $\int_2^3 \frac{2}{x - 1} + 4dx$
Indicate on your graph the region for which you have determined the area.

17 Sketch the graph of $y = \sqrt{2x - 4} + 1$ and evaluate $\int_2^3 \sqrt{2x - 4} + 1$
Indicate on your graph the region for which you have determined the area.

18 Evaluate each of the following:

a $\int_3^4 \sqrt{x - 2}dx$ **b** $\int_0^2 \sqrt{2 - x}dx$ **c** $\int_0^1 \frac{1}{3x + 1}dx$

d $\int_1^2 \frac{1}{2x - 1} + 3dx$ **e** $\int_{2.5}^3 \sqrt{2x - 5} - 6dx$ **f** $\int_3^4 \frac{1}{\sqrt{x - 2}}dx$

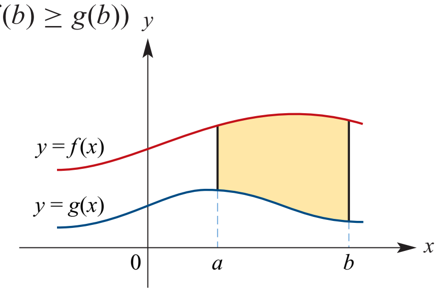
12.9 Area of a region between two curves

Let f and g be continuous on $[a, b]$ such that:

$$f(u) \geq g(u) \text{ for } u \in [a, b], (f(a) \geq g(a), f(b) \geq g(b))$$

Then the area of the region bounded by the curves and the lines $x = a$ and $x = b$ can be calculated by evaluating:

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$



Example 27

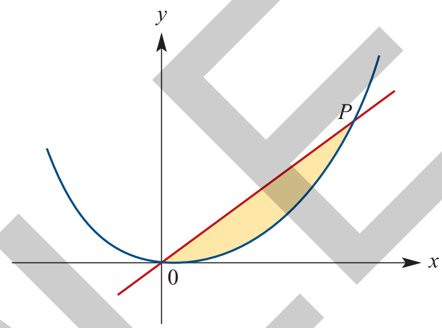
Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution

We first find the coordinates of the point P.

For $2x = x^2$, $x(x - 2) = 0$, which implies $x = 0$ or $x = 2$.

Therefore the coordinates of P are (2, 4).



$$\text{The required area} = \int_0^2 2x - x^2 dx$$
$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$
$$= 4 - \frac{8}{3}$$
$$= \frac{4}{3}$$

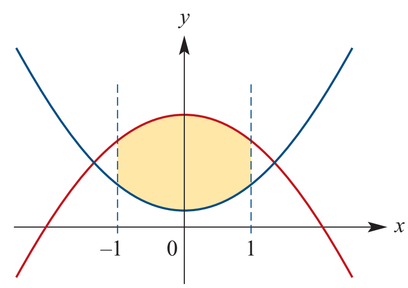
The area is $\frac{4}{3}$ square units.

Example 28

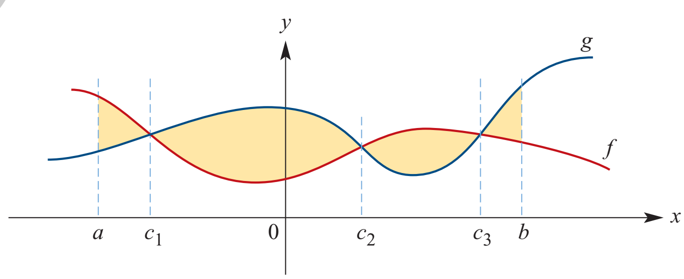
Calculate the area of the region enclosed between the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$, and the lines $x = 1$ and $x = -1$.

Solution

$$\text{Required area} = \int_{-1}^{1} 4 - x^2 - (x^2 + 1)\, dx$$
$$= \int_{-1}^{1} (3 - 2x^2)\, dx$$
$$= \left[3x - \frac{2x^3}{3}\right]_{-1}^{1}$$
$$= \frac{14}{3}$$



In Examples 27 and 28, the graph of one function is always 'above' the graph of the other for the intervals considered. What happens when the graphs cross?



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To find the area of the shaded region we must consider the intervals $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$ and $[c_3, b]$ separately. Thus, the shaded area is given by:

$$\int_a^{c_1} f(x) - g(x)dx + \int_{c_1}^{c_2} g(x) - f(x)dx + \int_{c_2}^{c_3} f(x) - g(x)dx + \int_{c_3}^{b} g(x) - f(x)dx$$

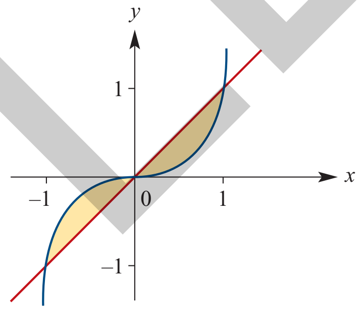
Example 29

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$

Solution

The graphs intersect where $f(x) = g(x)$

$$\begin{aligned} x^3 &= x \\ \therefore x^3 - x &= 0 \\ \therefore x(x^2 - 1) &= 0 \\ \therefore x = 0 \text{ or } x &= \pm 1 \end{aligned}$$



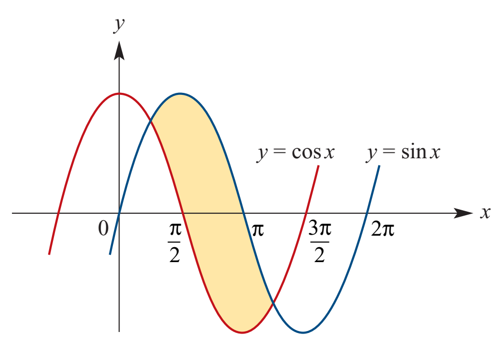
We see that $f(x) \geq g(x)$ for $-1 \leq x \leq 0$ and $f(x) \leq g(x)$ for $0 \leq x \leq 1$

Thus the area is given by:

$$\begin{aligned} \int_{-1}^{0} f(x) - g(x)dx + \int_0^1 g(x) - f(x)dx &= \int_{-1}^{0} x^3 - x dx + \int_0^1 x - x^3 dx \\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 \\ &= -\left(-\frac{1}{4}\right) + \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

Example 30

Find the area of the shaded region of the graph.



Solution

To find the coordinates of the point of intersection:

$$\sin x = \cos x, \tan x = 1, x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\begin{aligned}
\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx \\
&= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= -\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\
&= \frac{4}{\sqrt{2}} \\
&= 2\sqrt{2}
\end{aligned}$$

$\therefore$ area of shaded region $= 2\sqrt{2}$ square units

Exercise 12I

1 Find the exact area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = (x - 1)^2$

2 Find the exact area of the region bounded by the graphs of $y = 12 - x - x^2$ and $y = x + 4$

3 Find the exact area of the region bounded by the graphs with equations:

a $y = x + 3$ and $y = 12 + x - x^2$

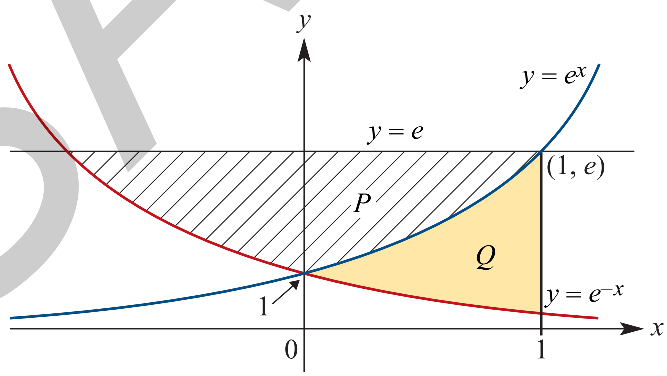
b $y = 3x + 5$ and $y = x^2 + 1$

c $y = 3 - x^2$ and $y = 2x^2$

d $y = x^2$ and $y = 3x$

e $y^2 = x$ and $x - y = 2$

4

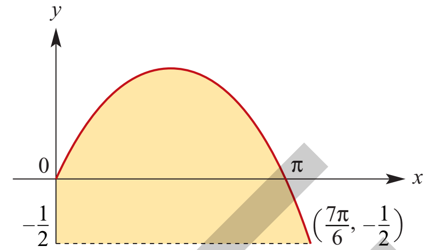


Find the areas of:

a the region P

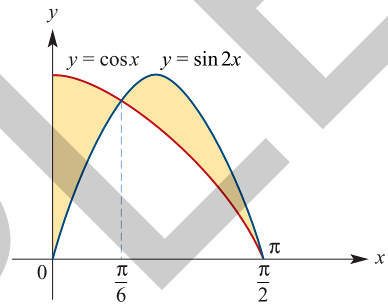
b the region Q

5 The figure shows part of the curve $y = \sin x$ Calculate the area of the shaded region, correct to three decimal places.



6 Using the same axes sketch the curves $y = \sin x$ and $y = \sin 2x$ for $0 \le x \le \pi$. Calculate the smaller of the two areas enclosed by the curves.

7 Find the area of the shaded region.



8 Find the coordinates of P, the point of intersection of the curves $y = e^x$ and $y = 2 + 3e^{-x}$. If these curves cut the y-axis at points A and B respectively, calculate the area bounded by AB and the arcs AP and BP. Give your answer correct to three decimal places.

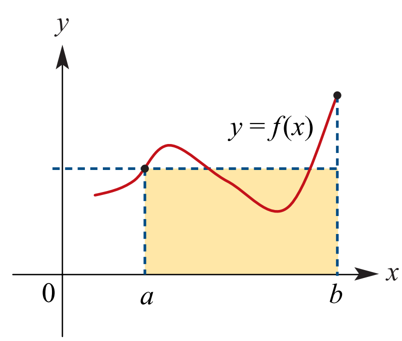
12.10 Applications of integration

Average value of a function

The average value of a function f with rule $y = f(x)$ for an interval $[a, b]$ is defined as:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

In terms of the graph of $y = f(x)$, the average is the height of a rectangle having the same area as the area under the graph for the interval $[a, b]$.

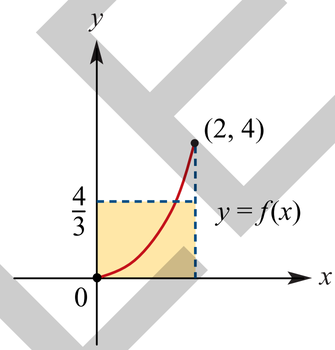


Example 31

Find the average value of $f(x) = x^2$ for the interval [0, 2]. Illustrate with a horizontal line determined by this value.

Solution

$$\begin{aligned}\text{Average} &= \frac{1}{2-0}\int_0^2 f(x)dx \\ &= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 \\ &= \frac{1}{2}\times\frac{8}{3} \\ &= \frac{4}{3}\end{aligned}$$



Note: The area of the rectangle $= \int_0^2 f(x)dx$

Kinematics

We may be given a rule for acceleration at time t and by the use of antidifferentiation with respect to t and some additional information we can deduce rules for both velocity and position.

Example 32

A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity (v cm/s) is given by $v = 2t - 4$. Find:

a its position x in terms of t
b its position after 3 seconds
c its average velocity in the first 3 seconds
d the distance travelled in the first 3 seconds
e its average speed in the first 3 seconds

Solution

a Antidifferentiate with respect to t to find the expression for position x m at time t seconds:

$$x = t^2 - 4t + c$$

When $t = 0$, $x = 0$ and therefore $c = 0$:

$$\therefore\ x = t^2 - 4t$$

b When $t = 3$, $x = -3$. The body is 3 units to the left of O.

c Average velocity $= \dfrac{-3-0}{3} = -1$ m/s

d $v = 0$ when $2t - 4 = 0$, i.e. when $t = 2$

When $t = 2$, $x = -4$

Therefore the body goes from $x = 0$ to $x = -4$ in the first 2 seconds and then back to -3 in the next second. It has travelled 5 m in the first 3 seconds.

e Average speed is $\frac{5}{3}$ m/s.

Example 33

A particle starts from rest 3 metres from a fixed point and moves in a straight line with an acceleration of $a = 6t + 8$. Find its position and velocity at time t seconds.

Solution

$$a = \frac{dv}{dt} = 6t + 8$$

By antidifferentiating $v = 3t^2 + 8t + c$

At $t = 0$, $v = 0$ and so $c = 0$

$$\therefore\ v = 3t^2 + 8t$$

By antidifferentiating again $x = t^3 + 4t^2 + d$

At $t = 0$, $x = 3$ and so $d = 3$

$$\therefore\ x = t^3 + 4t^2 + 3$$

Example 34

A stone is projected vertically upward from the top of a building 20 m high with an initial velocity of 15 m/s.

Find:

a the time taken for the stone to reach its maximum height

b the maximum height reached by the stone

c the time taken for the stone to reach the ground

d the velocity of the stone as it hits the ground

In this case we only consider the stone's motion in a vertical direction so we can consider it as rectilinear motion. Also we will assume that the acceleration due to gravity is approximately -10 m/s^2 (note that downward is considered the negative direction).

Solution

Given that $a = -10$

$$\therefore\ v = -10t + c$$

At $t = 0$, $v = 15$

$$\therefore\ v = -10t + 15$$

$$\therefore\ x = -5t^2 + 15t + d$$

At $t = 0$, $x = 20$

$$\therefore\ x = -5t^2 + 15t + 20$$

a The stone will reach its maximum height when $v = 0$

$$\therefore\ -10t + 15 = 0$$

which implies $t = 1.5$

b At $t = 1.5$, $x = -5(1.5)^2 + 15(1.5) + 20$
$= 31.25$

The maximum height reached by the stone is 31.25 metres.

c The stone reaches the ground when $x = 0$

$$-5t^2 + 15t + 20 = 0$$
$$-5(t^2 - 3t - 4) = 0$$
$$-5(t - 4)(t + 1) = 0$$
$$\therefore\ t = 4 \text{ (solution of } t = -1 \text{ is rejected since } t \geq 0)$$

i.e. the stone takes 4 seconds to reach the ground.

d At $t = 4$, $v = -10(4) + 15$
$= -25$

i.e. velocity on impact is -25 m/s.

Other applications

Example 35

The rate of change of temperature with respect to time of a liquid which has been boiled and then allowed to cool is given by $\dfrac{dT}{dt} = -0.5(T - 30)$, where T is the temperature (°C) at time t (min).

a Sketch the graph of $\dfrac{dT}{dt}$ against T for $T > 30$.

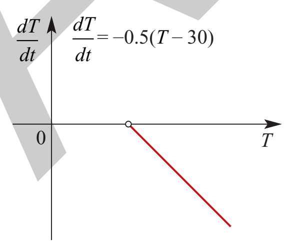
b Sketch the graph of $\dfrac{dt}{dT}$ against T for $T > 30$.

c **i** Find the area of the region enclosed between the graph of **b**, the x-axis and the lines $T = 35$ and $T = 120$. Give your answer correct to two decimal places.

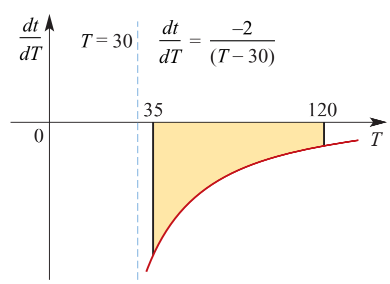
ii What does this area represent?

Solution

a



b



c **i** Area $= -\displaystyle\int_{35}^{120} \frac{-2}{(T - 30)}\,dt$
$= 5.78$

ii The area represents the time taken for the temperature to change from 35°C to 120°C.

Example 36

The function $f\colon R^+ \cup \{0\} \to R,\ f(x) = \log_e(x+1)$

a Find f^{-1} and sketch the graphs of f and f^{-1} on the one set of axes.

b Find the exact value of the area $\int_0^{\log_e 2} f^{-1}(x)dx$

c Find the exact value of $\int_0^1 f(x)dx$

Solution

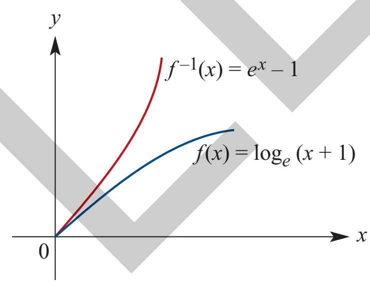
a Let $x = \log_e(y+1)$

$\therefore e^x = y + 1$

$\therefore y = e^x - 1$

$\therefore$ the inverse function is:

$f^{-1}\colon R^+ \cup \{0\} \to R,\ f^{-1}(x) = e^x - 1$

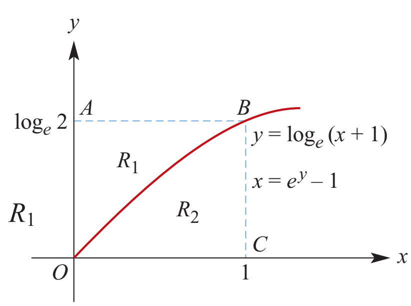


b
$$\begin{aligned}\int_0^{\log_e 2} f^{-1}(x)dx &= \int_0^{\log_e 2} e^x - 1dx\\ &= [e^x - x]_0^{\log_e 2}\\ &= e^{\log_e 2} - \log_e 2 - (e^0)\\ &= 2 - \log_e 2 - 1\\ &= 1 - \log_e 2\end{aligned}$$

c Area of rectangle $OABC = \log_e 2$

$$\begin{aligned}\text{Area of region } R_1 &= \int_0^{\log_e 2} e^y - 1dy\\ &= [e^y - y]_0^{\log_e 2}\\ &= 1 - \log_e 2 \text{ (see b)}\end{aligned}$$

$$\begin{aligned}\text{Area of region } R_2 &= \text{area of rectangle } OABC - \text{area of region } R_1\\ &= \log_e 2 - (1 - \log_e 2)\\ &= 2\log_e 2 - 1\end{aligned}$$



$$\therefore \int_0^1 f(x)dx = 2\log_e 2 - 1$$

Exercise 12J

1 Find the average value of each of the following functions for the stated interval:

a $f(x) = x(2-x),\ x \in [0, 2]$

b $f(x) = \sin(x),\quad x \in [0, \pi]$

c $f(x) = \sin(x),\ x \in \left[0, \frac{\pi}{2}\right]$

d $f(x) = \sin(nx),\quad x \in \left[0, \frac{2\pi}{n}\right]$

e $f(x) = e^x + e^{-x},\ x \in [-2, 2]$

2 A body is cooling and its temperature, T, in degrees Celsius after t minutes is given by $T = 50e^{\frac{-t}{2}}$. What is its average temperature over the first 10 minutes of cooling?

3 Find the average speed over the given interval for each of the following speed functions. For each of them sketch a graph and mark in the average as a horizontal line. Time is in seconds and speed in metres per second.

a $v = 20t$ $\quad [0, 5]$

b $v = 24\sin\left(\frac{1}{4}\pi t\right)$ $\quad [0, 4]$

c $v = 5(1 - e^{-t})$ $\quad [0, 5]$

4 A body falls from rest. Its velocity in metres per second at time t seconds is given by $v = 9.8t$. Find the average velocity of the body over the first 3 seconds of its motion.

5 Find the mean value of $x(a - x)$ from $x = 0$ to $x = a$

6 A quantity of gas expands under pressure p N/m^2 according to the law $pv^{0.9} = 300$ where v m^3 is the volume of gas under pressure p N/m^2.

a What is the average pressure as the volume changes from $\frac{1}{2}$ m^2 to 1 m^2?

b If the change in volume in terms of t is given by $v = 3t + 1$, what is the average pressure as the time changes from 0 to 1?

7 A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity (v cm/s) is given by $v = 2t - 3$. Find:

a its position x in terms of t

b its position after 3 seconds

c its average velocity in the first 3 seconds

d the distance travelled in the first 3 seconds

e its average speed in the first 3 seconds

8 The velocity (v m/s) at time t seconds ($t \geq 0$) of a particle is given by $v = 2t^2 - 8t + 6$. It is initially 4 m to the right of a point O. Find:

a its displacement and acceleration at any time

b its displacement when the velocity is zero

c its acceleration when the velocity is zero

9 A body moves in a straight line with an acceleration of 8 m/s^2. If after 1 second it passes through O and after 3 seconds is 30 metres from O, find its initial displacement relative to O.

10 A body moves in a straight line so that its acceleration (a m/s^2) after time t seconds ($t \geq 0$) is given by $a = 2t - 3$. If the initial position of the body is 2 m to the right of a point O and its velocity is 3 m/s, find the particle's position and velocity after 10 seconds.

11 A body is projected vertically upwards with a velocity of 25 m/s. (Its acceleration due to gravity is -10 m/s^2.) Find:

a the particle's velocity at any time

b its height above the point of projection at any time

c the time it takes to reach its maximum height

d the maximum height reached

e the time taken to return to the point of projection

12 The function $f\colon R^{+} \to R$, $f(x) = \log_e(2x)$

a Find f^{-1} and sketch the graphs of f and f^{-1} on the one set of axes.

b Find the exact value of the area $\displaystyle\int_0^{\log_e 4} f^{-1}(x)\,dx$.

c Find the exact value of $\displaystyle\int_{\frac{1}{2}}^{2} f(x)\,dx$.

13 Heat escapes from a storage tank so that the rate of heat loss C in kilojoules per day is given by $\dfrac{dH}{dt} = 1 + \dfrac{3}{4}\sin\left(\dfrac{\pi t}{60}\right)$ for $0 \le t \le 200$ where $H(t)$ is the total accumulated heat loss at time t days after midday on 1 April.

a Sketch the graph of $\dfrac{dH}{dt}$ against t for $0 \le t \le 200$.

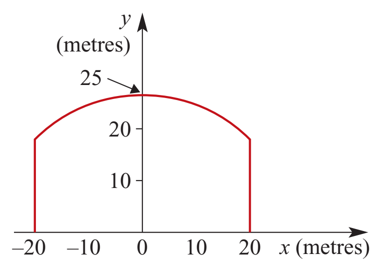
b Find the values of t for which the rate of heat loss, i.e. $\dfrac{dH}{dt}$, is greater than 1.375.

c Find the values of t for which the rate of heat loss reaches its maximum.

d Find the heat lost between

i $t = 0$ and $t = 120$ **ii** $t = 0$ and $t = 200$

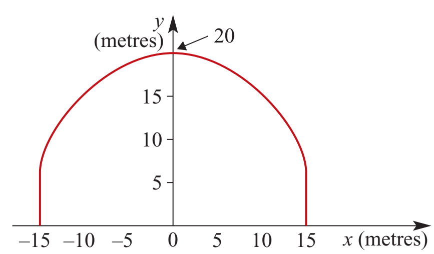
14 The roof of an exhibition hall has the shape of the function $f\colon [-20, 20] \to R$ where $f(x) = 25 - 0.02x^2$. The hall is 80 metres long. A cross-section of the hall is shown in the figure. An air conditioning company wishes to find the volume of the hall so that a suitable system may be installed. Find this volume.



15 An aircraft hangar has the cross-section illustrated. The roof has the shape of the function $f\colon [-15, 15] \to R$ where $f(x) = 20 - 0.06x^2$

a Find the area of the cross-section.

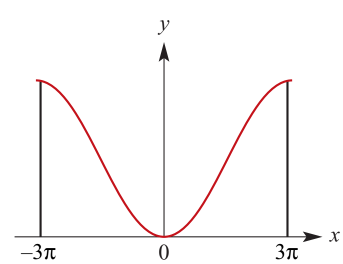
b Find the volume of the hangar if it is 100 metres long.



16 A long trough whose cross-section is parabolic is $1\frac{1}{2}$ metres wide at the top and 2 metres deep. Find the depth of water when it is half-full.

17 A sculpture has cross-section as shown. The equation of the curve is $y = 3 - 3\cos\left(\dfrac{x}{3}\right)$ for $x \in [-3\pi, 3\pi]$. All measurements are in metres.

a Find the maximum value of the function and hence the height of the sculpture.



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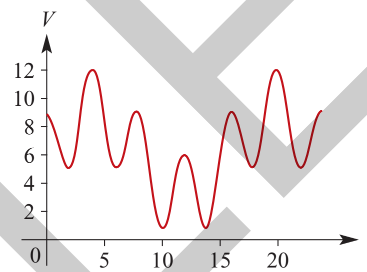
b The sculpture has a flat metal finish on one face, which in the diagram is represented by the region between the curve and the x-axis. Find the area of this region.

c There is a strut that meets the right side of the curve at right angles and passes through the point (9, 0).

i Find the equation of the normal to the curve where $x = a$.

ii Find, correct to three decimal places, the value of a if the normal passes through (9, 0).

18 The graph shows the number of litres/minute of water flowing through a pipe against number of minutes since the machine started. The pipe is attached to the machine which requires the water for cooling.



The curve has equation:

$$\frac{dV}{dt} = 3\left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2\right]$$

a What is the rate of flow of water when:

i $t = 0$? **ii** $t = 2$? **iii** $t = 4$?

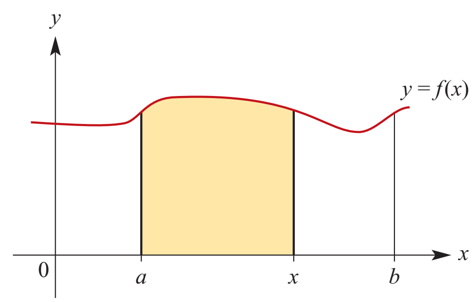
b Find, correct to three decimal places, the maximum and minimum flow through the pipe.

c Find the volume of water which flows through the pipe in the first 8 minutes.

12.11 The fundamental theorem of calculus

Let $f: [a, b] \rightarrow R$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$.

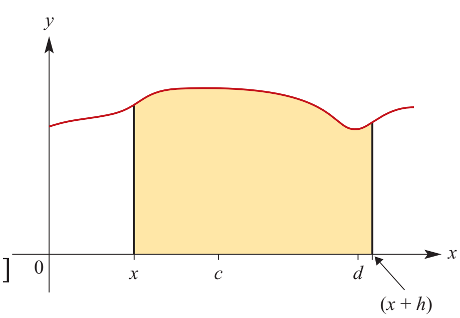
We define geometrically the function $F(x)$ by saying that it is the measure of the area under the curve between a and x. We thus have $F(a) = 0$. It will be shown that $F'(x) = f(x)$.



Consider the quotient $\dfrac{F(x+h) - F(x)}{h}$, $h > 0$.

By our definition of $F(x)$, $F(x+h) - F(x)$ is the area between x and $x + h$.

Let c be the point in $[x, x + h]$ such that $f(c) \geq f(z)$ for all $z \in [x, x + h]$ and let d be the point in the same interval such that $f(d) \leq f(z)$ for all $z \in [x, x + h]$.



Thus $\quad f(d) \leq f(z) \leq f(c)$ for all $z \in [x, x + h]$

Therefore $\quad hf(d) \leq F(x+h) - F(x) \leq hf(c)$

i.e. the shaded region has an area less than the area of the rectangle with base h and height $f(c)$ and an area greater than the area of the rectangle with base h and height $f(d)$.

Dividing by h $\quad f(d) \le \dfrac{F(x+h) - F(x)}{h} \le f(c)$

As $h \to 0$ both $f(c)$ and $f(d)$ approach $f(x)$. Thus we have shown $F'(x) = f(x)$.

We know that if $G(x)$ is an antiderivative of $f(x)$ then $F(x) = G(x) + k$, where k is a constant.

Let $x = a$. We then have:

$$0 = F(a) = G(a) + k, \text{ i.e. } k = -G(a)$$

Thus $F(x) = G(x) - G(a)$ and letting $x = b$ yields

$$F(b) = G(b) - G(a) \qquad (1)$$

The area under the curve $y = f(x)$ between a and b is $G(b) - G(a)$ where $G(x)$ is an antiderivative of $f(x)$.

A similar argument could be used if $f(x) < 0$ for all $x \in [a, b]$, but in this case we must take F to be the negative of the area under the curve.

We thus have by (1):

$$\int_a^b f(x)dx = G(b) - G(a) \text{ where } G \text{ is an antiderivative of } f$$

This result holds generally and is known as the **fundamental theorem of integral calculus**.

Finally the limit of a sum is considered in a special case. It is provided here as an indication of how this process can be undertaken in general.

We first introduce a notation to help us express sums. We do this through examples.

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

$$\sum_{i=1}^{3} y_i^2 = y_1^2 + y_2^2 + y_3^2$$

$$\sum_{i=1}^{4} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + x_4 f(x_4)$$

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n$$

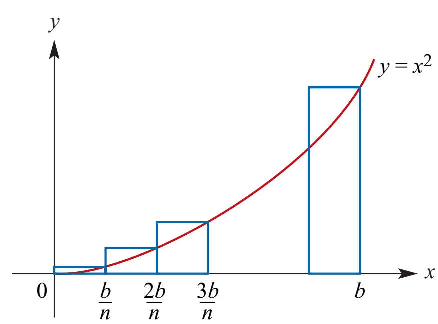
($\sum$ is the upper case of the Greek letter 'sigma' which is used in mathematics to denote *sum*.)

Consider the graph of $y = x^2$.

Divide the interval $[0, b]$ into n sub-intervals.

This gives a sequence of closed intervals.

$$\left[0, \frac{b}{n}\right], \left[\frac{b}{n}, \frac{2b}{n}\right], \left[\frac{2b}{n}, \frac{3b}{n}\right], \ldots, \left[\frac{(n-1)b}{n}, b\right]$$



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We consider the sum of the areas of the rectangles for which the height of each rectangle is defined by the right endpoint of the interval.

$$\begin{aligned}\text{Sum} &= \frac{b-0}{n}\left[f\left(\frac{b}{n}\right)+f\left(\frac{2b}{n}\right)+f\left(\frac{3b}{n}\right)+\cdots+f(b)\right]\\ &= \frac{b}{n}\left[\frac{b^2}{n^2}+\frac{4b^2}{n^2}+\frac{9b^2}{n^2}+\cdots+b^2\right]\\ &= \frac{b^3}{n^3}[1+4+9+\cdots+n^2]\\ &= \frac{b^3}{n^3}\sum_{i=1}^{n} i^2\end{aligned}$$

There is a rule for working out the sum of the first n square numbers. This is:

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\begin{aligned}\therefore\ \text{right endpoint estimate} &= \frac{b^3}{n^3}\times\frac{n}{6}(n+1)(2n+1)\\ &= \frac{b^3}{6n^2}(2n^2+3n+1)\\ &= \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\end{aligned}$$

As n becomes very large, the terms $\frac{3}{n}$ and $\frac{1}{n^2}$ become very small.

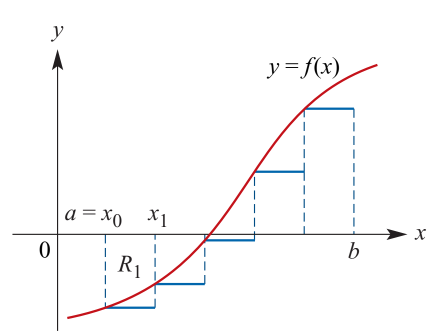
We write $\lim\limits_{h\to\infty}\frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)=\frac{b^3}{3}$.

We read this as 'the limit of the sum as n approaches infinity is $\frac{b^3}{3}$.

It can also be shown that working with the left endpoint rectangles for n rectangles and considering the limit as n approaches infinity gives $\frac{b^3}{3}$. (This result was first achieved by Archimedes.)

This may also be applied in general to a continuous function f, in an interval $[a, b]$. For convenience we will consider an increasing function.

We note that to calculate the area of rectangle R_1 we calculate $(x_1 - x_0)f(x_0)$ and as $f(x_0) < 0$ the result is a negative. We have found the signed area of R_1.



If an interval $[a, b]$ is divided into n sub-intervals and the rectangles to be used to find the area have their 'height' determined by the left endpoint of that interval then the area is given by:

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)$$

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If the limit as $n \to \infty$ exists, we can make the following definition:

$$\lim_{n \to \infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i) = \int_a^b f(x)dx$$

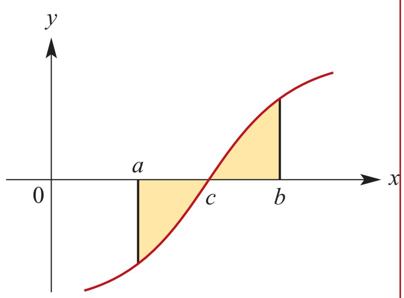
The procedure indicated could also have started with the right endpoint estimate being used, as the left and right endpoint estimates will converge to the one limit as n approaches infinity. Definite integrals may be defined as the limit of suitable sums and the fundamental theorem of calculus holds true under this definition.

MC TEST
D&D TEST

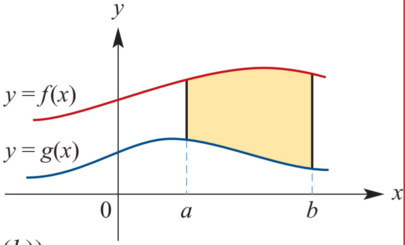
Review

Chapter summary

- To find the general antiderivative:
  If $F'(x) = f(x)$
  then $\int f(x)dx = F(x) + c$, where c is an arbitrary real number.
- $\int x^r dx = \frac{x^{r+1}}{r+1} + c, r \in Q\backslash\{-1\}$
- $\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$
- $\int kf(x)dx = k\int f(x)dx$ where k is a real number
- $\int e^{kx} dx = \frac{1}{k}e^{kx} + c, k \neq 0$
  $\int \sin kx dx = -\frac{1}{k} \cos kx + c, k \neq 0$
  $\int \cos kx dx = \frac{1}{k} \sin kx + c, k \neq 0$
  $\int \frac{1}{kx+b} dx = \frac{1}{k} \log_e |kx + b| + c, k \neq 0$
- The fundamental theorem of integral calculus states that:
  $\int_a^b f(x)dx = G(b) - G(a)$ where G is any antiderivative of f and $\int_a^b f(x)dx$ is called the definite integral from a to b.
  The number a is called the lower limit of integration and b is called the upper limit of integration. The function f is called the integrand.
- If $f(x) \geq 0$ for all $x \in [a, b]$, the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)dx$
  If $f(x) \leq 0$ for all $x \in [c, b]$ the area of the region contained between the curve, the x-axis and the lines $x = c$ and $x = b$ is given by $-\int_c^b f(x)dx$
  If $c \in [a, b]$, $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by:
  $$\int_c^b f(x)dx + \left\{-\int_a^c f(x)dx\right\}$$



- To find the area of the shaded region bounded by the curves and the lines $x = a$ and $x = b$ use
  $$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$
  where f and g are continuous on $[a, b]$ such that
  $$f(u) > g(u) \text{ for } u \in [a, b], (f(a) \geq g(a), f(b) \geq g(b))$$



- If the interval $[a, b]$ on the x-axis is divided into n equal sub-intervals

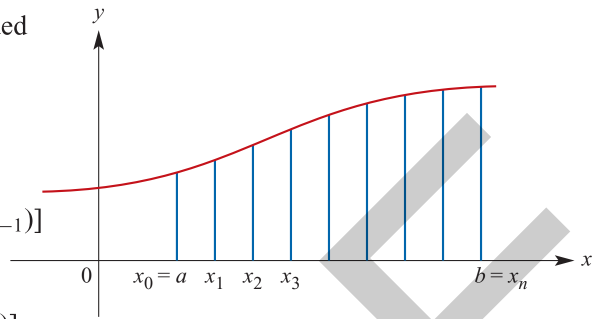
  $[a, x_1], [x_1, x_2][x_2, x_3], \ldots, [x_{n-1}, x_n]$

  the left endpoint estimate is:

  $$L_n = \frac{b-a}{n}[f(x_0) + f(x_1) + \cdots f(x_{n-1})]$$

  The right endpoint estimate is:

  $$R_n = \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots f(x_n)]$$



- The average value of an integrable function f for an interval $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Multiple-choice questions

1 $\displaystyle\int_0^2 3f(x) + 2dx =$

A $3\displaystyle\int_0^2 f(x)dx + 3x$ **B** $3\displaystyle\int_0^2 f(x)dx + x$ **C** $3\displaystyle\int_0^2 f(x)dx + 4$

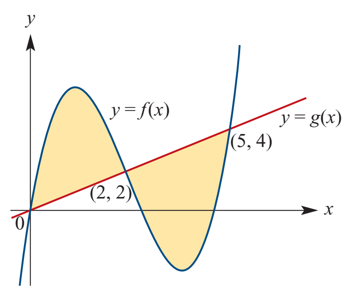
D $3f'(x) + 4$ **E** $\displaystyle\int_0^2 f(x)dx + 4$

2 For a and b positive real constants the antiderivative of $\sqrt{(ax-b)^3}$, $x > \dfrac{b}{a}$ is:

A $\dfrac{3}{2}(ax-b)^{\frac{1}{2}} + c$ **B** $\dfrac{1}{5\sqrt{(ax-b)}} + c$ **C** $\dfrac{2}{5a}(ax-b)^{\frac{5}{2}} + c$

D $\dfrac{1}{5a}\sqrt{(ax-b)^5} + c$ **E** $\dfrac{2}{5\sqrt{(ax-b)^5}} + c$

3 An expression using integral notation for the area of the shaded region shown is:



A $\displaystyle\int_0^5 f(x) - g(x)dx$

B $\displaystyle\int_0^2 f(x) - g(x)dx + \int_2^5 g(x) - f(x)dx$

C $\displaystyle\int_0^4 f(x) - g(x)dx$

D $\displaystyle\int_0^2 f(x) - g(x)dx + \int_2^4 f(x) - g(x)dx$

E $\displaystyle\int_0^2 f(x) - g(x)dx + \int_2^4 g(x) - f(x)dx$

4 $\displaystyle\int_a^b c\,dx$ in terms of a, b and c, where a, b and c are distinct real number constants, is equal to:

A ca **B** $cb - ca$ **C** $ca - b$ **D** $cb - a$ **E** $c(a + b)$

5 An expression for y if $\dfrac{dy}{dx} = \dfrac{ax}{2} + 1$ and $y = 1$ when $x = 0$ is:

A $y = \dfrac{ax^2}{4} + x + 1$ **B** $y = a$ **C** $y = ax^2 + x - 1$

D $y = ax^2 + x + a$ **E** $y = ax^2 + ax + a$

6 The function f such that $f'(x) = -6\sin 3x$ and $f\left(\dfrac{2\pi}{3}\right) = 3$ is:

A $-18\cos 3x + 21$ **B** $-2\cos 2x + 5$ **C** $-2\sin 3x + 1$

D $2\cos 3x + 1$ **E** $2\sin 4x + 3$

7 The area of the region enclosed by the curve $y = e^{5x} - 2\sin 4x$, the x-axis and the lines $x = -1$ and $x = 1$, correct to two decimal places, is:

A 0.17 **B** 29.55 **C** 29.68 **D** 29.85 **E** 30.02

8 Given $\dfrac{dy}{dx} = ae^{-x} + 2$ and that when $x = 0$, $\dfrac{dy}{dx} = 5$ and $y = 1$, then when $x = 2$, $y =$

A $-\dfrac{3}{e^2} + 2$ **B** $-\dfrac{3}{e^2} + 4$ **C** $-\dfrac{3}{e^2} + 8$ **D** $3e^2 + 4$ **E** $3e^2 + 8$

9 The rate of the flow of water from a tap follows the rule $R(t) = 5e^{-0.1t}$ where $R(t)$ litres per minute is the rate of flow after t minutes. The number of litres, to the nearest litre, which flowed out in the first 3 minutes is:

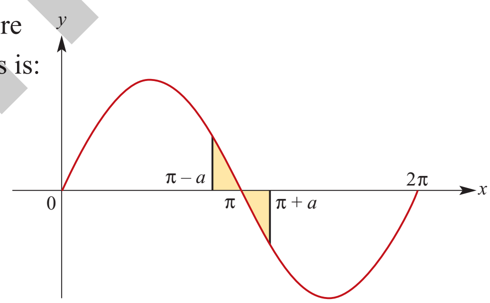
A 0 **B** 5 **C** 13 **D** 50 **E** 153

10 The graph represents the function $y = \sin x$ where $0 \le x \le 2\pi$. The total area of the shaded regions is:

A $\dfrac{1}{2}\cos a$ **B** $2\cos a$

C $\dfrac{1}{2}(1 - \cos a)$ **D** $2(1 - \cos a)$

E $2\sin^2 a$



Short-answer questions (technology-free)

1 Evaluate each of the following definite integrals:

a $\displaystyle\int_2^3 x^3\,dx$ **b** $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\theta$

c $\displaystyle\int_a^{4a} \left(a^{\frac{1}{2}} - x^{\frac{1}{2}}\right)dx$, where a is a positive constant.

d $\displaystyle\int_1^4 \frac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}}\,dx$ **e** $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2\theta\,d\theta$

f $\displaystyle\int_1^e \frac{1}{x}\,dx$ **g** $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2\left(\theta + \frac{\pi}{4}\right)d\theta$ **h** $\displaystyle\int_0^{\pi} \sin 4\theta\,d\theta$

2 Find $\displaystyle\int_{-1}^{2} x + 2f(x)\,dx$ if $\displaystyle\int_{-1}^{2} f(x)\,dx = 5$

Review

3 Find $\int_1^5 f(x)dx$ if $\int_0^1 f(x)dx = -2$ and $\int_0^5 f(x)dx = 1$

4 Find $\int_3^{-2} f(x)dx$ if $\int_{-2}^1 f(x)dx = 2$ and $\int_1^3 f(x)dx = -6$

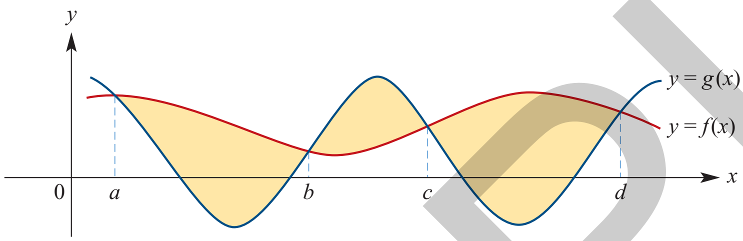
5 Evalutate $\int_0^2 (x+1)^7 dx$.

6 Evalute $\int_0^1 (3x+1)^3 dx$.

7 Find $\int_0^3 f(3x)dx$ if $\int_0^9 f(x)dx = 5$

8 Find $\int_0^1 f(3x+1)dx$ if $\int_1^4 f(x)dx = 5$

9 Set up a sum of definite integrals that represents the total shaded area between the curves $y = f(x)$ and $y = g(x)$

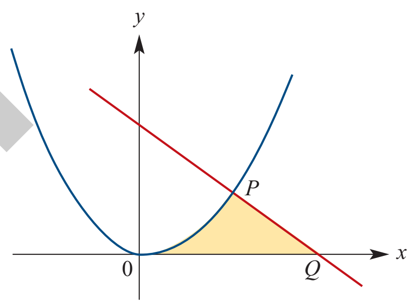


10 The figure shows the curve $y = x^2$ and the straight line $2x + y = 15$

Find:

a the coordinates of P and Q

b the area of the shaded region

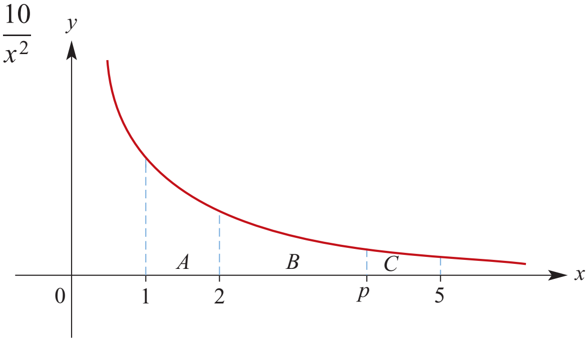


11 The figure shows part of the curve $y = \dfrac{10}{x^2}$

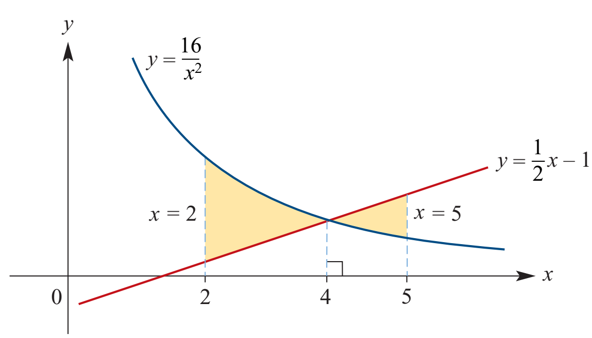
Find:

a the area of region A

b the value of p for which the regions B and C are of equal area

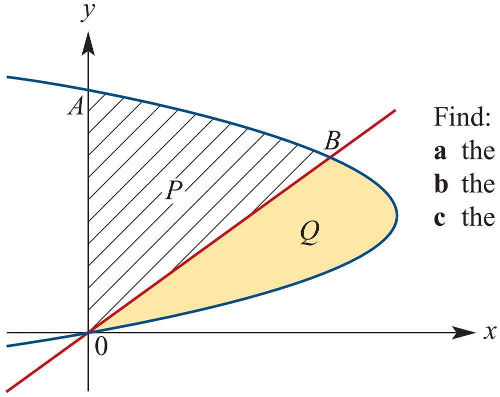


12 Find the area of the shaded region:



Review

13 The figure shows part of the curve $x = 6y - y^2$ and part of the line $y = x$



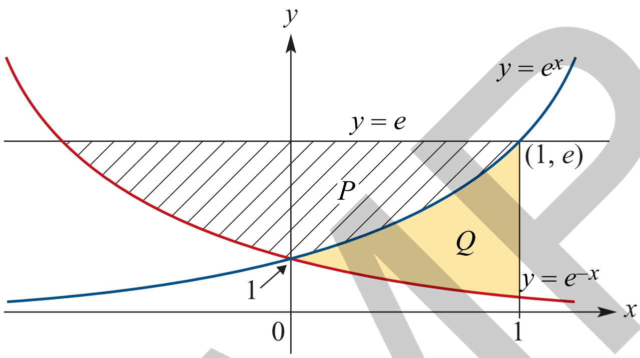
Find:

a the coordinates of A and B

b the area of region P

c the area of region Q

14



Find the areas of:

a the region P

b the region Q

15 **a** Sketch the graph of $y = e^x + 1$ and clearly indicate, by shading the region, the area given by $\int_0^2 e^x + 1dx$.

b Evaluate $\int_0^2 e^x + 1dx$.

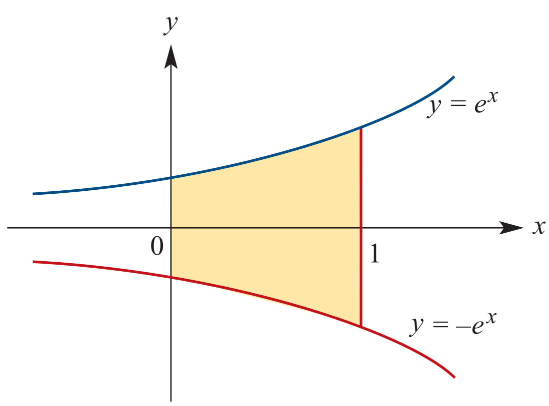
16 **a** Sketch the graph of $y = e^{-x}$ and $y = e^x$ on the one set of axes and clearly indicate, by shading the region, the area given by:

$$\int_0^2 e^{-x}dx + \int_{-2}^0 e^x dx$$

b Evaluate $\int_0^2 e^{-x}dx + \int_{-2}^0 e^x dx$.

17 **a** Evaluate $\int_0^1 e^x dx$.

b By symmetry find the area of the region shaded in the figure.



Review

18 Sketch the graph of $f(x) = 2e^{2x} + 3$ and find the area of the region enclosed between the curve, the axes and the line $x = 1$

19 Sketch the graph of $y = x(x - 2)(x + 1)$ and find the area of the region contained between the graph and the x-axis. (Do not attempt to find the coordinates of the turning points.)

20 Evaluate each of the following definite integrals:

a $\displaystyle\int_0^2 e^{-x} + x\,dx$ **b** $\displaystyle\int_{-2}^{-1} x + \frac{1}{x-1}\,dx$

c $\displaystyle\int_0^{\frac{\pi}{2}} \sin x + x\,dx$ **d** $\displaystyle\int_{-4}^{-5} e^x + \frac{1}{2-2x}\,dx$

Extended-response questions

1 The rate of flow of water from a reservoir is given by $\dfrac{dV}{dt} = 1000 - 30t^2 + 2t^3$, $0 \le t \le 15$ where V is measured in millions of litres and t is the number of hours after the sluice gates are opened.

a Find the rate of flow (in million litres/hour) when $t = 0$ and $t = 2$.

b Find:

i the times when the rate of flow is a maximum

ii the maximum flow

c Sketch the graph of $\dfrac{dV}{dt}$ against t for $0 \le t \le 15$.

d **i** Find the area beneath the graph between $t = 0$ and $t = 10$.

ii What does this area represent?

2 The population of penguins on an island off the coast of Tasmania is increasing steadily. The rate of growth is given by the function $R: [0, \infty) \to R$, $R(t) = 10 \log_e (t + 1)$. The rate is measured in number of penguins per year. The date 1 January 1875 coincides with $t = 0$.

a Find the rate of growth of penguins when $t = 5$, $t = 10$, $t = 100$.

b Sketch the graph of R for $[0, \infty)$.

c Find the inverse function R^{-1}.

d **i** Find the area under the graph of $y = R(t)$ between $t = 0$ and $t = 100$. (Use the inverse function to help find this area.)

ii What does this area represent?

3 The diagram shows part of the curve with equation

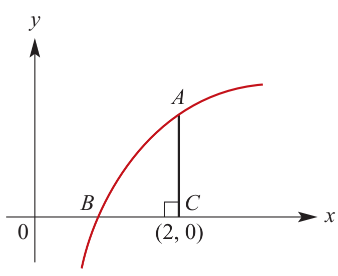
$$y = x - \frac{1}{x^2}$$

C is the point (2, 0).

Find:

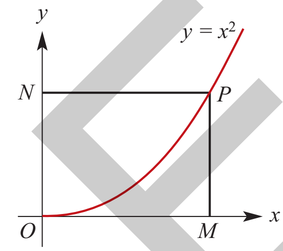
a the equation of the tangent to the curve at point A

b the coordinates of the point T where this tangent meets the x-axis

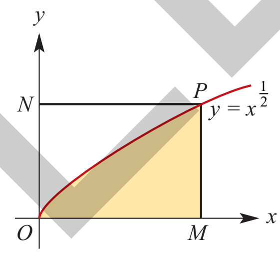


c the coordinates of the point B where the curve meets the x-axis
d the area of the region enclosed by the curve and the lines AT and BT
e the ratio of the area found in part **d** to the area of the triangle ATC

4 **a** In the figure, P is a point on the curve $y = x^2$. Prove that the curve divides the rectangle $OMPN$ into two regions whose areas are in the ratio $2 : 1$.

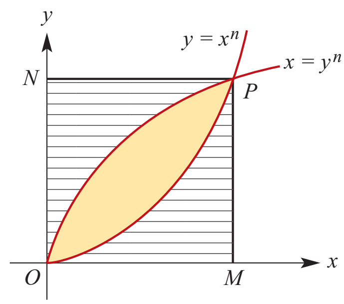


b In the figure, P is a point on the curve $y = x^{\frac{1}{2}}$. Prove that the area of the shaded region is two-thirds of the area of the rectangle $OMPN$.



c For the curve $y = x^n$, P is a point on the curve with PM and PN the perpendiculars from P to the x-axis and the y-axis respectively. Prove that the area of the region enclosed between PM, the x-axis and the curve is equal to $\dfrac{1}{n+1}$ of the area of the rectangle $OMPN$.

5 **a** Find the area enclosed between the parabolas $y^2 = x$ and $x^2 = y$.
b Show that the curves with equations $y = x^n$ and $y^n = x$ intersect at $(1, 1)$ (where $n = 1, 2, 3, \ldots$).
c Show that the area of the region contained between the curves $y = x^n$ and $y^n = x$ is $\dfrac{n-1}{n+1}$
d Find the area of the region indicated by horizontal shading in the diagram.
e Use your result from **c** to find the area of the region between the curves for $n = 10, 100, 1000$.
f Describe the result for n very large.



6 It is thought that the average temperature θ of a piece of charcoal in a barbecue will increase at a rate $\dfrac{d\theta}{dt}$ given by $\dfrac{d\theta}{dt} = e^{2.6t}$ where θ is in degrees and t is in minutes.
a If the charcoal starts at a temperature of 30°C, find the expected average temperature of the charcoal after 3 minutes.
b Sketch the graph of θ against t.
c At what time does the temperature of the charcoal reach 500°C?
d Find the average rate of increase of temperature from $t = 1$ to $t = 2$.

7 It is believed that the velocity of a certain atomic particle t microseconds after a collision will be given by the expression

$$\frac{dx}{dt} = ve^{-t}, \quad v = 5 \times 10^4 \text{ m/s}$$

where x stands for distance travelled in metres.

a What is the initial velocity of the particle?

b What happens to the velocity at $t \to \infty$ (i.e. t becomes very large)?

c How far will the particle travel between $t = 0$ and $t = 20$?

d Find an expression for x in terms of t. **e** Sketch the graph of x against t.

8 **a** Differentiate $e^{-3x} \sin 2x$ and $e^{-3x} \cos 2x$ with respect to x.

b Hence show that

$$e^{-3x} \sin 2x + c_1 = -3 \int e^{-3x} \sin 2x\, dx + 2 \int e^{-3x} \cos 2x\, dx$$

and $$e^{-3x} \cos 2x + c_2 = -3 \int e^{-3x} \cos 2x\, dx - 2 \int e^{-3x} \sin 2x\, dx$$

c Use the two equations from **b** to determine $\int e^{-3x} \sin 2x\, dx$.

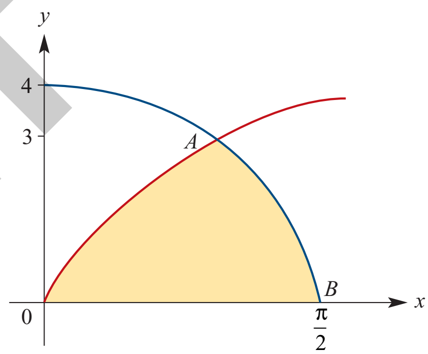
9 The curves $y = 3 \sin x$, $y = 4 \cos x$ $\left(0 \le x \le \frac{\pi}{2}\right)$ intersect at a point A.

a If $x = a$ at the point of intersection of the two curves:

i Find $\tan a$.

ii Hence find $\sin a$ and $\cos a$.

b Hence find the area of the shaded region in the diagram.



10 **a** If $y = x \log_e x$ find $\frac{dy}{dx}$. Hence find the value of $\int_1^e \log x\, dx$.

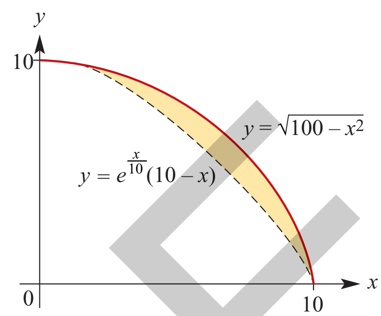
b If $y = x(\log_e x)^n$, where n is a positive integer, find $\frac{dy}{dx}$.

c Let $I_n = \int_1^e (\log_e x)^n dx$. For $n > 1$, show that $I_n + nI_{n-1} = e$

d Hence find the value of $\int_1^e (\log_e x)^3 dx$.

11 The curves $y^2 = ax$ and $x^2 = by$, where a and b are both positive, intersect at the origin and at the point (r, s). Find r and s in terms of a and b. Prove that the two curves divide the rectangle whose corners are the points $(0, 0)$, $(0, s)$, (r, s), $(r, 0)$ into three regions of equal area.

12 A teacher attempts to draw a quarter circle on the white board. The circle is of radius 10. However, the first attempt results in a curve with rule $y = e^{\frac{x}{10}}(10 - x)$. The circle has equation $y = \sqrt{100 - x^2}$



a Find $\dfrac{dy}{dx}$ for both functions.

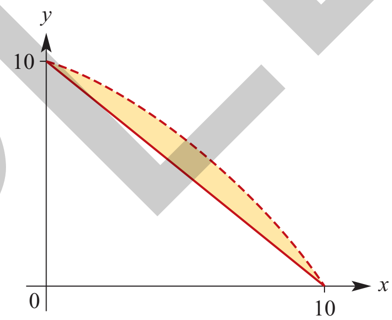
b Find the gradient of each of the functions when $x = 0$.

c Find the gradient of $y = e^{\frac{x}{10}}(10 - x)$ when $x = 10$.

d Find the area of the shaded region correct to two decimal places using a calculator.

e Find the percentage error for the calculation of the area of the quarter circle.

f The teacher draws in a chord from (0, 10) to (10, 0). Find the area of the shaded region using a calculator.



g **i** Use the result that the derivative of $e^{\frac{x}{10}}(10 - x)$ is $-e^{\frac{x}{10}} + \dfrac{1}{10}e^{\frac{x}{10}}(10 - x)$ to find $\displaystyle\int_0^{10} e^{\frac{x}{10}}(10 - x)\,dx$ by analytic techniques.

ii Find the exact area of the original shaded region and compare it to the answer of **d**.

13 A water-cooling device has a system of water circulation for the first 30 minutes of its operation. The circulation follows the following sequence:

- For the first three minutes water is flowing in.
- For the second three minutes water is flowing out.
- For the third three minutes water is flowing in.

This pattern is continued for the first 30 minutes. The rate of flow of water is given by the function

$$R(t) = 10e^{-\frac{t}{10}} \sin\left(\frac{\pi t}{3}\right)$$

where $R(t)$ litres/minute is the rate of flow at time t minutes. Initially there are 4 litres of water in the device.

a Find:

i $R(0)$ **ii** $R(3)$

b Find $R'(t)$

c **i** Solve the equation $R'(t) = 0$ for $t \in [0, 12]$.

ii Find the coordinates of the stationary points of $y = R(t)$ for $t \in [0, 12]$.

d Solve the equation $R(t) = 0$ for $t \in [0, 12]$.

e Sketch the graph of $y = R(t)$ for $t \in [0, 12]$.

f **i** How many litres of water flowed into the device for $t \in [0, 3]$?
ii How many litres of water flowed out of the device for $t \in [3, 6]$?
iii How many litres of water are in the device when $t = 6$? (Remember there are initially 4 litres of water.)

g How many litres of water are there in the container when $t = 30$?

14 **a** Sketch the graph of $f(x) = 2\sin x - 1, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

b Evaluate $\int_0^{\frac{\pi}{6}} f(x)dx$ and indicate the area given by this integral on the graph of **a**.

c Find the inverse function f^{-1}.

d Evaluate $\int_0^1 f^{-1}(x)dx$ and indicate the area given by this integral on the sketch graph of **a**.

15 **a** Use the identities $\cos 2x = 2\cos^2 x - 1$ and $\cos 2x = 1 - 2\sin^2 x$ to show that:

$$\frac{1 - \cos 2x}{1 + \cos 2x} = \sec^2 x - 1$$

b Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{1 + \cos 2x} dx$