Back to Menu >>>

# CHAPTER 8

# Applications of matrices and using parameters

## Objectives

- To review solving simultaneous equations in up to four unknowns using matrices
- To use matrices to define transformations and apply matrices
- To be able to use matrix equations in determining the equation of the image of a curve under linear transformations
- To be able to use parameters to describe families of curves

## 8.1 Systems of equations and using parameters

### Linear simultaneous equations with two unknowns

In Chapter 3 it was seen that simultaneous linear equations in two variables could be solved by using matrices. Using your CAS calculator will result in some outcomes that you need to be able to understand.

In this chapter, systems of equations for which the matrix of the coefficients is singular are also considered.

Remember that a $2 \times 2$ matrix is said to be singular if its determinant is equal to 0.

The matrix of coefficients being singular can correspond to one of two situations:

- There are infinitely many solutions.
- There is no solution.

**Example 1**

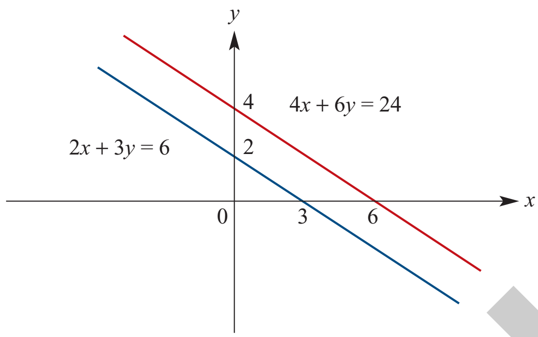
Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 24$ have no solution.

Back to Menu >>>



### Solution

The equations have no solution as they correspond to parallel lines and they are different lines.




Each of the lines has gradient $-\frac{2}{3}$.

The matrix of the coefficients of $x$ and $y$ is $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ and the determinant of this matrix is 0. That is, the matrix is singular.

## Example 2

The simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

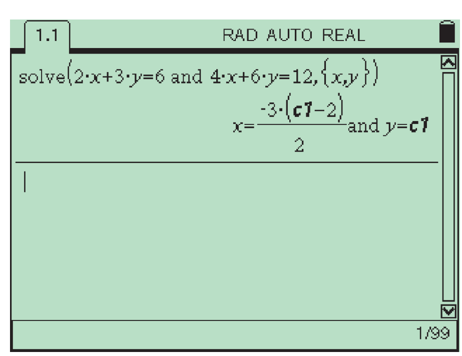
### Solution

## Using the TI-Nspire

The parameter is a third variable. Note that the two equations represent the same straight line. They both have gradient $-\frac{2}{3}$ and $y$-axis intercept 2.
Let $\lambda$ be this third variable.
In this case let $y = \lambda$. Then $x = \dfrac{-3(\lambda - 2)}{2}$ and the line can be described by

$$\left\{\left(\frac{-3(\lambda - 2)}{2}, \lambda\right): \lambda \in R\right\}.$$

This may seem to make the situation unnecessarily complicated, but it is the solution given by the calculator, as shown opposite. The variable $\boldsymbol{c}$ takes the place of $\lambda$.



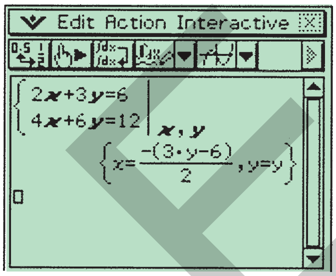
Back to Menu >>>



## Using the Casio ClassPad

Solving these equations simultaneously yields the answer shown.

Choose $y = \lambda$ to obtain the solution $x = \dfrac{-(3\lambda - 6)}{2}$, $y = \lambda$ where $\lambda \in R$.




**Example 3**

Consider the simultaneous linear equations $(m - 2)x + y = 2$ and $mx + 2y = k$. Find the values of $m$ and $k$ such that the system of equations has:

**a** a unique solution **b** no solution **c** infinitely many solutions.

**Solution**

where $m$ is a parameter.

**a** The solution is unique if $m \neq 4$ and $k$ is any real number.

**b** If $m = 4$, the equations become $2x + y = 2$ and $4x + 2y = k$. There is no solution if $m = 4$ and $k \neq 4$.

**c** If $m = 4$ and $k = 4$ there are infinitely many solutions, as the equations are the same.

Importantly, it is a method of expressing a solution which generalises to the more complicated situation in three dimensions. This is also discussed in this section.

Note that for a system of linear equations in two unknowns, the matrix of the coefficients of $x$ and $y$ is singular and corresponds to either no solutions (parallel lines) or infinitely many solutions (same line).

# Simultaneous linear equations in three unknowns

Consider the general linear system of three equations in three unknowns.

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ kx + my + nz &= p \end{aligned}$$

It can be written as a matrix equation:

$$\begin{bmatrix} a & b & c \\ e & f & g \\ k & m & n \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ h \\ p \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} a & b & c \\ e & f & g \\ k & m & n \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} d \\ h \\ p \end{bmatrix}$.

The equation is $\mathbf{AX} = \mathbf{B}$.

We recall that for $3 \times 3$ matrices $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{DI} = \mathbf{D} = \mathbf{ID}$ for all $3 \times 3$ matrices $\mathbf{D}$.

If the inverse $\mathbf{A}^{-1}$ exists, which is not always the case, the equation can be solved by multiplying $\mathbf{AX}$ and $\mathbf{B}$ on the left by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B} \text{ and } \mathbf{A}^{-1}(\mathbf{AX}) = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$$

where $\mathbf{I}$ is the identity matrix for $3 \times 3$ matrices.

Hence $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$, which is a formula for the solution of the system. Of course it depends on the inverse $\mathbf{A}^{-1}$ existing, but once $\mathbf{A}^{-1}$ is found then equations of the form $\mathbf{AX} = \mathbf{B}$ can be solved for all possible $3 \times 1$ matrices $\mathbf{B}$.

In this course you are not required to find the inverse of a $3 \times 3$ matrix 'by hand', but an understanding of matrix arithmetic is necessary. In this chapter we will restrict our attention to $2 \times 2$ and $3 \times 3$ matrices.

**Example 4**

Consider the system of three equations in three unknowns:

$$\begin{aligned} 2x + y + z &= -1 \\ 3y + 4z &= -7 \\ 6x + z &= 8 \end{aligned}$$

Use matrix methods to solve the system of equations.

Back to Menu >>>



**Solution**

Enter $3 \times 3$ matrix **A** and $3 \times 1$ matrix **B** into the calculator.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 4 \\ 6 & 0 & 1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 \\ -7 \\ 8 \end{bmatrix}$$

The equations can be written as a matrix equation:

$$\mathbf{AX} = \mathbf{B}$$

Multiply both sides by $\mathbf{A}^{-1}$.

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{AX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{X} &= \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} \end{aligned}$$

It should be noted that, just as for two equations in two unknowns, there is a geometric interpretation for three equations in three unknowns. There is only a unique solution if the equations represent three planes intersecting at a point.

A CAS calculator can be used to solve systems of three equations in the same way as was used for two simultaneous equations.

## Using the TI-Nspire

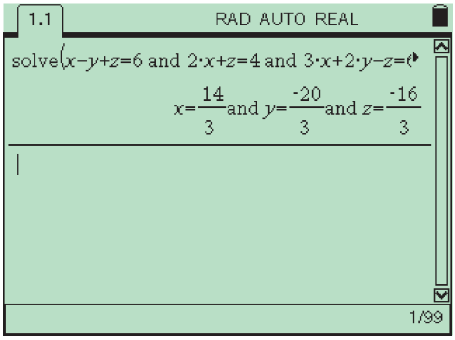
**Example 5**

Solve the following linear simultaneous equations for $x$, $y$ and $z$:

$$x - y + z = 6, 2x + z = 4, 3x + 2y - z = 6$$

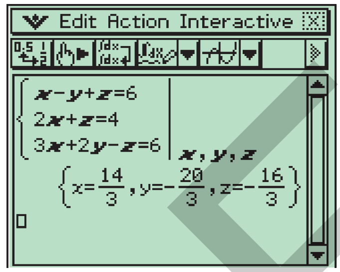
**Solution**

Use **solve(**$x - y + z = 6$ and $2x + z = 4$ and $3x + 2y - z = 6$, $\{x, y, z\}$).

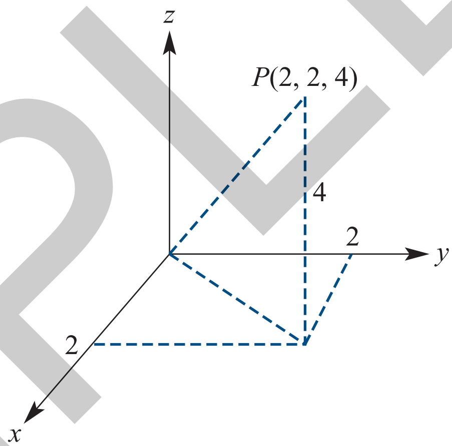


## Using the Casio ClassPad

Turn on the keyboard, from 2D press twice to create a template to solve three simultaneous equations (use if necessary to get the correct menu). Enter the equations using the variables (VAR) keyboard.




As a linear equation in two variables defines a line, a linear equation in three variables defines a plane. The coordinate axes in three dimensions are drawn as shown. The point $P(2, 2, 4)$ is as marked.



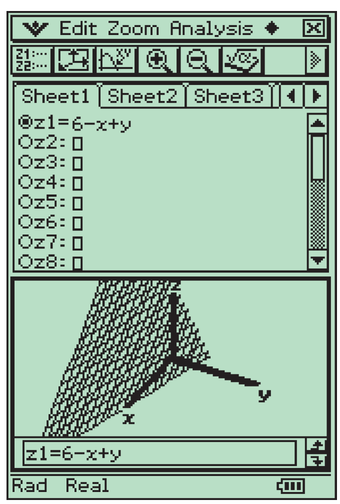

An equation of the form $ax + by + cy = e$ defines a plane.

For example, the equation $x - y + z = 6$ corresponds to the graph shown below.

When using a ClassPad, Use 3D Graph and enter the formula, as shown, for $z1$.

Use zoom and use your stylus to drag to view the graph from different perspectives.

**Note:** The equation has been rewritten with $z$ as the subject. If your graph does not show axes, you can alter the settings in —3D Format.



The solution of simultaneous linear equations in three variables can correspond to:

- a plane
- a point
- a line.

There also may be no solution.

The situations are as shown. Examples 4 and 5 provide examples of planes intersecting at a point (Diagram 1).

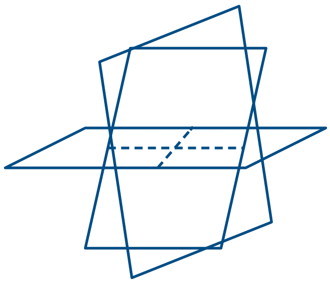


**Diagram 1**
Intersection at a point

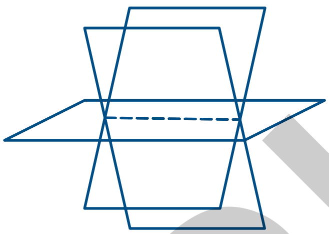


**Diagram 2**
Intersection a line

**Diagram 3**
No intersection

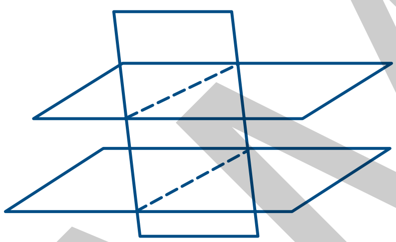


**Diagram 4**
No common intersection

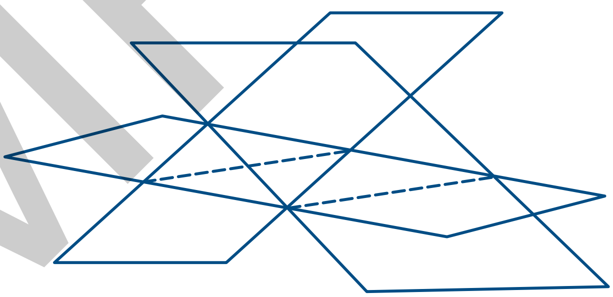


**Diagram 5**
No common intersection

Back to Menu >>>



**Example 6**

The simultaneous equations $x + 2y + 3z = 13$, $-x - 3y + 2z = 2$ and $-x - 4y + 7z = 17$ have infinitely many solutions. Describe these solutions through the use of a parameter.

**Solution**

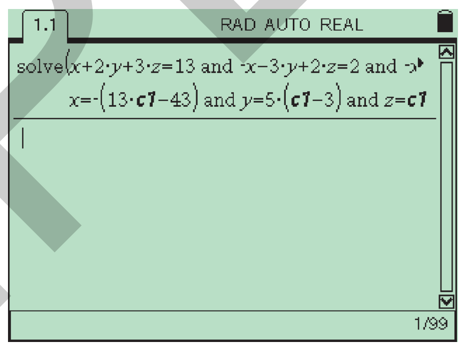
## Using the TI-Nspire

The equations have no unique solution. The point $(-9, 5, 4)$ satisfies all three equations but it is certainly not the only solution. We use a CAS calculator to find the solution in terms of a fourth variable, $\lambda$.

Let $z = \lambda$, then $y = 5(\lambda - 3)$ and $x = 43 - 13\lambda$.

If $\lambda = 4$, $x = -9$, $y = 5$ and $z = 4$.

Note that the matrix $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \\ -1 & -4 & 7 \end{bmatrix}$ does not have an inverse.



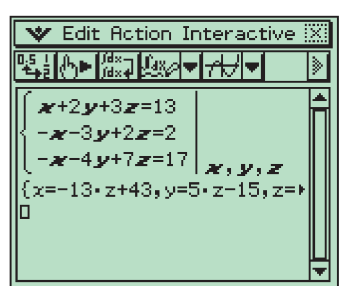

Note that as $z$ increases by 1, $y$ increases by 5 and $x$ decreases by 13. All of the points that satisfy the equations lie on a straight line. The situation is similar to that shown in Diagram 2.

The calculator uses the parameter $\boldsymbol{c}$ for the parameter $\lambda$.

See Question 5 in Extended-response questions 8 for a 'by hand' approach.

## Using the Casio ClassPad

The Casio ClassPad calculator gives the solutions $x = -13z + 43$, $y = 5z - 15$, $z = z$.



# Exercise 8A

Example 1

1 Explain why the simultaneous equations $x + y = 6$ and $2x + 2y = 13$ have no solution.

Example 2

2 The simultaneous equations $x + y = 6$ and $2x + 2y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Example 3

3 Find the value of $m$ for which the following simultaneous equations have no solution.

$$(m + 2)x + my = 12$$
$$(m - 1)x + (m - 2)y = 7$$

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4 Find the value of $m$ for which the simultaneous equations $3x + my = 5$ and $(m + 2)x + 5y = m$ have:

**a** an infinite number of solutions

**b** no solutions.

5 The following is a pair of simultaneous equations:

$$mx + 2y = 8$$
$$4x - (2 - m)y = 2m$$

**a** Find the values of $m$ for which there are:

**i** no solutions **ii** infinitely many solutions.

**b** Solve the equations in terms of $m$, for suitable values of $m$.

Example 4

6 Solve each of the following sets of simultaneous equations using matrix methods.

**a** $2x + 3y - z = 12$
$2y + z = 7$
$2y - z = 5$

**b** $x + 2y + 3z = 13$
$-x - y + 2z = 2$
$-x + 3y + 4z = 26$

**c** $x + y = 5$
$y + z = 7$
$z + x = 12$

7 Use a matrix method to solve the following system of equations.

$$x - y - z = 0,\ 5x + 20z = 50 \text{ and } 10y - 20z = 30$$

Example 6

8 The following system of equations has infinitely many solutions.

$$x + y + z + w = 4,\ x + 3y + 3z = 2,\ x + y + 2z - w = 6$$

Describe this family of solutions and give the unique solution when $w = 6$.

9 The quadratic with equation $y = ax^2 + bx + c$ passes through the points with coordinates $(1, 2)$, $(-1, 6)$ and $(2, 3)$. Use a matrix method to find the values of $a$, $b$ and $c$.

10 The cubic with equation $y = x^3 + bx^2 + cx + d$ passes through the points with coordinates $(-2, -3)$, $(-1, 3)$ and $(1, 9)$. Use a matrix method to find the values of $b$, $c$ and $d$.

Example 5 **11** Solve the following simultaneous linear equations for $x$, $y$ and $z$.

$$2x + 3y + z = 5, x + 2y = 1, x + y - 2z = 1$$

**12** The cubic with equation $y = x^3 + bx^2 + cx + d$ passes through the points with coordinates $(-2, -13)$, $(-1, 0)$ and $(1, 2)$. Use a matrix method to find the values of $b$, $c$ and $d$.

## 8.2 Using matrices with transformations

A summary of some of the transformations and their rules, which were introduced in Chapter 6, is presented here. Two new transformations, rotation by $\frac{\pi}{2}$ about the origin, $O$, and expansion of factor $k$ from the origin, are introduced. Suppose $(x', y')$ is the image of $(x, y)$ under the mapping in the first column of the table below.

| *Mapping* | *Rule* | |
|---|---|---|
| Reflection in the $x$-axis | $x' = x$<br>$y' = -y$ | $= x + 0y$<br>$= 0x + -y$ |
| Reflection in the $y$-axis | $x' = -x$<br>$y' = y$ | $= -x + 0y$<br>$= 0x + y$ |
| Dilation by factor $k$ from the $y$-axis | $x' = kx$<br>$y' = y$ | $= kx + 0y$<br>$= 0x + y$ |
| Dilation by factor $k$ from the $x$-axis | $x' = x$<br>$y' = ky$ | $= x + 0y$<br>$= 0x + ky$ |
| Rotation of $\frac{\pi^c}{2}$ about $O$ in an anticlockwise direction | $x' = -y$<br>$y' = x$ | $= 0x + -y$<br>$= x + 0y$ |
| Expansion of factor $k$ from the origin | $x' = kx$<br>$y' = ky$ | $= kx + 0y$<br>$= 0x + ky$ |
| Reflection in the line $y = x$ | $x' = y$<br>$y' = x$ | $= 0x + y$<br>$= x + 0y$ |
| Translation defined by a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ | $x' = x + a$<br>$y' = y + b$ | |

The first seven mappings are special cases of a general kind of mapping defined by

$$x' = ax + by$$
$$y' = cx + dy$$

where $a$, $b$, $c$, $d$ are real numbers.

These equations can be rewritten as

$$x' = a_{11}x + a_{12}y$$
$$y' = a_{21}x + a_{22}y$$

to yield the equivalent matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A transformation of the form

$$(x, y) \to (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$$

is called a **linear transformation**.

These first seven transformations can be defined by a $2 \times 2$ matrix. This is shown in the following table.

| *Mapping* | *Matrix* |
|---|---|
| Reflection in the $x$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| Reflection in the $y$-axis | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Dilation by factor $k$ from the $y$-axis | $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ |
| Dilation of factor $k$ from the $x$-axis | $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ |
| Rotation of $\frac{\pi^c}{2}$ about $O$ in an anticlockwise direction | $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ |
| Expansion of factor $k$ from the origin | $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ |
| Reflection in the line $y = x$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |

## Example 7

Find the image of the point (2, 3) under:

**a** a reflection in the $x$-axis **b** a dilation of factor $k$ from the $y$-axis

### Solution

**a** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Therefore $(2, 3) \to (2, -3)$.

**b** $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2k \\ 3 \end{bmatrix}$. Therefore $(2, 3) \to (2k, 3)$.

**Example 8**

Consider a linear transformation such that $(1, 0) \to (3, -1)$ and $(0, 1) \to (-2, 4)$. Find the image of $(-3, 5)$.

**Solution**

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\therefore \qquad a_{11} = 3, a_{21} = -1 \text{ and } a_{12} = -2, a_{22} = 4$$

The transformation can be defined by the $2 \times 2$ matrix $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$.

Let $(-3, 5) \to (x', y')$.

$$\begin{aligned} \therefore \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 \times -3 + -2 \times 5 \\ -1 \times -3 + 4 \times 5 \end{bmatrix} \\ &= \begin{bmatrix} -19 \\ 23 \end{bmatrix} \end{aligned}$$

$$\therefore \qquad (-3, 5) \to (-19, 23)$$

The image of $(-3, 5)$ is $(-19, 23)$.

Note that non-linear mappings cannot be represented by a matrix in the way indicated above.

Thus for the translation defined by $(0, 0) \to (a, b)$:

$$x' = x + a$$
$$y' = y + b$$

While this cannot be represented by a square matrix, the defining equations suggest that:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

where the ‘sum’ has the following definition:

For each $x, y, a, b$ in $R$, $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \end{bmatrix}$.

## Composition of mappings

Consider a linear transformation defined by the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ composed with a linear transformation defined by the matrix $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

The composition consists of the transformation of **A** being applied first and then the transformation of **B**.

The matrix of the resulting composition is the product **BA**:

$$\mathbf{BA} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

### Example 9

Find the image of the point $(2, -3)$ under a reflection in the $x$-axis followed by a dilation of factor $k$ from the $y$-axis.

#### Solution

Matrix multiplication gives the matrix of the composition of transformations. Let **A** be the matrix of the transformation reflection in the $x$-axis and **B** the matrix of the transformation dilation of factor $k$ from the $y$-axis. Then the required transformation is defined by the product $\mathbf{BA} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & -1 \end{bmatrix}$ and

$\mathbf{BA}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2k \\ 3 \end{bmatrix}$.

### Example 10

Express the composition of the transformations, dilation of factor $k$ from the $y$-axis followed by a translation defined by the matrix $\mathbf{C} = \begin{bmatrix} a \\ b \end{bmatrix}$, mapping a point $(x, y)$ to a point $(x', y')$ as a matrix equation. Hence find $x$ and $y$ in terms of $x'$ and $y'$ respectively.

#### Solution

Let **A** be the matrix of the dilation transformation, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{X}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

The equation is $\mathbf{AX} + \mathbf{C} = \mathbf{X}'$.

Then $\mathbf{AX} = \mathbf{X}' - \mathbf{C}$ and hence $\mathbf{X} = \mathbf{A}^{-1}(\mathbf{X}' - \mathbf{C})$.

Now $\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$.

$\det(\mathbf{A}) = k$ and therefore $\mathbf{A}^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathbf{X} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x'-a \\ y'-b \end{bmatrix} = \begin{bmatrix} \frac{1}{k}(x'-a) \\ y'-b \end{bmatrix}$$

Hence $x = \frac{1}{k}(x' - a)$ and $y = y' - b$.

## Exercise 8B

**1** Find:

**a** $\begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **b** $\begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

**2** If a linear transformation is defined by the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$, find the image of (1, 0), (0, 1) and (3, 2) under this transformation.

**3** Find the images of (1, 0) and (−1, 2) under the linear transformation whose matrix is

**a** $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$

Example 7

**4** Using matrix methods find the image of the point (6, 7) under each of the following transformations.

**a** dilation of factor 3 from the $x$-axis
**b** dilation of factor 2 from the $y$-axis
**c** reflection in the $x$-axis
**d** reflection in the $y$-axis
**e** reflection in the line $y = x$

Example 8

**5** **a** Find the matrix of the linear transformation that maps $(1, -2) \rightarrow (-4, 5)$ and $(3, 4) \rightarrow (18, 5)$.

**b** The images of two points are given for a linear transformation. Investigate whether this is sufficient information to determine the matrix of the transformation.

**c** Find the matrix of the linear transformation such that $(1, 0) \rightarrow (1, 1)$ and $(0, 1) \rightarrow (2, 2)$.

Example 9

**6** Find the matrix that determines the composition of transformations, in the given order:

- reflection in the $x$-axis
- dilation of factor 2 from the $x$-axis.

Example 9

**7** Write down the matrix of each of the following transformations.

**a** reflection in the line $x = 0$
**b** reflection in the line $y = x$
**c** reflection in the line $y = -x$
**d** dilation of factor 2 from the $x$-axis
**e** expansion of factor 3 from the origin
**f** dilation of factor 3 from the $y$-axis

Back to Menu >>>



**8** A transformation $T$ is equivalent to an expansion from $O$ by a factor 2, followed by a reflection in the line $y = -x$.

**a** What matrix defines $T$?

**b** Find $T(3, 2)$.

**c** If $T(a, b) = (6, 2)$, find the values of $a$ and $b$.

Example 10

**9** Express as a matrix equation the composition of the transformations dilation of factor 2 from the $x$-axis followed by a translation defined by the matrix $\mathbf{C} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, mapping a point $(x, y)$ to a point $(x', y')$. Hence find $x$ and $y$ in terms of $x'$ and $y'$ respectively.

Skillsheet TEACHER

**10** A linear transformation $T$ maps the points $(1, 3)$ and $(-2, -3)$ to the points $(2, 4)$ and $(-3, -11)$ respectively. Find the matrix of the transformation.

## 8.3 Using parameters to describe families of curves

This section demonstrates a different use of parameters. They can be used to discuss families of relations.

Here are some familiar families of relations.

- $f: R \to R, f(x) = mx$, where $m \in R$. The graphs of these functions are the straight lines through the origin.
- $f: R \to R, f(x) = ax^2$, where $a \in R \setminus \{0\}$. The graphs of these functions are the parabolas with vertex at the origin.
- $f: R \to R, f(x) = mx + 2$, where $m \in R$. The graphs of these functions are the straight lines with $y$-axis intercept 2.
- $x^2 + y^2 = a^2$, where $a \in R \setminus \{0\}$. The graphs of these relations are the circles that have the origin as their centre.

The use of parameters makes it possible to describe general properties.

What can be said in general about each of these? The family of functions of the form $f: R \to R, f(x) = mx + 2$, where $m \in R$, is explored in Example 11.

### Example 11

Let $f: R \to R, f(x) = mx + 2$, where $m \in R \setminus \{0\}$.

**a** Find the $x$-axis intercept.

**b** For which values of $m$ is the $x$-axis intercept greater than 3?

**c** Find the inverse function of $f$.

**d** Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

**e** Find the equation of the line perpendicular to the line at the point with coordinates (0, 2).

## Solution

**a** $mx + 2 = 0$ implies $mx = -2$ and $x = -\frac{2}{m}$. The $x$-axis intercept is $-\frac{2}{m}$.
Note that if $m$ is positive the intercept is negative, and if $m$ is negative the intercept is positive.

**b** If $-\frac{2}{m} > 3$, then $\frac{2}{m} < -3$.
Two cases should be considered, although it is clear that only one case will be possible.

**Case 1**: $m > 0$

Multiply both sides of the inequality by $m$.

$$2 < -3m$$

Divide both sides by $-3$.

$$m < -\frac{2}{3}$$

But this is impossible as $m > 0$.

**Case 2**: $m < 0$

Multiply both sides of the inequality by $m$.

$$2 > -3m \qquad \text{(as } m \text{ is negative the inequality is reversed)}$$

Divide both sides by $-3$.

$$m > -\frac{2}{3}$$

Therefore the $x$-axis intercept is greater than 3 for $-\frac{2}{3} < m < 0$.
It is worthwhile doing this question by considering the graphs of the form $f(x) = mx + 2$. First the intercept is 3 when $-\frac{2}{m} = 3$. That is, when $m = -\frac{2}{3}$. As the magnitude of $m$ increases the $x$-axis intercept becomes closer to the $y$-axis. As the magnitude decreases the intercept goes further from the $y$-axis.

**c** Consider $x = my + 2$ and solve for $y$.

$$my = x - 2 \text{ and } y = \frac{x-2}{m}$$

Therefore $y = \frac{x}{m} - \frac{2}{m}$.

The inverse function $f^{-1}(x) = \frac{x}{m} - \frac{2}{m}$.

**d** Consider the pair of equations $y = x$ and $y = mx + 2$.
To first determine the value of $x$, solve $x = mx + 2$.

$$x - mx = 2$$

$$x(1 - m) = 2$$

$$x = \frac{2}{1-m}$$

The graphs intersect at the point $\left(\frac{2}{1-m}, \frac{2}{1-m}\right)$ for $m \neq 1$.

**e** The perpendicular line has gradient $-\frac{1}{m}$.

The equation is determined as $y - 2 = -\frac{1}{m}x$ and the gradient–intercept form is $y = -\frac{1}{m}x + 2$.

## Exercise 8C

Example 11

**1** Let $f : R \to R$, $f(x) = mx - 3$, where $m \in R \setminus \{0\}$.

**a** Find the $x$-axis intercept.

**b** For which values of $m$ is the $x$-axis intercept less than or equal to 1?

**c** Find the inverse function of $f$.

**d** Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

**e** Find the equation of the line perpendicular to the line at the point with coordinates $(0, -3)$.

**2** Let $f : R \to R$, $f(x) = 2x + c$, where $c \in R$.

**a** Find the $x$-axis intercept.

**b** For which values of $c$ is the $x$-axis intercept less than or equal to 1?

**c** Find the inverse function of $f$.

**d** Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

**e** Find the equation of the line perpendicular to $y = f(x)$ at the point with coordinates $(0, c)$.

**3** Consider the family of quadratics with rule of the form $y = x^2 + bx$, where $b$ is a non-zero real number.

**a** Find the $x$-axis intercepts.

**b** Find the coordinates of the vertex of the parabola.

**c** **i** Find the coordinates of the points of intersection of the graph of $y = x^2 + bx$ with the line $y = x$, in terms of $b$.

**ii** For what value(s) of $b$ is there one point of intersection?

**iii** For what value(s) of $b$ are there two points of intersection?

**4** A circle has equation $(x - a)^2 + y^2 = a^2$, where $a$ is a positive real number.

**a** Sketch the graph of the circle when $a = 2$.

**b** Find the equation of the tangents to the circle at the point where $x = a$.

**c** Find the coordinates of the points of intersection of the line with equation $y = x - 4$ and the circle $(x - a)^2 + y^2 = a^2$.

# 8.4 Transformation of graphs of functions with matrices

Matrix notation for transformations was introduced in Section 8.2. In this section the notation is applied to transforming graphs.

The notation used here is consistent with the notation introduced in Chapter 6.

**Example 12**

A transformation is defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Find the equation of the image of the graph of the quadratic equation $y = x^2 + 2x + 3$ under this transformation.

**Solution**

As before, the transformation maps $(x, y) \rightarrow (x', y')$.
Using matrix notation

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

It can be written as the matrix equation $\mathbf{TX} = \mathbf{X}'$.
Now multiply both sides of the equation by $\mathbf{T}^{-1}$.
Therefore $\mathbf{T}^{-1}\mathbf{TX} = \mathbf{T}^{-1}\mathbf{X}'$
and $\mathbf{X} = \mathbf{T}^{-1}\mathbf{X}'$
Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ \frac{1}{2}y' \end{bmatrix}$$

So $x = x'$ and $y = \dfrac{y'}{2}$.
The curve with equation $y = x^2 + 2x + 3$ is mapped to the curve with equation $\dfrac{y'}{2} = (x')^2 + 2x' + 3$.

This makes quite hard work of an easy problem, but it demonstrates a procedure that can be used for any transformation defined by a $2 \times 2$ non-singular matrix.

Back to Menu >>>



**Example 13**

A transformation is described by the equation $\mathbf{T}(\mathbf{X}+\mathbf{B}) = \mathbf{X}'$, where $\mathbf{T} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Find the image of the straight line with equation $y = 2x + 5$ under the transformation.

**Solution**

First solve the matrix equation for **X**.

$$\mathbf{T}^{-1}\mathbf{T}(\mathbf{X}+\mathbf{B}) = \mathbf{T}^{-1}\mathbf{X}'$$
$$\mathbf{X}+\mathbf{B} = \mathbf{T}^{-1}\mathbf{X}'$$

and $\mathbf{X} = \mathbf{T}^{-1}\mathbf{X}' - \mathbf{B}$

Therefore $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{y'}{2} - 1 \\ -\frac{x'}{3} - 2 \end{bmatrix}$

So $x = \frac{y'}{2} - 1$ and $y = -\frac{x'}{3} - 2$.

The straight line with equation $y = 2x + 5$ is transformed to the straight line with equation $-\frac{x'}{3} - 2 = 2\left(\frac{y'}{2} - 1\right) + 5$.

Rearranging gives $y' = -\frac{x'}{3} - 5$.

## Exercise 8D

Example 12

1 A transformation is defined by the matrix $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. Find the equation of the image of the graph of the quadratic equation $y = x^2 + x + 2$ under this transformation.

2 A transformation is defined by the matrix $\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$. Find the equation of the image of the graph of the cubic equation $y = x^3 + 2x$ under this transformation.

3 A transformation is defined by the matrix $\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$. Find the equation of the image of the graph of the straight line with equation $y = 2x + 3$ under this transformation.

4 A transformation is defined by the matrix $\begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix}$. Find the equation of the image of the graph of the straight line with equation $y = -2x + 4$ under this transformation.

Example 13

**5** A transformation is described by the equation $\mathbf{T}(\mathbf{X}+\mathbf{B}) = \mathbf{X}'$, where $\mathbf{T} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find the image of the straight line with equation $y = -2x + 6$ under the transformation.

**6** A transformation is described by the equation $\mathbf{TX}+\mathbf{B} = \mathbf{X}'$, where $\mathbf{T} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find the image of the straight line with equation $y = -2x + 6$ under the transformation.

**7** A transformation is described by the equation $\mathbf{TX}+\mathbf{B} = \mathbf{X}'$, where $\mathbf{T} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Find the image of the curve with equation $y = -2x^3 + 6x$ under the transformation.

**8** A transformation is described by the equation $\mathbf{TX}+\mathbf{B} = \mathbf{X}'$, where $\mathbf{T} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Find the image of the curve with equation $y = -2x^3 + 6x^2 + 2$ under the transformation.

**9** A cubic polynomial, $P$, has rule $P(x) = ax^3 + bx^2 + cx + d$. $P(0) = -1$, $P(1) = 1$, $P(2) = 1$, $P(3) = 5$

- **a** Write four equations in terms of $a$, $b$, $c$ and $d$.
- **b** Write these simultaneous equations as a matrix equation.
- **c** Solve the equations by a matrix method.
- **d** Find the equation of the image of the graph of $y = P(x)$ under a reflection in the $x$-axis followed by a dilation of factor 2 from the $x$-axis.

**10** A cubic polynomial, $P$, has rule $P(x) = ax^3 + bx^2 + cx + d$.
It satisfies the following: $P(-x) = -P(x)$, $P(1) = -2$ and $P(2) = 8$.

- **a** Find the values of $a$, $b$, $c$ and $d$.
- **b** Find the equation of the image of the graph of $y = P(x)$ under a reflection in the $y$-axis.

MC TEST D&D TEST

Review

## Chapter summary

- The linear simultaneous equations $ax + by = c$ and $dx + ey = f$ can also be written as the matrix equation $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} c \\ f \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

  If $\mathbf{A}^{-1}$ exists, there is a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. The inverse exists if the determinant is not equal to zero (Chapter 3).
  If $\mathbf{A}^{-1}$ does not exist (i.e. $\mathbf{A}$ is singular) then either no solution exists or there are infinitely many solutions.
  If infinitely many solutions exist then the corresponding lines are the same line.
  If there is no solution the corresponding lines are parallel.
  The converse of each of these statements also holds.
- If infinitely many solutions exist they can be described by a third variable called a parameter. For example, if $x + y = 6$ and $2x + 2y = 12$ are the simultaneous equations, the solution is $\{(x, y) : x = \lambda, y = 6 - \lambda, \text{ where } \lambda \in R\}$.
- Consider the general linear system of three equations in three unknowns.

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ kx + my + nz &= p \end{aligned}$$

  It can be written as a matrix equation:

$$\begin{bmatrix} a & b & c \\ e & f & g \\ k & m & n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ h \\ p \end{bmatrix} \text{ or } \mathbf{AX} = \mathbf{B}$$

  If the inverse of $\mathbf{A}$ exists, the solution is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.
- The solution of simultaneous linear equations in three variables can correspond to:
  - a point
  - a line
  - a plane.

  There also may be no solution.
  The situations are as shown.

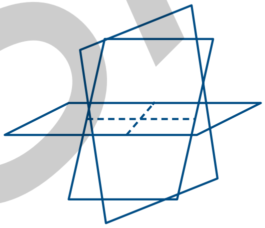


**Diagram 1**
Intersection at a point

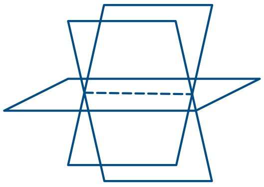


**Diagram 2**
Intersection a line

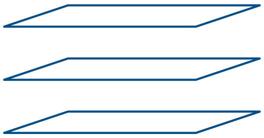


**Diagram 3**
No intersection

Back to Menu >>>





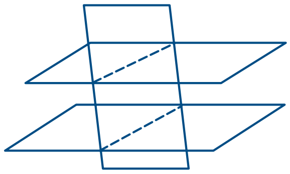


**Diagram 4**
No common intersection

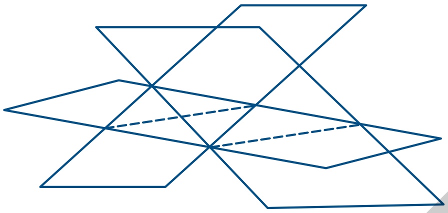


**Diagram 5**
No common intersection

| *Mapping* | *Rule* | *Matrix* |
|---|---|---|
| Reflection in the $x$-axis | $x' = x$, $y' = -y$ | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| Reflection in the $y$-axis | $x' = -x$, $y' = y$ | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Dilation by factor $k$ from the $y$-axis | $x' = kx$, $y' = y$ | $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ |
| Dilation by factor $k$ from the $x$-axis | $x' = x$, $y' = ky$ | $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ |
| Rotation of $\frac{\pi^c}{2}$ about $O$ in an anticlockwise direction | $x' = -y$, $y' = x$ | $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ |
| Expansion of factor $k$ from the origin | $x' = kx$, $y' = ky$ | $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ |
| Reflection in the line $y = x$ | $x' = y$, $y' = x$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |

- Consider a linear transformation defined by the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ composed with a linear transformation defined by the matrix $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.
  The composition consists of the transformation of **A** being applied first and then the transformation of **B**. The matrix of the resulting composition is the product **BA**:

$$\mathbf{BA} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

- Parameters are used to describe families of functions. For example:

  $f: R \to R,\ f(x) = mx$, where $m \in R$. The graphs of these functions are the straight lines through the origin.

  $f: R \to R,\ f(x) = ax^2$, where $a \in R$. The graphs of these functions are the parabolas with vertex at the origin.

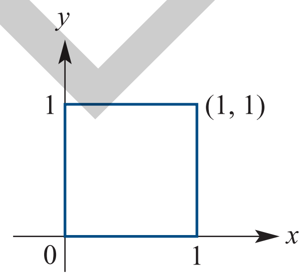
  $f: R \to R,\ f(x) = mx + 2$, where $m \in R$. The graphs of these functions are the straight lines with $y$-axis intercept 2.

  $x^2 + y^2 = a^2$ where $a \in R \setminus \{0\}$. The graphs of these relations are the circles with centre the origin.

## Multiple-choice questions

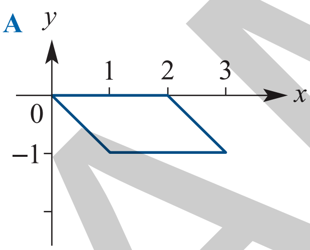
**1** The square shown is subject to successive transformations.

The first transformation has matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and the second transformation has matrix $\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$.

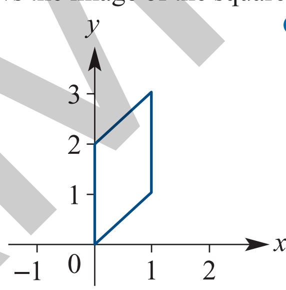



Which one of the following shows the image of the square after these two transformations?

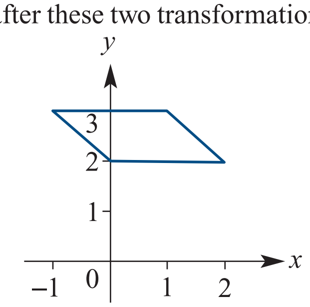
**A**



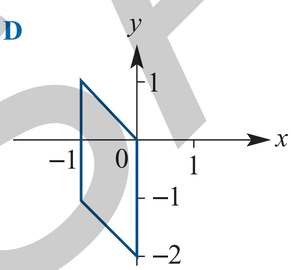

**B**



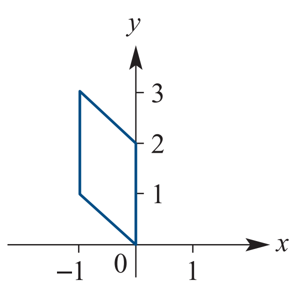

**C**




**D**




**E**




**2** The turning point of a quadratic with rule $y = x^2 - ax$ has coordinates

**A** $(0, a)$ **B** $\left(\dfrac{a}{2}, \dfrac{-a^2}{4}\right)$ **C** $(a, 0)$ **D** $\left(a, \dfrac{-a^2}{2}\right)$ **E** $\left(\dfrac{-a}{2}, \dfrac{-a^2}{4}\right)$

**3** The inequality $ax + b > 0$, where $b < 0$ and $a < 0$, has solution

**A** $x > \dfrac{a}{b}$ **B** $x < \dfrac{-b}{a}$ **C** $x > \dfrac{-b}{a}$ **D** $x < \dfrac{b}{a}$ **E** $x > \dfrac{a}{b}$

Back to Menu >>>



Review

**4** The solution of the inequality $x^2 > b^2$, where $b < 0$, is

**A** $x > b$ or $x < -b$ **B** $x > b$ **C** $b < x < -b$

**D** $-b < x < b$ **E** $x < b$ or $x > -b$

**5** The matrix which determines the transformation, dilation from the $x$-axis of factor 2 followed by a dilation of factor 3 from the $y$-axis, is

**A** $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$

**6** The matrix which determines the transformation, dilation from the $x$-axis of factor 2 followed by reflection in the line $y = x$, is

**A** $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

**7** The simultaneous equations $(m-2)x + 3y = 6$ and $2x + (m+2)y = m$, have a unique solution for

**A** $m \in R \setminus \{0\}$ **B** $m \in R \setminus \{-1, 1\}$ **C** $m \in R \setminus \{-\sqrt{10}, \sqrt{10}\}$

**D** $m \in R \setminus [-1, 1]$ **E** $m \in R$

**8** The solution of the two simultaneous equations $2ax + 2by = 3$ and $3ax - 2by = 7$ for $x$ and $y$ is

**A** $x = 2a,\ y = \dfrac{3-4a^2}{2b}$ **B** $x = 2,\ y = \dfrac{3-4a}{2b}$

**C** $x = \dfrac{2}{a},\ y = -\dfrac{1}{2b}$ **D** $x = 0,\ y = 0$

**E** $x = 3a,\ y = 7b$

**9** The quadratic equation $x^2 - 2ax + b = 0$, where $a$ and $b$ are positive constants, has one solution when

**A** $b = a$ and $a \neq 1$ **B** $b = \sqrt{a}$ or $b = -\sqrt{a}$

**C** $b = 1$ and $a \neq 1$ **D** $a = \sqrt{b}$ or $a = -\sqrt{b}$

**E** $b = a = 2$

**10** The matrix which determines the transformation, dilation from the $x$-axis of factor 2 followed by reflection in the $x$-axis is

**A** $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$

## Short-answer questions (technology-free)

**1** Using matrix methods, find the image of the point $(-1, 3)$ under each of the following transformations and give the corresponding transformation matrix.

**a** dilation of factor 4 from the $x$-axis

**b** dilation of factor 3 from the $y$-axis

**c** reflection in the $x$-axis

**d** reflection in the $y$-axis

**e** reflection in the line $y = x$

Review

**2** Solve the system of equations $x + y = 3$, $x - y = 5$, $x + y + z = 10$ for $x$, $y$ and $z$.

**3** A family of straight lines satisfy the rule $y = ax + 2$.

**a** Find the equation of the straight line in this family for which $y = 6$ when $x = 2$.

**b** **i** Find the $x$-axis intercept of the line with equation $y = ax + 2$.

**ii** If $a < 0$, find the values of $a$ for which the $x$-axis intercept is greater than 1.

**c** Find the coordinates of the point of intersection of the line with equation $y = x + 3$ and the line with equation $y = ax + 2$, given that $a \neq 1$.

**4** A family of parabolas satisfies the rule $y = ax^2 + 2x + a$.

**a** Express $ax^2 + 2x + a$ in the form $a(x + b)^2 + c$ for real numbers $b$ and $c$.

**b** Give the coordinates of the turning point of the graph of $y = ax^2 + 2x + a$ in terms of $a$.

**c** For which values of $a$ is $ax^2 + 2x + a$ a perfect square?

**d** For which values of $a$ are there two $x$-axis intercepts of the graph of $y = ax^2 + 2x + a$?

**5** Express the composition of the transformations, dilation of factor 2 from the $x$-axis followed by a translation defined by the matrix $\mathbf{C} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, mapping a point $(x, y)$ to a point $(x', y')$ as a matrix equation. Hence find $x$ and $y$ in terms of $x'$ and $y'$ respectively.

**6** Express the composition of the transformations, reflection in the $x$-axis followed by a dilation of factor 3 from the $y$-axis and then by a translation defined by the matrix $\mathbf{C} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, mapping a point $(x, y)$ to a point $(x', y')$ as a matrix equation. Hence find $x$ and $y$ in terms of $x'$ and $y'$ respectively.

## Extended-response questions

**1** **a** The graph of $f(x) = x^2$ is translated to the graph of $y = f(x + h)$. Find the possible values of $h$ if $f(1 + h) = 8$.

**b** The graph of $f(x) = x^2$ is transformed to the graph of $y = f(ax)$. Find the possible values of $a$ if the graph of $y = f(ax)$ passes through the point with coordinates (1, 8).

**c** The quadratic with equation $y = ax^2 + bx$ has vertex with coordinates (1, 8). Find the values of $a$ and $b$.

**2** The general equation of the circle can be written as $x^2 + y^2 + bx + cy + d = 0$. A circle passes through the points with coordinates (−4, 5), (−2, 7) and (4, −3).

**a** Write three simultaneous equations in $b$, $c$ and $d$.

**b** Determine the equation of the circle.

**3** A circle passes through the origin. It has equation $x^2 + y^2 + bx + cy = 0$. The circle also passes through the point (4, 4).

**a** Find $c$ in terms of $b$.

**b** Find the $x$-axis intercepts in terms of $b$.

**c** **i** Find the $y$-axis intercepts in terms of $b$.

**ii** For what value of $b$ does the circle touch the $y$-axis?

**4** A family of functions has rule of the form $f(x) = \sqrt{a - x}$, where $a$ is a positive real number.

- **a** State the maximal domain of $f$.
- **b** Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.
- **c** For what value of $a$ does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates (1, 1)?
- **d** For what value of $a$ does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates (2, 2)?
- **e** For what value of $a$ does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(c, c)$ where $c$ is a positive real number?

**5** Consider the following simultaneous equations:

$$\begin{aligned} x + 2y + 3z &= 13 && (1)\\ -x - 3y + 2z &= 2 && (2)\\ -x - 4y + 7z &= 17 && (3) \end{aligned}$$

- **a** Add equation (2) to equation (1) and subtract equation (2) from equation (3).
- **b** Comment on the equations obtained in **a**.
- **c** Let $z = \lambda$ and find $y$ in terms of $\lambda$.
- **d** Substitute for $z$ and $y$, in terms of $\lambda$, in equation (1) to find $x$ in terms of $\lambda$.

**6** Consider the simultaneous equations $x + 2y - 3z = 4$ and $x + y + z = 6$.

- **a** Subtract the second equation from the first to find $y$ in terms of $z$.
- **b** Let $z = \lambda$. Solve the equations to give the solution in terms of $\lambda$.

**7** If $\mathbf{A} = \begin{bmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$ and $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ u & 1 & 0 \\ v & 0 & 1 \end{bmatrix}$, express $u$ and $v$ in terms of the elements of $\mathbf{A}$.