CHAPTER 7

Cubic and Quartic Functions

Objectives

- To recognise and sketch the graphs of **cubic and quartic functions**.
- To **divide** polynomials.
- To use the **remainder theorem** and the **factor theorem** to solve cubic equations.
- To find equations for given cubic graphs.
- To apply cubic and quartic functions to solving problems.
- To use **finite difference tables** to find rules of sequences generated by polynomial functions.

In Chapter 4 we looked at second degree polynomials or quadratics.

A third degree polynomial is called a cubic and is a function, f, with rule

$$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$$

A fourth degree polynomial is called a quartic and is a function, f, with rule

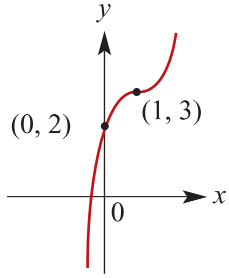
$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$$

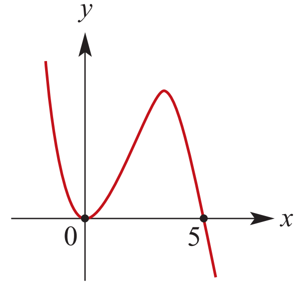
In Chapter 4 it was shown that all quadratic functions could be written in 'perfect square' form and that the graph of a quadratic has one basic form, the parabola.

This is not true of cubic or quartic functions.

Two examples of graphs of cubic functions and two examples of quartic functions are shown.

Cubic functions

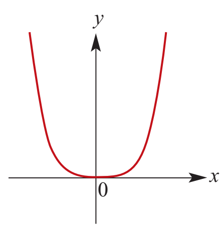


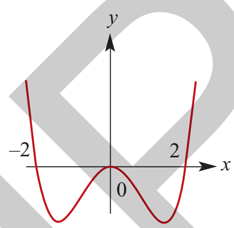


$$f(x) = (x-1)^3 + 3 = x^3 - 3x^2 + 3x + 2$$

$$f(x) = x^2(5-x) = -x^3 + 5x^2$$

Quartic functions





$y = 2x^4$

$y = x^4 - 4x^2 = x^2(x^2 - 4)$

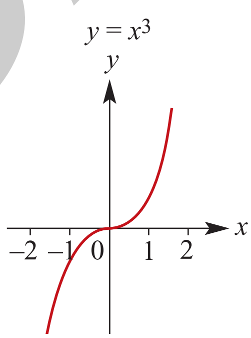
7.1 Functions of the form $f: R \to R$, $f(x) = a(x-h)^n + k$

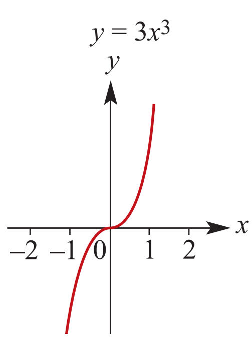
Cubic functions of this form

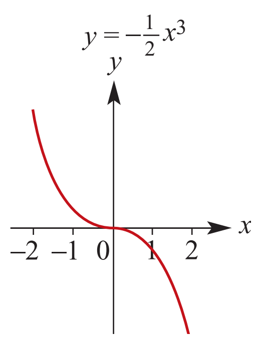
The graph of $f(x) = (x-1)^3 + 3$ is obtained from the graph of $y = x^3$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

As with other graphs it has been seen that changing a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$ the graph is inverted.

For example:







The significant feature of the graph of cubics of this form is the **point of inflexion** (a point of zero gradient).

The point of inflexion of $y = x^3$ is at the origin (0, 0).

It should be noted that the implied **domain** of all cubics is $\boldsymbol{R}$ and the **range** is also $\boldsymbol{R}$.

Translations of basic graphs will be considered in the same way as other types of graphs seen in chapters 4, 5 and 6.

Vertical translations

By adding or subtracting a constant term to $y = x^3$, the graph moves either 'up' or 'down'.

$y = x^3 + k$ is the basic graph moved k units up ($k > 0$). The point of inflexion becomes $(0, k)$. In this case the graph of $y = x^3$ is translated k units in the positive direction of the y-axis.

Horizontal translations

The graph of $y = (x - h)^3$ is simply the basic graph moved h units to the 'right' for $h > 0$.

The point of inflexion is at $(h, 0)$. In this case the graph of $y = x^3$ is translated h units in the positive direction of the x-axis.

> The general form of cubics of this form is
> $$y = a(x - h)^3 + k$$
> The point of inflexion is at (h, k).

When sketching cubic graphs which are of the form $y = a(x - h)^3 + k$, first identify the point of inflexion. To add further detail to the graph, the x-axis and y-axis intercepts are found.

Example 1

Sketch the graph of the function $y = (x - 2)^3 + 4$.

Solution

The graph of $y = x^3$ is translated 2 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.

Point of inflexion is (2, 4).

x-axis intercept:

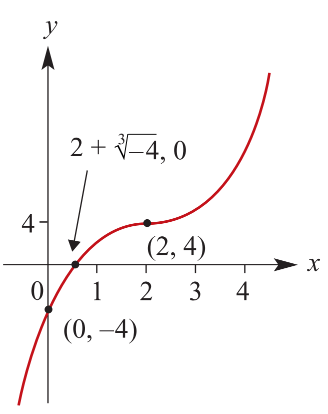
let $y = 0$

$$0 = (x - 2)^3 + 4$$
$$-4 = (x - 2)^3$$
$$\sqrt[3]{-4} = x - 2$$
$$x = 2 + \sqrt[3]{-4}$$
$$\approx 0.413$$

y-axis intercept:

let $x = 0$

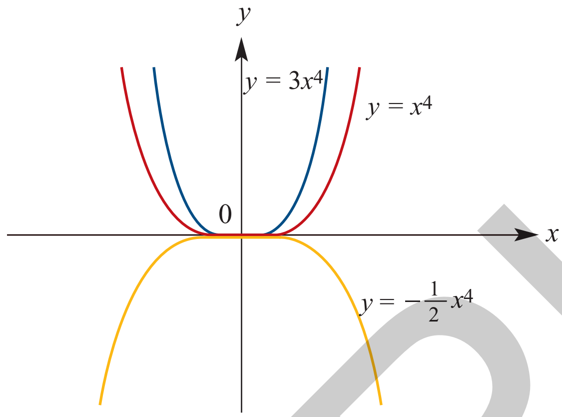
$$y = (0 - 2)^3 + 4$$
$$y = -8 + 4$$
$$y = -4$$



Quartic graphs

The graph of $f(x) = (x - 1)^4 + 3$ is obtained from the graph of $y = x^4$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

As with other graphs it has been seen that changing a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$ the graph is inverted.



The significant feature of the graph of quartics of this form is the **turning point** (a point of zero gradient).

The turning point of $y = x^4$ is at the origin (0, 0).

It should be noted that the implied **domain** of all quartics is R, but unlike cubics the range is not R.

Vertical translations

By adding or subtracting a constant term to $y = x^4$, the graph moves either up or down.

$y = x^4 + k$ is the basic graph moved k units up ($k > 0$). The turning point becomes $(0, k)$. The graph of $y = x^4$ is translated k units in the positive direction of the y-axis.

Horizontal translations

The graph of $y = (x - h)^4$ is simply the basic graph moved h units to the right for $h > 0$.

The turning point is at $(h, 0)$. The graph of $y = x^4$ is translated h units in the positive direction of the x-axis.

> The general form of quartics of this form is
> $$y = a(x - h)^4 + k$$
> The turning point is at (h, k).

When sketching quartic graphs of the form $y = a(x - h)^4 + k$, first identify the turning point. To add further detail to the graph, the x-axis and y-axis intercepts are found.

Example 2

Sketch the graph of the function $y = (x - 2)^4 - 1$.

Solution

Turning point is $(2, -1)$.

x-axis intercept:

let $y = 0$

$0 = (x - 2)^4 - 1$

$1 = (x - 2)^4$

$\pm\sqrt[4]{1} = x - 2$

$x = 2 + 1$ or $x = 2 - 1$

$x = 3$ or $x = 1$

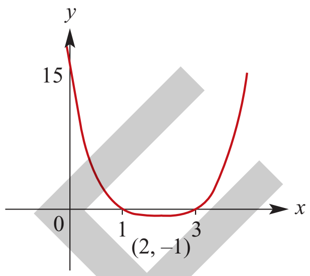
y-axis intercept:

let $x = 0$

$y = (0 - 2)^4 - 1$

$y = 16 - 1$

$y = 15$



Power functions of higher degree

Even degree

Functions with rules $f(x) = x^2$ and $f(x) = x^4$ are examples of even degree functions.

The following are properties of all even degree power functions:

- $f(-x) = f(x)$ for all x
- $f(0) = 0$
- As $x \to \pm\infty, y \to \infty$

Note that m and n are positive even integers and if $m > n$ then:

- $x^m > x^n$ for $x > 1$ or $x < -1$
- $x^m < x^n$ for $-1 < x < 1$ but x not 0
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$

Odd degree

The functions with rules $f(x) = x^3$ and $f(x) = x$ are examples of odd degree functions.

The following are properties of all odd degree power functions:

- $f(-x) = -f(x)$ for all x
- $f(0) = 0$
- As $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$

Note that m and n are positive odd integers and if $m > n$ then:

- $x^m > x^n$ for $x > 1$ and $-1 < x < 0$
- $x^m < x^n$ for $x < -1$ and $0 < x < 1$
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$

Exercise 7A

Example 1

1 Using the method of horizontal and vertical translations, sketch the graphs of each of the following:

a $y = (x + 2)^3 - 1$ **b** $y = (x - 1)^3 - 1$ **c** $y = (x + 3)^3 + 2$

d $y = (x - 2)^3 + 5$ **e** $y = (x + 2)^3 - 5$

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2 Sketch the graphs of the following functions:

a $y = 2x^3 + 3$ **b** $y = 2(x - 3)^3 + 2$ **c** $3y = x^3 - 5$
d $y = 3 - x^3$ **e** $y = (3 - x)^3$ **f** $y = -2(x + 1)^3 + 1$
g $y = \frac{1}{2}(x - 3)^3 + 2$

Example 2 **3** Using the method of horizontal and vertical translations, sketch the graphs of each of the following:

a $y = (x + 2)^4 - 1$ **b** $y = (x - 1)^4 - 1$ **c** $y = (x + 3)^4 + 2$
d $y = (x - 2)^4 + 5$ **e** $y = (x + 2)^4 - 5$

4 Sketch the graphs of the following functions:

a $y = 2x^4 + 3$ **b** $y = 2(x - 3)^4 + 2$ **c** $y = x^4 - 16$
d $y = 16 - x^4$ **e** $y = (3 - x)^4$ **f** $y = -2(x + 1)^4 + 1$

5 Use a CAS calculator to compare the graphs of higher degree power functions.

7.2 Division of polynomials

Not all cubics can be written in the form $y = a(x - h)^3 + k$.

When sketching the graphs of cubics which are not of the form $y = a(x - h)^3 + k$ begin by finding the x-axis intercepts. All cubics will have at least one x-axis intercept. Some will have two and others will have three.

As with quadratics, finding x-axis intercepts can be done by factorising and solving the resulting equation using the null factor law.

We shall first look at the techniques required for factorising cubics.

Reviewing the process of long division, for example $172 \div 13$, gives

$$\begin{array}{r} 13 \\ 13\,\overline{)172} \\ \underline{13} \\ 42 \\ \underline{39} \\ 3 \end{array}$$

$\therefore \frac{172}{13} = 13$ and 3 remainder, i.e $13\frac{3}{13}$

We note that as there is a remainder, it can be seen that 13 is not a factor of 172.

The process of dividing a polynomial by a linear factor follows very similar steps.

For example, $x^2 + 7x + 11 \div (x - 2)$ gives

$$\begin{array}{rl} x + 9 & \\ x - 2\,\overline{)\,x^2 + 7x + 11} & \text{Divide } x^2 \text{ by } x. \\ \underline{x^2 - 2x } & \text{Multiply } (x - 2) \text{ by } x \text{ and subtract from } x^2 + 7x + 11. \\ 9x + 11 & \text{This leaves } 9x + 11,\ x \text{ into } 9x \text{ goes 9 times.} \\ \underline{9x - 18} & \text{Multiply } (x - 2) \text{ by 9 and subtract from } 9x + 11. \\ 29 & \text{This leaves 29 remainder.} \end{array}$$

Thus $x^2 + 7x + 11 \div (x - 2) = x + 9$ with remainder 29.

$$\therefore \quad \frac{x^2 + 7x + 11}{x - 2} = x + 9 + \frac{29}{x - 2}$$

We can see in this example that $x - 2$ is *not* a factor of $x^2 + 7x + 11$.

Example 3

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$\begin{array}{r|l}
 & x^2 - x - 12 \\
x + 2 & x^3 + x^2 - 14x - 24 \\
 & x^3 + 2x^2 \\
\hline
 & \quad -x^2 - 14x \\
 & \quad -x^2 - 2x \\
\hline
 & \qquad -12x - 24 \\
 & \qquad -12x - 24 \\
\hline
 & \qquad\qquad 0
\end{array}$$

In this example we see that $x + 2$ *is* a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero.

Thus $x^3 + x^2 - 14x - 24 \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \quad \frac{x^3 + x^2 - 14x - 12}{x + 2} = x^2 - x - 12$$

Example 4

Divide $3x^3 + x - 3$ by $x - 2$.

Solution

Note: Here there is no term in x^2, however we can rewrite the polynomial as $3x^3 + 0x^2 + x - 3$.

$$\begin{array}{r|l}
 & 3x^2 + 6x + 13 \\
x - 2 & 3x^3 + 0x^2 + x - 3 \\
 & 3x^3 - 6x^2 \\
\hline
 & \qquad 6x^2 + x \\
 & \qquad 6x^2 - 12x \\
\hline
 & \qquad\qquad 13x - 3 \\
 & \qquad\qquad 13x - 26 \\
\hline
 & \qquad\qquad\qquad 23
\end{array}$$

$$\text{Thus} \quad \frac{3x^3 + x - 3}{x - 2} = 3x^2 + 6x + 13 \text{ with a remainder of } 23$$

$$= 3x^2 + 6x + 13 + \frac{23}{x - 2}$$

Example 5

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

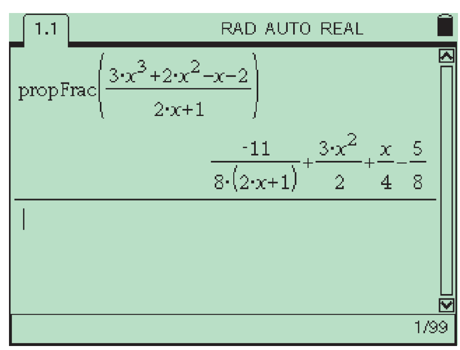
Solution

$$
\begin{array}{r l}
 & \frac{3}{2}x^2 + \frac{1}{4}x - \frac{5}{8} \\
2x + 1 \,) & \overline{3x^3 + 2x^2 - x - 2} \\
 & \underline{3x^3 + \frac{3}{2}x^2} \\
 & \frac{1}{2}x^2 - x \\
 & \underline{\frac{1}{2}x^2 + \frac{1}{4}x} \\
 & -\frac{5}{4}x - 2 \\
 & \underline{-\frac{5}{4}x - \frac{5}{8}} \\
 & -1\frac{3}{8}
\end{array}
$$

$$\therefore \frac{3x^3 + 2x^2 - x - 2}{2x + 1} = \frac{3x^2}{2} + \frac{x}{4} - \frac{5}{8} - \frac{11}{8(2x + 1)}$$

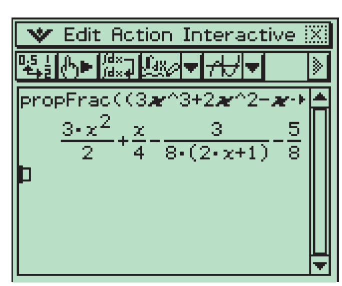
Using the TI-Nspire

Use **PropFrac( )** from the **Fraction Tools** menu ((menu) (3) (7) (1)) as shown.



Using the Casio ClassPad

Select **Interactive—Transformation—propFrac**, then enter the expression $(3x\text{^}2 + 2x\text{^}2 - x - 2) / (2x + 1)$. Remember to close the propFrac brackets and to be careful of the brackets when entering fractions.



Exercise 7B

1 For each of the following divide the polynomial by the accompanying linear expression:

Example 3, Example 4

a $x^3 + x^2 - 2x + 3,\ x - 1$ **b** $2x^3 + x^2 - 4x + 3,\ x + 1$
c $3x^3 - 4x^2 + 2x + 1,\ x + 2$ **d** $x^3 + 3x - 4,\ x + 1$
e $2x^3 - 3x^2 + x - 2,\ x - 3$ **f** $2x^3 + 3x^2 + 17x + 15,\ x + 4$
g $x^3 + 4x^2 + 3x + 2,\ x + 3$

Example 5

2 For each of the following divide the polynomial by the accompanying linear expression:

a $x^3 + 6x^2 + 8x + 11,\ 2x + 5$ **b** $2x^3 + 5x^2 - 4x - 5,\ 2x + 1$
c $x^3 - 3x^2 + 1,\ 3x - 1$ **d** $x^3 - 3x^2 + 6x + 5,\ x - 2$
e $2x^3 + 3x^2 - 32x + 15,\ 2x - 1$ **f** $x^3 + 2x^2 - 1,\ 2x + 1$

3 **a** Write $\dfrac{x^3 + 2x^2 + 5x + 1}{x - 1}$ in the form $P(x) + \dfrac{a}{x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

b Write $\dfrac{2x^3 - 2x^2 + 5x + 3}{2x - 1}$ in the form $P(x) + \dfrac{a}{2x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

7.3 Factorisation of polynomials

Remainder theorem

Example 6

Let $P(x) = x^3 + 3x^2 + 2x + 1$.

a **i** Divide $P(x)$ by $(x - 1)$. **ii** Evaluate $P(1)$.
b **i** Divide $P(x)$ by $(x - 2)$. **ii** Evaluate $P(2)$.

Solution

a **i**

$$\begin{array}{r} x^2 + 4x + 6 \\ x - 1 \overline{)\, x^3 + 3x^2 + 2x + 1} \\ \underline{x^3 - x^2 \qquad\qquad} \\ 4x^2 + 2x \quad \\ \underline{4x^2 - 4x \quad} \\ 6x + 1 \\ \underline{6x - 6} \\ 7 \end{array}$$

Thus $\dfrac{x^3 + 3x^2 + 2x + 1}{x - 1} = x^2 + 4x + 6 + \dfrac{7}{x - 1}$

ii $P(1) = 1^3 + 3(1)^2 + 2 + 1$
$= 7$

b i

$$\begin{array}{r} x^2 + 5x + 12 \\ x - 2 \overline{\big) x^3 + 3x^2 + 2x + 1} \\ \underline{x^3 - 2x^2 } \\ 5x^2 + 2x \\ \underline{5x^2 - 10x } \\ 12x + 1 \\ \underline{12x - 24} \\ 25 \end{array}$$

Thus $\dfrac{x^3 + 3x^2 + 2x + 1}{x - 2} = x^2 + 5x + 12 + \dfrac{25}{x - 2}$

ii $P(2) = (2)^3 + 3(2)^2 + 2(2) + 1$

$= 8 + 12 + 4 + 1$

$= 25$

The results indicate that, when $P(x)$ is divided by $(x - a)$, the remainder is equal to $P(a)$. This is in fact true, and the result is called **remainder theorem**.

It is proved as follows. Suppose that, when the polynomial $P(x)$ is divided by $(x - a)$, the quotient is $Q(x)$ and the remainder is R, then

$$P(x) = (x - a)Q(x) + R$$

Now, as the two expressions are equal for all values of x, they are equal for $x = a$.

$$\therefore P(a) = (a - a)Q(a) + R \qquad \therefore R = P(a)$$

i.e. the remainder when $P(x)$ is divided by $(x - a)$ is equal to $P(a)$. We therefore have

$$P(x) = (x - a)Q(x) + P(a)$$

More generally:

> When $P(x)$ is divided by $ax + b$ the remainder is $P\left(-\dfrac{b}{a}\right)$.

Example 7

Use the remainder theorem to find the value of the remainder when $P(x) = x^3 - 2x + 4$ is divided by $2x + 1$.

Solution

$$P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right) + 4 = -\frac{1}{8} + 1 + 4 = \frac{39}{8}$$

The remainder is $\dfrac{39}{8}$.

Example 8

When $P(x) = x^3 + 2x + a$ is divided by $x - 2$ the remainder is 4. Find the value of a.

Solution

$$P(2) = 8 + 4 + a = 4.$$

Therefore $a = -8$.

Exercise 7C

Example 7 1 Without dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 - x^2 - 3x + 1, x - 1$

b $x^3 - 3x^2 + 4x - 1, x + 2$

c $2x^3 - 2x^2 + 3x + 1, x - 2$

d $x^3 - 2x + 3, x + 1$

e $x^3 + 2x - 5, x - 2$

f $2x^3 + 3x^2 + 3x - 2, x + 2$

g $6 - 5x + 9x^2 + 10x^3, 2x + 3$

h $10x^3 - 3x^2 + 4x - 1, 2x + 1$

i $108x^3 - 27x^2 - 1, 3x + 1$

Example 8 2 Find the values of a in the expressions below when the following conditions are satisfied:

a $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$.

b $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$.

c $x^3 - 3x^2 + ax + 5$ has remainder 17 when divided by $x - 3$.

d $x^3 + x^2 + ax + 8$ has remainder 0 when divided by $x - 1$.

7.4 Factor theorem

Now, in order for $(x - a)$ to be a factor of the polynomial $P(x)$, the remainder must be zero. We state this result as the **factor theorem**.

> If for a polynomial, $P(x)$, $P(a) = 0$ then $x - a$ is a factor.
> Conversely, if $x - a$ is a factor of $P(x)$ then $P(a) = 0$.

More generally:

> If $ax + b$ is a factor of $P(x)$ then $P\left(-\frac{b}{a}\right) = 0$.
>
> Conversely, if $P\left(-\frac{b}{a}\right) = 0$ then $ax + b$ is a factor of $P(x)$.

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Example 9

Show that $(x + 1)$ is a factor of $x^3 - 4x^2 + x + 6$ and hence find the other linear factors.

Solution

$$\begin{aligned} \text{Let } P(x) &= x^3 - 4x^2 + x + 6 \\ P(-1) &= (-1)^3 - 4(-1)^2 + (-1) + 6 \\ &= 0 \end{aligned}$$

$\therefore$ From the factor theorem $(x - (-1)) = (x + 1)$ is a factor.

Using division of polynomials, the other factor can be found. This will be a quadratic factor.

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ \hline x + 1 & x^3 - 4x^2 + x + 6 \\ & x^3 + x^2 \\ \hline & \quad -5x^2 + x \\ & \quad -5x^2 - 5x \\ \hline & \qquad\qquad 6x + 6 \\ & \qquad\qquad 6x + 6 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

$\therefore x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$

Now factorising the quadratic factor we have

$$P(x) = (x + 1)(x - 3)(x - 2)$$

$\therefore$ the linear factors of $x^3 - 4x^2 + x + 6$ are $(x + 1)$, $(x - 3)$ and $(x - 2)$.

Example 10

Factorise $x^3 - 2x^2 - 5x + 6$.

Solution

Let us assume there are three linear factors

$$\begin{aligned} \text{i.e. } x^3 - 2x^2 - 5x + 6 &= (x - a)(x - b)(x - c) \\ &= x^3 - (a + b + c)x^2 + (ab + bc + ac)x - abc \end{aligned}$$

By considering the constant term it can be seen that $abc = -6$.

Thus only the factors of -6 need be considered (i.e. $\pm1, \pm2, \pm3, \pm6$).

Try these in turn until a value for a makes $P(a) = 0$:

$$P(1) = 1 - 2(1) - 5 + 6 = 0$$

$\therefore (x - 1)$ is a factor.

Now divide to find the other factors.

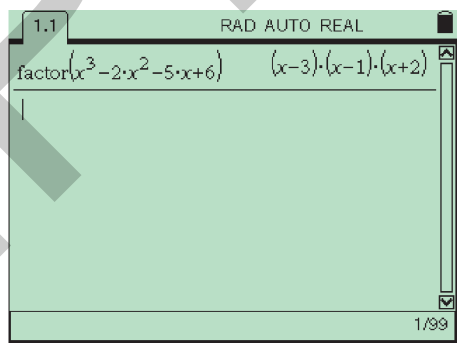
$$
\begin{array}{r}
x^2 - x - 6 \\
x - 1 \overline{\smash{\big)}\, x^3 - 2x^2 - 5x + 6} \\
\underline{x^3 - x^2 \phantom{{}- 5x + 6}} \\
-x^2 + 5x \phantom{{}+ 6} \\
\underline{-x^2 + x \phantom{{}+ 6}} \\
-6x + 6 \\
\underline{-6x + 6} \\
0
\end{array}
$$

$$
\begin{aligned}
\therefore x^3 - 2x^2 - 5x + 6 &= (x - 1)(x^2 - x - 6) \\
&= (x - 1)(x - 3)(x + 2)
\end{aligned}
$$

$\therefore$ the factors of $x^3 - 2x^2 - 5x + 6$ are $(x - 1)$, $(x - 3)$ and $(x + 2)$.

Using the TI-Nspire

Use **Factor( )** from the **Algebra** menu (menu 3 2) to factorise the expression $x^3 - 2x^2 - 5x + 6$.

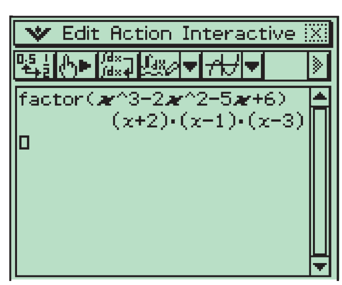


Using the Casio ClassPad

Enter and highlight the expression

$x\^3 - 2x\^2 - 5x + 6$

Select **Interactive**—**Transformation**—**factor**.



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Special cases: sums and differences of cubes

Example 11

Factorise $x^3 - 27$.

Solution

Let $P(x) = x^3 - 27$
$P(3) = 27 - 27 = 0$
$\therefore (x - 3)$ is a factor.

$$
\begin{array}{r|l}
 & x^2 + 3x + 9 \\
\hline
x - 3 & x^3 + 0x^2 + 0x - 27 \\
 & x^3 - 3x^2 \\
\hline
 & \quad 3x^2 \\
 & \quad 3x^2 - 9x \\
\hline
 & \qquad 9x - 27 \\
 & \qquad 9x - 27 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$\therefore x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

In general, if $P(x) = x^3 - a^3$ then $(x - a)$ is a factor and by division

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$ then

$$x^3 - (-a)^3 = (x - (-a))(x^2 + (-a)x + (-a)^2)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 12

Factorise $8x^3 + 64$.

Solution

$$
\begin{aligned}
8x^3 + 64 &= (2x)^3 + (4)^3 \\
&= (2x + 4)(4x^2 - 8x + 16)
\end{aligned}
$$

Exercise 7D

Example 9

1 Without dividing, show that the first polynomial is exactly divisible by the second polynomial:

a $x^3 - x^2 + x - 1,\ x - 1$

b $x^3 + 3x^2 - x - 3,\ x - 1$

c $2x^3 - 3x^2 - 11x + 6,\ x + 2$

d $2x^3 - 13x^2 + 27x - 18,\ 2x - 3$

2 Find the value of m if the first polynomial is exactly divisible by the second:

a $x^3 - 4x^2 + x + m, x - 3$ **b** $2x^3 - 3x^2 - (m + 1)x - 30, x - 5$

c $x^3 - (m + 1)x^2 - x + 30, x + 3$

Examples 9,10

3 Factorise each of the following:

a $2x^3 + x^2 - 2x - 1$ **b** $x^3 + 3x^2 + 3x + 1$ **c** $6x^3 - 13x^2 + 13x - 6$

d $x^3 - 21x + 20$ **e** $2x^3 + 3x^2 - 1$ **f** $x^3 - x^2 - x + 1$

g $4x^3 + 3x - 38$ **h** $4x^3 + 4x^2 - 11x - 6$

Examples 11,12

4 Factorise each of the following:

a $x^3 - 1$ **b** $x^3 + 64$ **c** $27x^3 - 1$

d $64x^3 - 125$ **e** $1 - 125x^3$ **f** $8 + 27x^3$

g $64m^3 - 27n^3$ **h** $27b^3 + 8a^3$

5 Factorise each of the following:

a $x^3 + x^2 - x + 2$ **b** $3x^3 - 7x^2 + 4$

c $x^3 - 4x^2 + x + 6$ **d** $6x^3 + 17x^2 - 4x - 3$

6 Find the values of a and b and factorise the polynomial $P(x) = x^3 + ax^2 - x + b$, given that $P(x)$ is divisible by $x - 1$ and $x + 3$.

7 **a** Show that, for any constant a and any natural number n, $x - a$ is a factor of $x^n - a^n$.

b Find conditions (if any) on n that are required in order that:

i $x + a$ is a factor of $x^n + a^n$ **ii** $x + a$ is a factor of $x^n - a^n$

8 The polynomial $P(x)$ has a remainder of 2 when divided by $x - 1$ and a remainder of 3 when divided by $x - 2$. The remainder when $P(x)$ is divided by $(x - 1)(x - 2)$ is $ax + b$, i.e. $P(x)$ can be written as $P(x) = (x - 1)(x - 2)Q(x) + ax + b$.

a Find the values of a and b.

b **i** Given that $P(x)$ is a cubic polynomial with coefficient of x^3 being 1, and -1 is a solution of the equation $P(x) = 0$, find $P(x)$.

ii Show that the equation $P(x) = 0$ has no other real roots.

7.5 Solving cubic equations

Example 13

Solve $(x - 2)(x + 1)(x + 3) = 0$.

Solution

By the null factor law $(x - 2)(x + 1)(x + 3) = 0$

implies $x - 2 = 0$ or $x + 1 = 0$ or $x + 3 = 0$.

Thus the solutions are $x = 2, -1$ and -3.

Example 14

Solve $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

Let

$$P(x) = x^3 - 4x^2 - 11x + 30$$
$$P(1) = 1 - 4 - 11 + 30 \neq 0$$
$$P(-1) = -1 - 4 + 11 + 30 \neq 0$$
$$P(2) = 8 - 16 - 22 + 30 = 0$$

$\therefore (x - 2)$ is a factor.

By division

$$x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15)$$
$$= (x - 2)(x - 5)(x + 3)$$

$\therefore\ (x - 2)(x - 5)(x + 3) = 0$

$\therefore\ x - 2 = 0$ or $x - 5 = 0$ or $x + 3 = 0$

$\therefore\ x = 2, 5$ or -3.

Example 15

Solve $2x^3 - 5x^2 + x + 2 = 0$.

Solution

Let

$$P(x) = 2x^3 - 5x^2 + x + 2$$
$$P(1) = 2 - 5 + 1 + 2 = 0$$

$\therefore (x - 1)$ is a factor

By division

$$2x^3 - 5x^2 + x + 2 = (x - 1)(2x^2 - 3x - 2)$$
$$= (x - 1)(2x + 1)(x - 2)$$

$\therefore (x - 1)(2x + 1)(x - 2) = 0$

$\therefore\ x = 1, -\frac{1}{2},$ or 2

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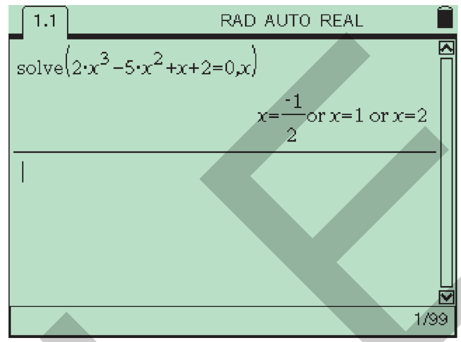
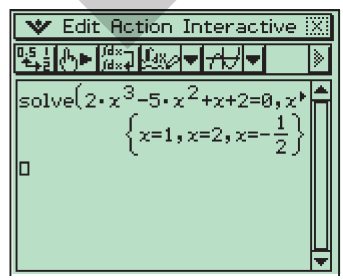
Using the TI-Nspire

Use **Solve( )** from the **Algebra** menu (menu 3 1) to solve the equation $2x^3 - 5x^2 + x + 2 = 0$.



Using the Casio ClassPad

Enter and highlight the equation
$2x\text{^}3 - 5x\text{^}2 + x + 2 = 0$.
Select **Interactive—Equation/Inequality—solve** and ensure the variable is set to x.



Example 16

Solve each of the following equations for x:

a $2x^3 - x^2 - x = 0$ **b** $x^3 + 2x^2 - 10x = 0$

Solution

a
$$2x^3 - x^2 - x = 0$$
$$x(2x^2 - x - 1) = 0$$
$$x(2x + 1)(x - 1) = 0$$
$$\therefore \quad x = 0 \text{ or } x = -\frac{1}{2} \text{ or } x = 1$$

b
$$x^3 + 2x^2 - 10x = 0$$
$$x(x^2 + 2x - 10) = 0$$

Completing the square gives

$$x(x^2 + 2x + 1 - 10 - 1) = 0$$
$$x(x^2 + 2x + 1 - (\sqrt{11})^2) = 0$$
$$x((x + 1)^2 - (\sqrt{11})^2) = 0$$
$$x(x + 1 - \sqrt{11})(x + 1 + \sqrt{11}) = 0$$
$$x = 0 \text{ or } x + 1 = \sqrt{11} \text{ or } x + 1 = -\sqrt{11}$$
$$\therefore \quad x = 0 \text{ or } x = -1 + \sqrt{11} \text{ or } x = -1 - \sqrt{11}$$

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Example 17

Solve each of the following equations for x:

a $x^3 - 4x^2 - 11x + 44 = 0$ **b** $x^3 - ax^2 - 11x + 11a = 0$

Solution

Grouping of terms sometimes facilitates factorisation.

a $x^3 - 4x^2 - 11x + 44 = 0$

$x^2(x - 4) - 11(x - 4) = 0$

Therefore $(x - 4)(x^2 - 11) = 0$

And $x = 4$ or $x = \pm\sqrt{11}$

b $x^3 - ax^2 - 11x + 11a = 0$

$x^2(x - a) - 11(x - a) = 0$

Therefore $(x - a)(x^2 - 11) = 0$

And $x = a$ or $x = \pm\sqrt{11}$

Exercise 7E

1 Solve each of the following. Remember to check your solutions in the original equations.

Example 13 Example 14 Example 15

a $(x - 1)(x + 2)(x - 4) = 0$ **b** $(x - 4)(x - 4)(x - 6) = 0$
c $(2x - 1)(x - 3)(3x + 2) = 0$ **d** $x^3 + 2x^2 - x - 2 = 0$
e $x^3 - 19x + 30 = 0$ **f** $3x^3 - 4x^2 - 13x - 6 = 0$
g $x^3 - x^2 - 2x + 2 = 0$ **h** $5x^3 + 12x^2 - 36x - 16 = 0$
i $6x^3 - 5x^2 - 2x + 1 = 0$ **j** $2x^3 - 3x^2 - 29x - 30 = 0$

2 Solve each of the following for x:

a $x^3 + x^2 - 24x + 36 = 0$ **b** $6x^3 + 13x^2 - 4 = 0$
c $x^3 - x^2 - 2x - 12 = 0$ **d** $2x^3 + 3x^2 + 7x + 6 = 0$
e $x^3 - x^2 - 5x - 3 = 0$ **f** $x^3 + x^2 - 11x - 3 = 0$

Example 16

3 Solve each of the following equations for x:

a $x^3 - 2x^2 - 8x = 0$ **b** $x^3 + 2x^2 - 11x = 0$
c $x^3 - 3x^2 - 40x = 0$ **d** $x^3 + 2x^2 - 16x = 0$

4 Solve each of the following equations for x:

a $2x^3 = 16x$ **b** $2(x - 1)^3 = 32$ **c** $x^3 + 8 = 0$
d $2x^3 + 250 = 0$ **e** $1000 = \frac{1}{x^3}$

Example 17

5 Use grouping to solve each of the following:

a $x^3 - x^2 + x - 1 = 0$ **b** $x^3 + x^2 + x + 1 = 0$
c $x^3 - 5x^2 - 10x + 50 = 0$ **d** $x^3 - ax^2 - 16x + 16a = 0$

6 Factorise each of the following cubic expressions, using a calculator to help find at least one linear factor:

a $2x^3 - 22x^2 - 250x + 2574$ **b** $2x^3 + 27x^2 + 52x - 33$
c $2x^3 - 9x^2 - 242x + 1089$ **d** $2x^3 + 51x^2 + 304x - 165$

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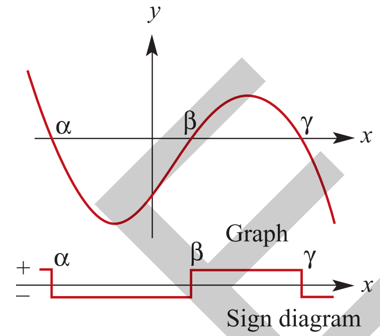
7.6 Graphs of cubic functions

We will look at a technique which assists in sketching graphs of cubic functions.

A sign diagram is a number line diagram which shows when an expression is positive or negative. Shown is a sign diagram for the cubic function, the graph of which is also shown.

The factorisation method requires that factors, and hence the x-axis intercepts, be found. The y-axis intercept and sign diagram can then be used to complete the graph.



Example 18

Sketch the graph of $y = x^3 + 2x^2 - 5x - 6$.

Solution

Let $P(x) = x^3 + 2x^2 - 5x - 6$

$P(1) = 1 + 2 - 5 - 6 \neq 0$

$P(-1) = -1 + 2 + 5 - 6 = 0$

$\therefore (x + 1)$ is a factor.

By division

$y = (x + 1)(x - 2)(x + 3)$

For $(x + 1)(x - 2)(x + 3) = 0$

$x + 1 = 0$ or $x - 2 = 0$ or $x + 3 = 0$

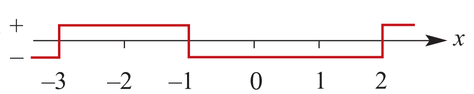
$\therefore x = -1, 2,$ or -3

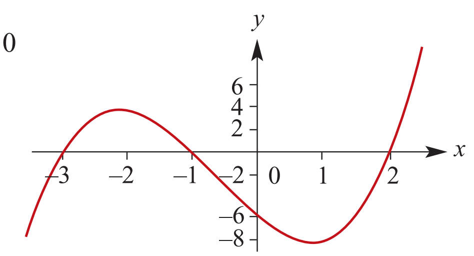
When $x < -3$, y is negative

$-3 < x < -1$, y is positive

$-1 < x < 2$, y is negative

giving the sign diagram.



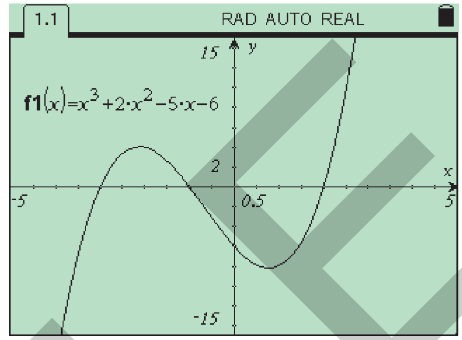


At this stage the location of the turning points is unspecified. It is important, however, to note that, unlike quadratic graphs, the turning points are not symmetrically located between x-axis intercepts. How to determine the exact values of the coordinates of the turning point will be shown later in the course.

Using the TI-Nspire

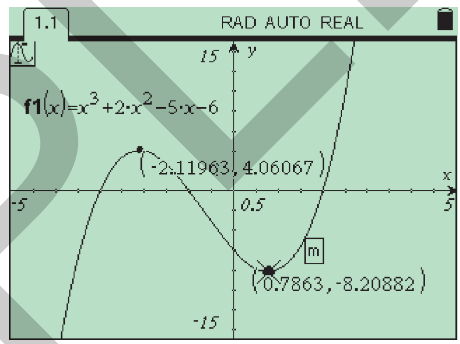
In order to provide more detail, the coordinates of the turning points can be found on the calculator. Enter $f_1(x) = x^3 + 2x^2 - 5x - 6$ in a **Graphs & Geometry** application and choose a suitable **Window** (menu 4 1).

The **Entry Line** can be hidden by pressing ctrl G.



Use **Graph Trace** (menu 5 1) to find the coordinates of the two turning points. A m will appear near a turning point to indicate that the calculator has found a local maximum or a local minimum.

Note that trace can be used with a combination of the Nav Pad and typing in a x-value followed by enter.

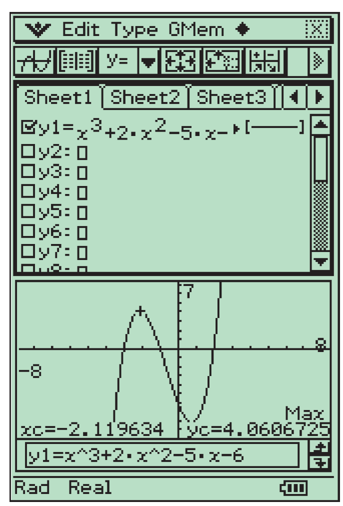


Using the Casio ClassPad

In order to provide more detail, the coordinates of the turning points can be found with a CAS calculator. In Graph&Tab enter $y1 = x^\wedge 3 + 2x^\wedge 2 - 5x - 6$. Tick the box and select the graph icon to produce the graph. Choose a suitable window using the window icon or a combination of **Zoom Out** and **Zoom Box**.

Click in the graph box to select it (bold border), then select **Analysis—G-Solve—Max** to find the local maximum and **Min** to find the local minimum points.

Note: The maximum and minimum points must be visible on the screen before carrying out the analysis step.



Example 19

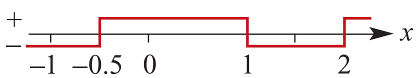
Sketch the graph of $y = 2x^3 - 5x^2 + x + 2$.

Solution

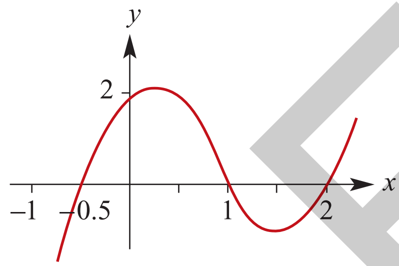
From Example 15 it can be seen that

$$2x^3 - 5x^2 + x + 2 = (x - 1)(2x + 1)(x - 2).$$

The y-axis intercept is (0, 2). The sign diagram and graph are shown.



Sign diagram



Example 20

Sketch the graph of $y = (x - 1)(x + 2)(x + 1)$. Do not give coordinates of turning points.

Solution

The cubic has been given in factorised form, so the x-axis intercepts can be easily found.

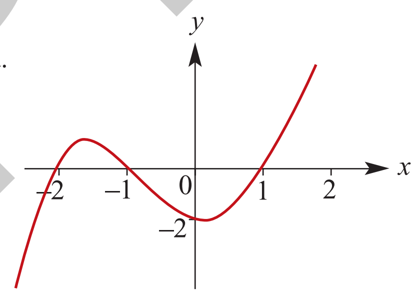
Let $y = 0$

$$0 = (x - 1)(x + 2)(x + 1)$$

Using the null factor law, the x-axis intercepts are 1, −1, and −2.

To find the y-axis intercept, let $x = 0$

$$\begin{aligned} y &= (0 - 1)(0 + 2)(0 + 1) \\ &= -2 \end{aligned}$$



It is noted that the implied domain of all cubics is R and that the range is also R.

If the factorised cubic has a repeated factor there are only two x-axis intercepts and the repeated factor corresponds to one of the turning points.

Example 21

Sketch the graph of $y = x^2(x - 1)$.

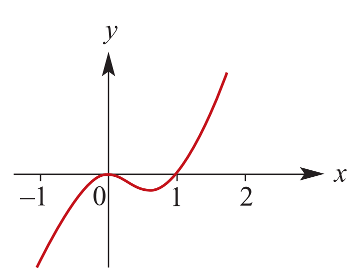
Solution

To find the x-axis intercepts, let $y = 0$.

Then $\quad x^2(x - 1) = 0$

x-axis intercepts are at $x = 0$ and 1 and, because the repeated factor is x^2, there is also a turning point at $x = 0$.

y-axis intercept (letting $x = 0$) is at $y = 0$.



Some cubics will only have one x-axis intercept. This is because, when they are factorised, they are found to have only one linear factor, with the remaining quadratic factor unable to be factorised further.

Example 22

Sketch the graph of $y = -(x - 1)(x^2 + 4x + 5)$.

Solution

To find the x-axis intercept, let $y = 0$.

First, we note that the factor of $x^2 + 4x + 5$ cannot be factorised further.

$$\Delta = b^2 - 4ac$$
$$\Delta = 4^2 - 4(1)(5)$$
$$= -4$$

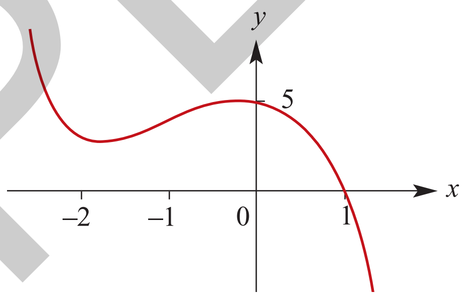
$\therefore$ there are no linear factors.

Hence, when solving the equation $-(x - 1)(x^2 + 4x + 5) = 0$, there is only one solution.

$\therefore$ x-axis intercept is $x = 1$.

To find the y-axis intercept, let $x = 0$.

$$-((0) - 1)((0)^2 + 4(0) + 5) = 5$$



On a CAS calculator it is found that the turning points are at (0, 5) and (−1.82, 2.91), where the values for the coordinates of the second point are given to 2 decimal places.

Exercise 7F

Examples 18–22

1 Sketch the graphs for each of the following and draw a sign diagram. Label your sketch graph showing the points of intersection with the axes. (Do not determine coordinates of turning points.)

- **a** $y = x(x - 1)(x - 3)$
- **b** $y = (x - 1)(x + 1)(x + 2)$
- **c** $y = (x - 1)(x - 2)(x - 3)$
- **d** $y = (2x - 1)(x - 2)(x + 3)$
- **e** $y = x^3 - 9x$
- **f** $y = x^3 + x^2$
- **g** $y = x^3 - 5x^2 + 7x - 3$
- **h** $y = x^3 - 4x^2 - 3x + 18$
- **i** $y = -x^3 + x^2 + 3x - 3$
- **j** $y = 3x^3 - 4x^2 - 13x - 6$
- **k** $y = 6x^3 - 5x^2 - 2x + 1$

2 Sketch the graphs of each of the following, using a CAS calculator to find the coordinates of axes intercepts and local maximum and local minimum values:

- **a** $y = -4x^3 - 12x^2 + 37x - 15$
- **b** $y = -4x^3 + 19x - 15$
- **c** $y = -4x^3 + 0.8x^2 + 19.8x - 18$
- **d** $y = 2x^3 + 11x^2 + 15x$
- **e** $y = 2x^3 + 6x^2$
- **f** $y = 2x^3 + 6x^2 + 6$

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3 Show that the graph of f, where $f(x) = x^3 - x^2 - 5x - 3$, cuts the x-axis at one point and touches it at another. Find the values of x at these points.

7.7 Solving cubic inequations

As was done with quadratic inequations, cubic inequations can be solved by considering the graph of the corresponding equation and determining the solution of the inequation from the graph.

Example 23

Find $\{x: x^3 + x^2 - 5x + 3 \le 0\}$

Solution

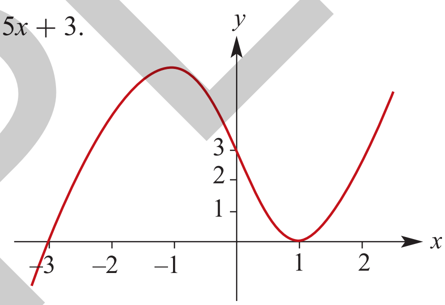
Start by sketching the graph of $y = x^3 + x^2 - 5x + 3$.

Let $P(x) = x^3 + x^2 - 5x + 3$

$P(1) = 1 + 1 - 5 + 3 = 0$

$\therefore$ $(x - 1)$ is a factor.

By division $y = (x - 1)^2(x + 3)$



There are only two x-axis intercepts, (1, 0) and (−3, 0).

The y-axis intercept is (0, 3).

From the graph we can see that $y \le 0$ when x is ≤ -3 or when $x = 1$.

$\therefore \{x: x^3 + x^2 - 5x + 3 \le 0\} = \{x: x \le -3\} \cup \{x: x = 1\}$

Exercise 7G

Example 23

Solve the following cubic inequations:

a $(x - 1)(x + 2)(x - 3) \le 0$

b $(x + 1)(x + 2)(x - 4) \ge 0$

c $(x - 1)(x - 2)^2 < 0$

d $x(x + 2)(x - 3) > 0$

e $(x - 1)^3 + 8 \le 0$

f $x^3 - 1 \ge 0$

g $x^2(x - 4) > 0$

h $(x + 3)(x^2 + 2x + 5) \le 0$

7.8 Finding equations for given cubic graphs

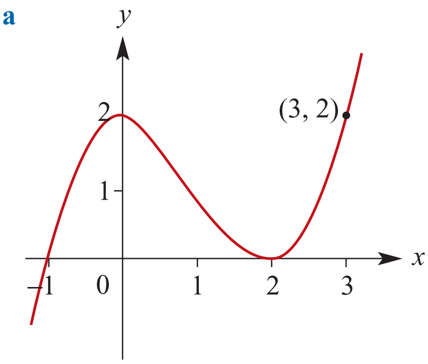
Example 24

Excel

Determine the cubic rule for graphs **a** and **b**.

Solution

a



The x-axis intercepts are -1 and 2 and the graph touches the x-axis at 2, therefore the form of the rule appears to be $y = a(x+1)(x-2)^2$.

Put (3, 2) in the equation:

$$2 = a(4)(1)$$
$$\frac{1}{2} = a$$
$$\therefore y = \frac{1}{2}(x+1)(x-2)^2 \text{ is the rule.}$$

Alternatively:

Using the general form of a cubic equation $y = ax^3 + bx^2 + cx + d$.

From the graph $d = 2$. Putting each of the ordered pairs $(-1, 0)$, $(2, 0)$ and $(3, 2)$ in the general equation will give three simultaneous equations in three unknowns:

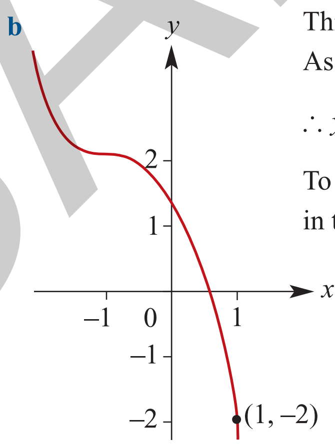
$$-2 = -a + b - c \qquad (1)$$
$$-1 = 4a + 2b + c \qquad (2)$$
$$0 = 9a + 3b + c \qquad (3)$$

Solving these gives the values

$$a = \frac{1}{2}, \quad b = -1\frac{1}{2} \quad \text{and} \quad c = 0$$
$$\therefore y = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 2$$

This is equivalent to $y = \frac{1}{2}(x+1)(x-2)^2$.

b



This graph appears to be of the form $y = a(x-h)^3 + k$.

As can be seen from the graph, $k = 2$, $h = -1$.

$$\therefore y - 2 = a(x+1)^3$$

To determine a, put one of the known points, say $(1, -2)$, in the equation.

$$-2 - 2 = a(2)^3$$
$$-4 = 8a$$
$$-\frac{1}{2} = a$$

$\therefore$ The rule is $y - 2 = \frac{-1}{2}(x+1)^3$.

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Using the TI-Nspire

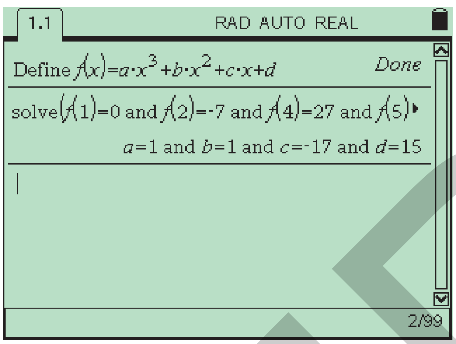
We can find the values of a, b, c and d of the cubic equation $y = ax^3 + bx^2 + cx + d$ given that the points (1, 0), (2, −7), (4, 27) and (5, 80) lie on its graph.

Define $f(x) = ax^3 + bx^2 + cx + d$.

Now write the information using function notation:

$f(1) = 0,\ f(2) = -7,\ f(4) = 27$ and $f(5) = 80$.

Use **Solve** ($f(1) = 0$ and $f(2) = -7$ and $f(4) = 27$ and $f(5) = 80$, $\{a, b, c, d\}$)



Using the Casio ClassPad

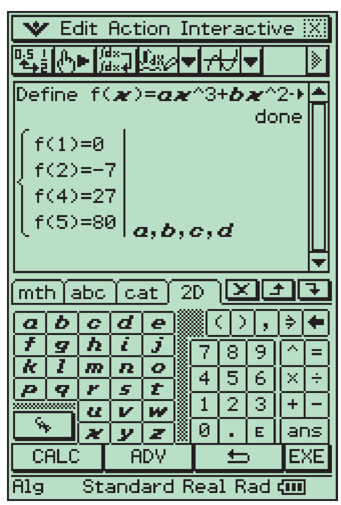
It is known that the points with coordinates (1, 0), (2, −7), (4, 27), (5, 80) lie on a curve with equation $y = f(x)$ where $f(x)$ is a cubic.

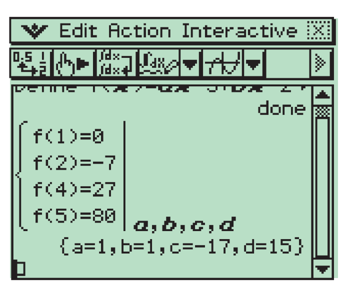
Define $f(x) = ax^3 + bx^2 + cx + d$.

Use **Interactive—Define** with function name **f**, variable x and enter the expression. Remember to use the **Keyboard—mth—VAR** to enter variables a, b, c, d and the **abc** tab to enter the function name, f.

Use the simultaneous equation solver.

In **Math—2D** (click it twice more to expand for four simultaneous equations) and enter the known values $f(1) = 0$, $f(2) = -7$, $f(4) = 27$ and $f(5) = 80$ into the four lines and the variables a, b, c, d in the bottom right text box.



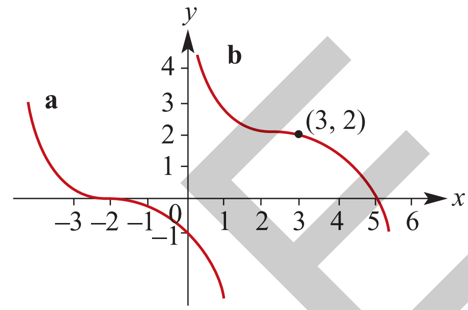


Skillsheet TEACHER

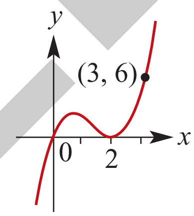
Exercise 7H

Example 24

1 The graphs shown are similar to the basic curve $y = -x^3$. Find possible cubic functions which define each of the curves.



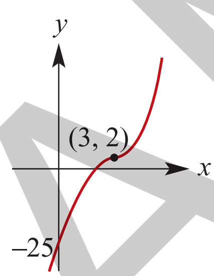
2 Find the equation of the cubic function for which the graph is shown.



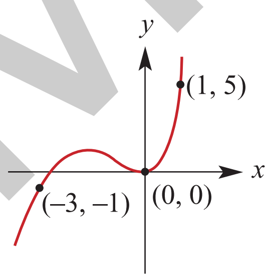
3 Find a cubic function whose graph touches the x-axis at $x = -4$, cuts it at the origin, and has a value 6 when $x = -3$.

4 The graphs below have equations of the form shown. In each case, determine the equation.

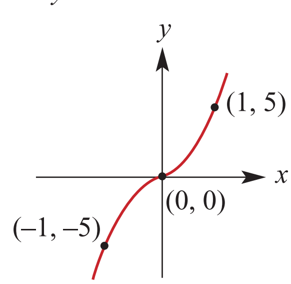
a $y = a(x - h)^3 + k$



b $y = ax^3 + bx^2$

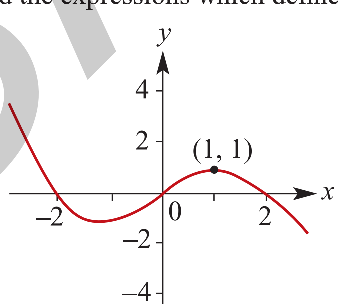


c $y = ax^3$

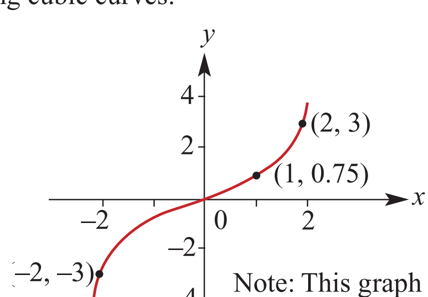


5 Find the expressions which define the following cubic curves:

a



b



Note: This graph is not of the form $y = a(x - h)^3 + k$.

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6 For each of the following, use a CAS calculator to find the values of a, b, c, d of the cubic equation $y = ax^3 + bx^2 + cx + d$, given that the following points lie on its graph:

a (0, 270) (1, 312) (2, 230) (3, 0)
b (−2, −406) (0, 26) (1, 50) (2, −22)
c (−2, −32) (2, 8) (3, 23) (8, 428)
d (1, −1) (2, 10) (3, 45) (4, 116)
e (−3, −74) (−2, −23) (−1, −2) (1, −2)
f (−3, −47) (−2, −15) (1, −3) (2, −7)
g (−4, 25) (−3, 7) (−2, 1) (1, −5)

7.9 Graphs of quartic functions

The techniques for graphing quartic functions are very similar to those employed for cubic functions. In this section solving simple quartic equations is first considered. A CAS calculator is to be used in the graphing of these functions. Great care needs to be taken in this process as it is easy to miss key points on the graph using these techniques.

Example 25

Solve each of the following equations for x:

a $x^4 - 8x = 0$ **b** $2x^4 - 8x^2 = 0$ **c** $x^4 - 2x^3 - 24x^2 = 0$

Solution

a $x^4 - 8x = 0$

Factorise to obtain

$x(x^3 - 8) = 0$

$\therefore x = 0$ or $x^3 - 8 = 0$

Thus $x = 2$ or $x = 0$.

b $2x^4 - 8x^2 = 0$

Factorise to obtain

$2x^2(x^2 - 4) = 0$

$\therefore 2x^2 = 0$ or $x^2 - 4 = 0$

Thus $x = 2$ or $x = -2$ or $x = 0$.

c $x^4 - 2x^3 - 24x^2 = 0$

Factorise to obtain

$x^2(x^2 - 2x - 24) = 0$

$\therefore x^2 = 0$ or $x^2 - 2x - 24 = 0$

i.e. $x = 0$ or $(x - 6)(x + 4) = 0$

Thus $x = 0$ or $x = 6$ or $x = -4$.

Quartic equations in general can be solved by techniques similar to those used for solving cubic functions.

Exercise 7I

Example 25

1 Solve each of the following equations for x:

a $x^4 - 27x = 0$
b $(x^2 - x - 2)(x^2 - 2x - 15) = 0$
c $x^4 + 8x = 0$
d $x^4 - 6x^3 = 0$
e $x^4 - 9x^2 = 0$
f $81 - x^4 = 0$
g $x^4 - 16x^2 = 0$
h $x^4 - 7x^3 + 12x^2 = 0$

i $x^4 - 9x^3 + 20x^2 = 0$ **j** $(x^2 - 4)(x^2 - 9) = 0$
k $(x - 4)(x^2 + 2x + 8) = 0$ **l** $(x + 4)(x^2 + 2x - 8) = 0$

2 Use a CAS calculator to help draw the graphs of each of the following. Give x-axis intercepts and coordinates of turning points. (Values of coordinates of turning points to be given correct to 2 decimal places.)

a $y = x^4 - 125x$ **b** $y = (x^2 - x - 20)(x^2 - 2x - 24)$
c $y = x^4 + 27x$ **d** $y = x^4 - 4x^3$
e $y = x^4 - 25x^2$ **f** $y = 16 - x^4$
g $y = x^4 - 81x^2$ **h** $y = x^4 - 7x^3 + 12x^2$
i $y = x^4 - 9x^3 + 20x^2$ **j** $y = (x^2 - 16)(x^2 - 25)$
k $y = (x - 2)(x^2 + 2x + 10)$ **l** $y = (x + 4)(x^2 + 2x - 35)$

7.10 Finite differences for sequences generated by polynomials

The following are examples of sequences:

A 1, 3, 5, 7, . . . B 1, 4, 9, 16, . . .
C 1, 3, 6, 10, . . . D 1, 5, 14, 30, 55, 9, . . .

Sequences can be defined by a function $f: N \to R$, where N is the set of natural numbers.

For sequence A: $f(n) = 2n - 1$ (linear function)

For sequence B: $f(n) = n^2$ (quadratic function)

In this section we consider those sequences that correspond to polynomial functions of degree less than or equal to 3 which have domain N.

For sequence C: $f(n) = \dfrac{n(n+1)}{2}$ (quadratic function)

For sequence D: $f(n) = \dfrac{n(n+1)(2n+1)}{6}$ (cubic function)

$f(n)$ is called the $\boldsymbol{n}^{\text{th}}$ term of the sequence.

In the study of these functions we will assume that the pattern continues infinitely. For example, for sequence A the sequence consists of all odd numbers and for sequence B the sequence continues to give all the square numbers.

We can construct a difference table for the sequences, A, B, and D given above. The difference table is continued until a constant is obtained. We will use the following notations for differences:

$$\Delta_1^1 = f(2) - f(1); \quad \Delta_1^2 = f(3) - f(2); \quad \Delta_1^n = f(n+1) - f(n)$$

$$\Delta_2^1 = \Delta_1^2 - \Delta_1^1; \quad \Delta_2^2 = \Delta_1^3 - \Delta_1^2; \quad \Delta_2^n = \Delta_1^{n+1} - \Delta_1^n$$

and in general $\Delta_{n+1}^m = \Delta_n^{m+1} - \Delta_n^m$

Difference table A

n	$f(n)$	Δ_1
1	1	
		2 (= 3 – 1)
2	3	
		2 (= 5 – 3)
3	5	
		2 (= 7 – 5)
4	7	
		2 (= 9 – 7)
5	9	

Difference table B

n	$f(n)$	Δ_1	Δ_2
1	1		
		3 (= 4 – 1)	
2	4		2 (= 5 – 3)
		5 (= 9 – 4)	
3	9		2 (= 7 – 5)
		7 (= 16 – 9)	
4	16		2 (= 9 – 7)
		9 (= 25 – 16)	
5	25		

Difference table C

n	$f(n)$	Δ_1	Δ_2	Δ_3
1	1			
		4		
2	5		5	
		9		2
3	14		7	
		16		2
4	30		9	
		25		2
5	55		11	
		36		
6	91			

From these tables we conjecture, but do not prove, that for a sequence generated by a polynomial function:

> If the column Δ_1 is constant and non zero the function is linear.
> If the column Δ_2 is constant and non zero the function is quadratic.
> If the column Δ_3 is constant and non zero the function is cubic.

The converse result also holds:

i.e. If f is linear $\Delta_1^n = c$, a constant, for all n.
If f is quadratic $\Delta_2^n = k$, a constant, for all n.
If f is cubic $\Delta_3^n = l$, a constant, for all n.

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The following three difference tables are for the general linear, quadratic and cubic functions. We will use these tables throughout the remainder of this section.

Linear: $f(n) = an + b$

n	$f(n)$	Δ_1
1	$a + b$	
		a
2	$2a + b$	
		a
3	$3a + b$	

Quadratic: $f(n) = an^2 + bn + c$

n	$f(n)$	Δ_1	Δ_2
1	$a + b + c$		
		$3a + b$	
2	$4a + 2b + c$		$2a$
		$5a + b$	
3	$9a + 3b + c$		$2a$
		$7a + b$	
4	$16a + 4b + c$		$2a$
		$9a + b$	
5	$25a + 5b + c$		

Cubic: $f(n) = an^3 + bn^2 + cn + d$

n	$f(n)$	Δ_1	Δ_2	Δ_3
1	$a + b + c + d$			
		$7a + 3b + c$		
2	$8a + 4b + 2c + d$		$12a + 2b$	
		$19a + 5b + c$		$6a$
3	$27a + 9b + 3c + d$		$18a + 2b$	
		$37a + 7b + c$		$6a$
4	$64a + 16b + 4c + d$		$24a + 2b$	
		$61a + 9b + c$		$6a$
5	$125a + 25b + 5c + d$		$30a + 2b$	
		$91a + 11b + c$		
6	$216a + 36b + 6c + d$			

These tables may be used to determine the rules for a given sequence. This is illustrated by the following example. We note that they do not constitute a proof that the rule is the one for the given sequence.

Example 26

Find the rule for each of the following sequences, using finite difference tables:

a 5, 11, 19, 29, 41, . . . **b** 6, 26, 64, 126, 218, 346 . . .

Solution

a

n	$f(n)$	Δ_1	Δ_2
1	5		
		6	
2	11		2
		8	
3	19		2
		10	
4	29		2
		12	
5	41		

The function is a quadratic of the form $f(n) = an^2 + bn + c$.
From the difference table for the quadratic

$2a = 2$ and therefore $a = 1$ (using column Δ_2)

Also $3a + b = 6$ and therefore $b = 3$ (using column Δ_1)

Finally $a + b + c = 5$ and therefore $c = 1$ (using column $f(n)$).

$\therefore \quad f(n) = n^2 + 3n + 1$

b

n	$f(n)$	Δ_1	Δ_2	Δ_3
1	6			
		20		
2	26		18	
		38		6
3	64		24	
		62		6
4	126		30	
		92		6
5	218		36	
		128		
6	346			

Therefore $f(n) = an^3 + bn^2 + cn + d$.

$\therefore \quad 6a = 6 \quad \therefore\ a = 1$ (from column Δ_3)

$12a + 2b = 18 \quad \therefore\ b = 3$ (from column Δ_2)

$7a + 3b + c = 20 \quad \therefore\ c = 4$ (from column Δ_1)

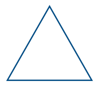
$a + b + c + d = 6 \quad \therefore\ d = -2$ (from column $f(n)$)

$\therefore \quad f(n) = n^3 + 3n^2 + 4n - 2$

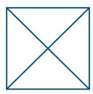
Example 27

How many diagonals are there in a convex polygon with n sides?

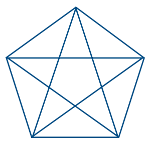
Solution



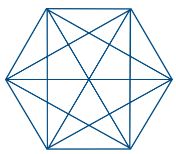
For a triangle, i.e. $n = 3$, there are no diagonals.



For a quadrilateral, i.e. $n = 4$, there are two diagonals.



For a pentagon, i.e. $n = 5$, there are five diagonals.



For a hexagon, i.e. $n = 6$, there are nine diagonals.

This gives a difference table:

n	$f(n)$	Δ_1	Δ_2
3	0		
		2	
4	2		1
		3	
5	5		1
		4	
6	9		

$\therefore$ The rule is given by a quadratic of the form $f(n) = an^2 + bn + c$,

where n is the number of sides.

We refer to the general difference table for a quadratic:

$$2a = 1 \quad \text{and} \quad \therefore a = \frac{1}{2}$$

$$7a + b = 2 \quad \text{and} \quad \therefore b = -\frac{3}{2}$$

and $\quad 9a + 3b + c = 0$

which implies $c = 0$

$$\therefore \ f(n) = \frac{1}{2}n^2 - \frac{3}{2}n = \frac{n}{2}(n - 3)$$

Example 28

Find the sum of n terms of $2 + 6 + 10 + 14 + 18 \ldots$

Solution

Consider the sequence of sums

$$S_1 = 2,\ S_2 = 8,\ S_3 = 18,\ S_4 = 32,\ S_5 = 50$$

where S_i is the sum of the first i terms. Consider the difference table

i	S_i	1	2
1	2		
		6	
2	8		4
		10	
3	18		4
		14	
4	32		4
		18	
5	50		

The rule for S_n is given by a quadratic of the form $S_n = an^2 + bn + c$.
The general difference table for a quadratic now gives

$$2a = 4 \quad \text{i.e. } a = 2$$
$$3a + b = 6 \quad \text{i.e. } b = 0$$
$$a + b + c = 2 \quad \text{i.e. } c = 0$$

and $S_n = 2n^2$

Exercise 7J

Example 26 **1** Find an expression to determine the n^{th} term of each of the following sequences:

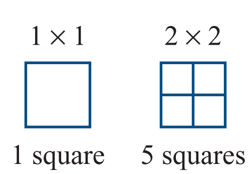
a 4, 7, 12, 19, 28, . . .
b 3, 3, 5, 9, 15, . . .
c 1, 4, 10, 20, 35, 56, . . .
d 1, 5, 14, 30, 55, 91, . . .
e −3, 11, 49, 123, 245, 427, 681, . . .

Example 28 **2** Find the sum to n terms of:

a $1 + 3 + 5 + 7 + \ldots$
b $2 + 4 + 6 + 8 + \ldots$
c $1^2 + 2^2 + 3^2 + 4^2 + \ldots$
d $1^2 + 3^2 + 5^2 + 7^2 + \ldots$
e $1 \times 3 + 2 \times 4 + 3 \times 5 + 4 \times 6 + \ldots$
f $2 \times 3 + 4 \times 5 + 6 \times 7 + 8 \times 9 + \ldots$

3 Given two points on a circle, one chord can be drawn. How many different chords can be drawn when there are n points?

4 A large square can be divided into smaller squares as shown. How many squares are there in an $n \times n$ square?



5 How many rectangles are there in an $n \times n$ square?

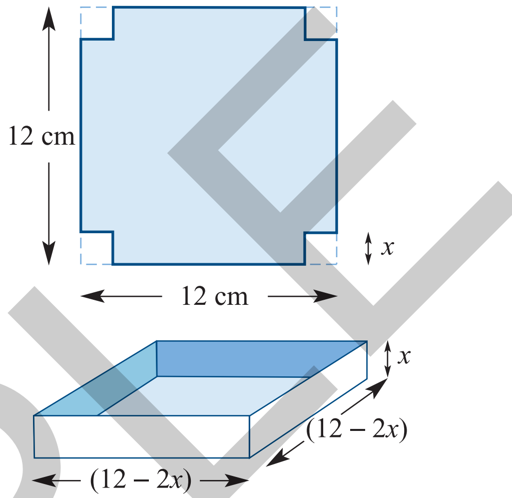
7.11 Applications of polynomial functions

Example 29

A square sheet of tin measures 12 cm × 12 cm. Four equal squares of edge x cm are cut out of the corners and the sides are turned up to form an open rectangular box. Find:

a the values of x for which the volume is 100 cm^3

b the maximum volume.



Solution

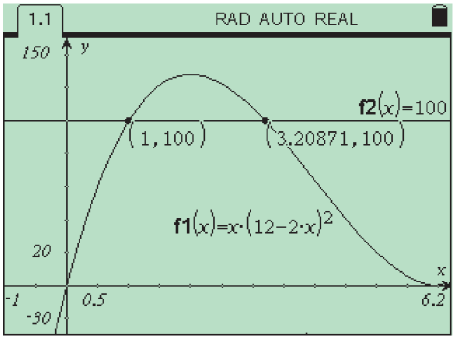
The figure shows how it is possible to form many open rectangular boxes with dimensions $12 - 2x$, $12 - 2x$ and x.

Volume of the box is $V = x(12 - 2x)^2$, $0 < x < 6$, a cubic model.

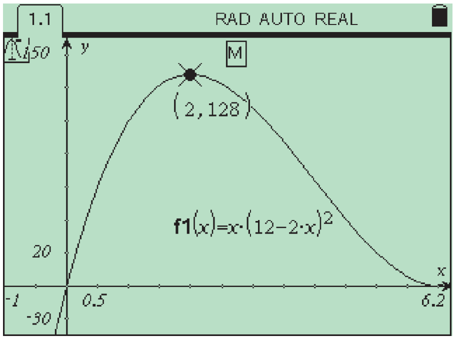
Using the T1-Nspire

This graph can be plotted on the T1-Nspire.

a The values of x for which $V = 100$ can be found by plotting the graph of $V = 100$ on the same screen and finding the **Intersection Point(s)** (menu 6 3).

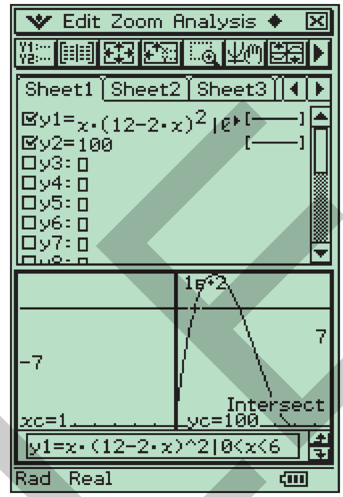


b Use **Graph Trace** (menu 5 1) to find the maximum volume.

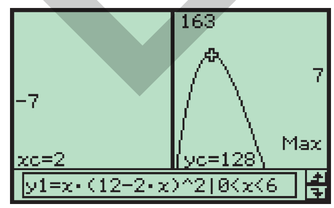


Using the Casio ClassPad

a The values of x for which $V = 100$ may be found by plotting the graph of $V = 100$ on the same screen. You may need to vary your window settings using Zoom Auto or [icon]. Then select the graph window and select **Analysis—G-Solve—Intersect**. To find the second value, use Zoom box so that only that intersection is shown and repeat.



b The maximum volume of the box may be found using **Analysis—G-Solve—Max** (you must first remove the tick for $y2$ and re-draw the graph).

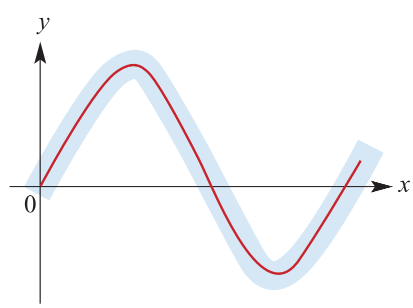


Example 30

It is found that 250 metres of the path of a stream can be modelled by a cubic function. The cubic passes throught the points (0, 0), (100, 22), (150, −10), (200, −20).

a Find the equation of the cubic function.

b Find the maximum deviation of the graph from the x-axis for $x \in [0, 250]$.



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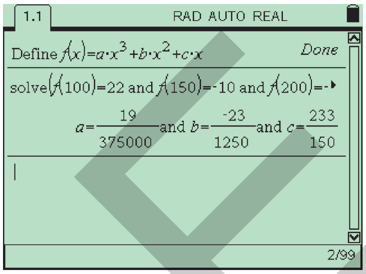
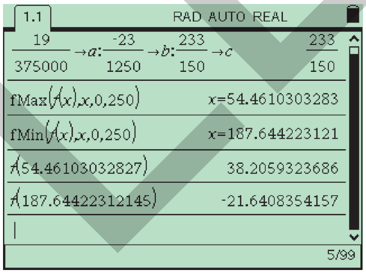
Solution

Using the T1-Nspire

a **Define** $f(x) = ax^3 + bx^2 + cx$, then write the information using function notation: $f(100) = 22$, $f(150) = -10$, $f(200) = -20$. Use **solve(**$f(100) = 22$ and $f(150) = -10$ and $f(200) = -20$, $\{a, b, c\}$).



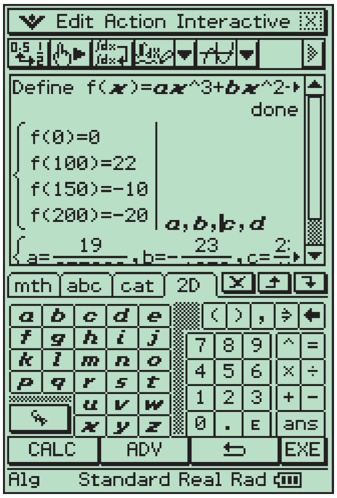
b **Store** these values as a, b and c respectively. Use **fmax( )** from the **Calculus** menu (menu 5 7) to find where f obtains its maximum value. Use **fmin( )** from the **Calculus** menu (menu 5 6) to find where f obtains its minimum value. The maximum deviation is 38.20 metres.

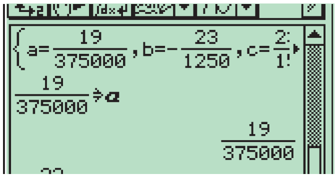


Using the Casio ClassPad

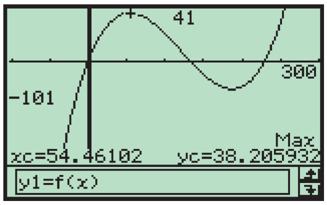
a Define $f(x) = ax^3 + bx^2 + cx + d$, then enter the four equations shown into simultaneous equations with variables set as a, b, c, d. The values are

$$a = \frac{19}{375000},\ b = -\frac{23}{1250},\ c = \frac{233}{150} \text{ and } d = 0$$





b The graph may be drawn by using entering the variable values for a, b, c, d as demonstrated (Edit copy and paste is useful here). In Graph&Tab enter and graph $y1 = f(x)$ and the use **Analysis—G-Solve—Max and Min** to solve. The answer is 38.21 metres.



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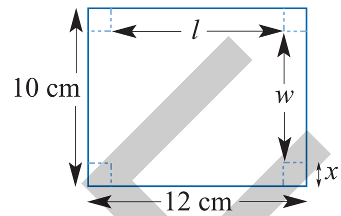
Exercise 7K

1 A rectangular sheet of metal measuring 10 cm × 12 cm is to be used to construct an open rectangular tray. The tray will be constructed by cutting out four equal squares from each corner of the sheet as shown in the diagram.



a If the edge of each cut-out square is x cm, express l and w in terms of x.

b Write down a rule for the volume, V cm^3, of the open tray in terms of x.

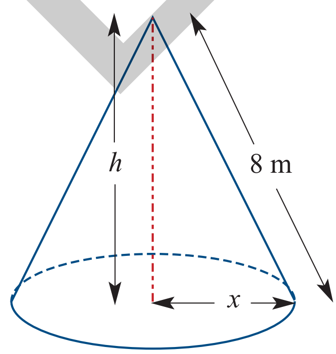
c Use a CAS calculator to help draw the graph of V against x for suitable values of x.

d Find the value of V when $x = 1$.

e Find the values of x for which $V = 50$.

f Find the maximum volume of the box and the value of x for which this occurs.

2 The diagram shows a conical heap of gravel. The slant height of the heap is 8 m, the radius of the base x m, and the height h m.



a Express x in terms of h.

b Construct a function which expresses V, the volume of the heap in m^3, in terms of h.

c Use a CAS calculator to help draw the graph of V against h.

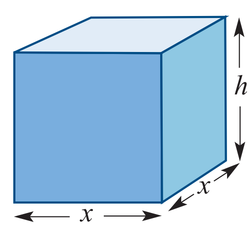
d State the domain for the function.

e Find the value of V when $h = 4$.

f Find the values of h for which $V = 150$.

g Find the maximum volume of the cone and the corresponding value of h.

3 The figure shows a rectangular prism with a square cross-section.



a If the sum of the dimensions, length plus width plus height, is 160 cm, express the height, h, in terms of x.

b Write down an expression for the volume, V cm^3, for the prism in terms of x. State the domain.

c Use a CAS calculator to help draw the graph of V against x.

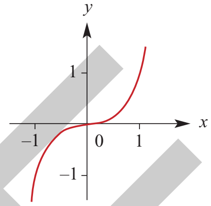
d Find the value(s) of x for which $V = 50\,000$.

e Find the maximum volume of the box.

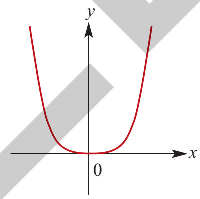
MC TEST D&D TEST

Chapter summary

- The basic shape of the curve defined by $y = x^3$ is shown in the graph.



- The graph of $y - k = a(x - h)^3$ has the same shape as $y = ax^3$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction. (h and k are positive constants.)
- The basic shape of the curve defined by $y = x^4$ is shown in the graph.



- The graph of $y - k = a(x - h)^4$ has the same shape as $y = ax^4$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction. (h and k are positive constants.)
  The turning point is at (h, k).
  When sketching quartic graphs which are of the form $y = a(x - h)^4 + k$, first identify the turning point. To add further detail to the graph the x- and y-axis intercepts are found.
- **Factorisation of polynomials**
  - **a** Remainder theorem
    When $P(x)$ is divided by $(x - a)$ then the remainder is equal to $P(a)$.
    e.g. If $P(x) = x^3 + 3x^2 + 2x + 1$ is divided by $x - 2$ then the remainder is
    $$\begin{aligned} P(2) &= (2)^3 + 3(2)^2 + 2(2) + 1 \\ &= 25 \end{aligned}$$
  - **b** Factor theorem
    If for a polynomial $P(x)$, $P(a) = 0$ then $x - a$ is a factor and, conversely, if $x - a$ is a factor of $P(x)$ then $P(a) = 0$.
    e.g. If $P(x) = x^3 - 4x^2 + x + 6$
    $$P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$
    then $(x + 1)$ is a factor or $P(x)$.
  - **c** Sums and differences of cubes
    $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
    e.g. $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$
    $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$
    e.g. $$\begin{aligned} 8x^3 + 64 &= (2x)^3 + 4^3 \\ &= (2x + 4)(4x^2 - 8x + 16) \end{aligned}$$

- **Solving cubic equations**

  The following are the steps in the process of solving cubic equations:

  i Determine factors by using the factor theorem and dividing.

  ii Use the null factor law to determine solutions.

  e.g. Solving $x^3 - 4x^2 - 11x + 30 = 0$

  $P(2) = 8 - 16 - 22 + 30 = 0 \quad \therefore x - 2$ is a factor

  Dividing $x - 2$ into $x^3 - 4x^2 - 11x + 30$ gives

  $$x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15)$$
  $$= (x - 2)(x - 5)(x + 3)$$
  $$\therefore (x - 2)(x - 5)(x + 3) = 0$$
  $$\therefore x = 2, 5 \text{ and } -3$$

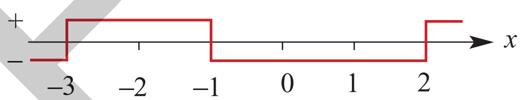
- Sign diagrams assist in sketching graphs of cubic functions.

  e.g. $y = x^3 + 2x^2 - 5x - 6$

  $= (x + 1)(x - 2)(x + 3)$

  When $x < -3$, y is negative.
  When $-3 < x < -1$, y is positive.
  When $-1 < x < 2$, y is negative.
  When $x > 2$, y is positive.

  Sign diagram



- Sketching the graph of cubic functions $y = ax^3 + bx^2 + cx + d$

  The steps in the process are:

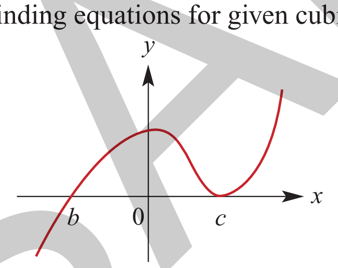
  i Use the factor theorem and division to determine the x-axis intercepts.

  ii d gives the y-axis intercept.

  iii Draw a sign diagram.

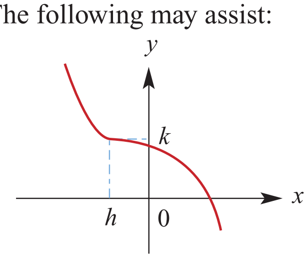
- Finding equations for given cubic graphs. The following may assist:

i



Form: $y = a(x - b)(x - c)^2$
Assume b and c are known, substitute another known point to calculate a.

ii



Form: $y - k = a(x - h)^3$
Substitute known values to determine a.

Alternatively, use the general form $y = ax^3 + bx^2 + cx + d$ and the known points to determine a, b, c and d.

- Finite difference tables can be used to find the polynomial rule for a polynomial generated sequence.

Multiple-choice questions

1 If $P(x) = x^3 + 3x^2 + x - 3$, then $P(-2) =$

A 1 **B** -1 **C** -25 **D** 3 **E** -5

2 For $a > b > c$ and $P(x) = (x - a)^2(x - b)(x - c)$, $P(x) < 0$ for $x \in$

A $(-\infty, a)$ **B** $(-\infty, b)$ **C** $(-\infty, c)$ **D** (c, b) **E** (b, a)

3 The image of the graph of $y = x^3$ under a dilation of factor 2 from the y-axis followed by a reflection in the y-axis and then a translation of 4 units in the negative direction of the y-axis is

A $y = -\frac{x^3}{8} - 4$ **B** $y = -\frac{x^3}{2} - 4$ **C** $y = -8x^3 - 4$

D $y = -\frac{x^3}{2} + 4$ **E** $y = \frac{x^3}{8} + 4$

4 The equation $x^3 + 5x - 10 = 0$ has only one solution. This solution lies between:

A -2 and -1 **B** -1 and 0 **C** 0 and 1 **D** 1 and 2 **E** 2 and 8

5 The polynomial $P(x) = x^4 + ax^2 - 4$ has zeros at $x = \sqrt{2}$ and $x = -\sqrt{2}$. The value of a is

A 0 **B** 2 **C** -2 **D** -3 **E** 3

6 The polynomial $P(x) = x^3 + ax^2 + bx - 9$ has zeros at $x = 1$ and $x = -3$. The values of a and b are

A $a = 1, b = -3$ **B** $a = -1, b = 3$ **C** $a = 5, b = 3$

D $a = -5, b = -3$ **E** $a = 0, b = 0$

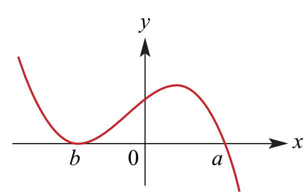
7 If $ax^3 + 2x^2 + 5$ is exactly divisible by $x + 1$, the value of a is

A 1 **B** 7 **C** -1 **D** 3 **E** -7

8 When the polynomial $P(x) = x^3 + 2x^2 - 5x + d$ is divided by $x - 2$, the remainder is 10. The value of d is

A 10 **B** 4 **C** -10 **D** -4 **E** 3

9 The diagram shows part of the graph of a polynomial function.

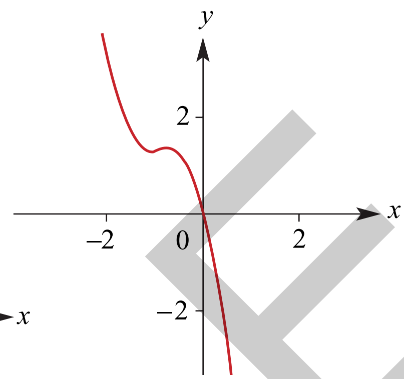


A possible equation for the rule of the function is

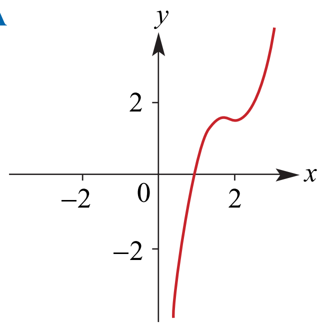
A $y = (x - b)^2(x - a)$ **B** $y = (x - a)^2(x - b)$ **C** $y = -(x + b)^2(x - a)$

D $y = (x - b)^2(a - x)$ **E** $y = (x + b)^2(a - x)$

10 The function with rule $y = f(x)$ is shown:
Which one of the following could be the graph of the function with rule $y = 1 - f(x)$?



A



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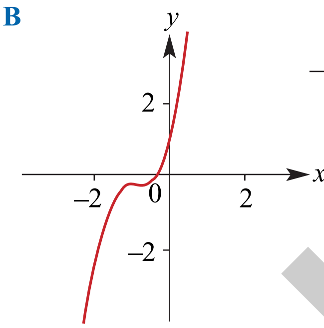

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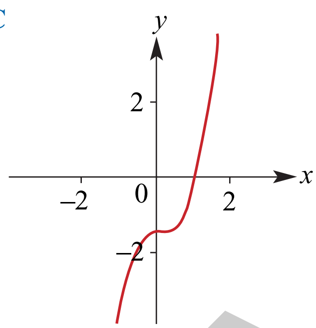

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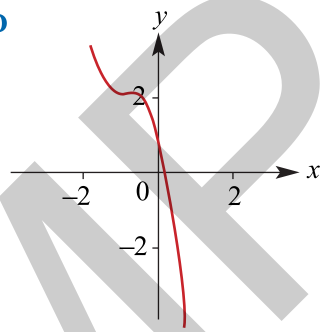
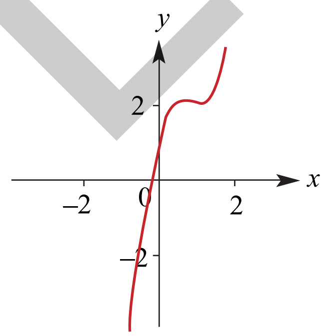


E



Short-answer questions (technology-free)

1 Sketch the graphs of each of the following:

a $y = (x - 1)^3 - 2$ **b** $y = (2x - 1)^3 + 1$ **c** $y = 3(x - 1)^3 - 1$
d $y = -3x^3$ **e** $y = -3x^3 + 1$ **f** $y = -3(x - 2)^3 + 1$
g $y = 4(x + 2)^3 - 3$ **h** $y = 1 - 3(x + 2)^3$

2 Sketch the graphs of each of the following:

a $y = (x - 1)^4$ **b** $y = (2x - 1)^4 + 1$ **c** $y = (x - 1)^4 - 1$
d $y = -2x^4$ **e** $y = -3x^4 + 1$ **f** $y = -(x - 2)^4 + 1$
g $y = 2(x + 1)^4 - 3$ **h** $y = 1 - 2(x + 2)^4$

3 **a** Show by use of the factor theorem that $2x - 3$ and $x + 2$ are factors of $6x^3 + 5x^2 - 17x - 6$. Find the other factor.

b Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$.

c Solve the equation $x^3 + x^2 - 11x - 3 = 8$.

d **i** Show that $3x - 1$ is a factor of $3x^3 + 2x^2 - 19x + 6$.

ii Find the factors of $3x^3 + 2x^2 - 19x + 6$.

4 Let $f(x) = x^3 - kx^2 + 2kx - k - 1$.

a Show that $f(x)$ is divisible by $x - 1$. **b** Factorise $f(x)$.

5 Find the values of a and b for which $x^3 + ax^2 - 10x + b$ is divisible by $x^2 + x - 12$.

6 Draw a sign diagram for each of the following and hence sketch the graph:

a $y = (x + 2)(3 - x)(x + 4)$ **b** $y = (x - 2)(x + 3)(x - 4)$

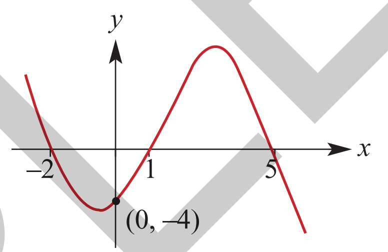
c $y = 6x^3 + 13x^2 - 4$ **d** $y = x^3 + x^2 - 24x + 36$

7 Without actually dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 + 4x^2 - 5x + 1, x + 6$ **b** $2x^3 - 3x^2 + 2x + 4, x - 2$

c $3x^3 + 2x + 4, 3x - 1$

8 Find the rule of the cubic for which the graph is shown.



9 Find a cubic function whose graph touches the x-axis at $x = -4$, passes through the origin and has a value of 10 when $x = 5$.

10 The function $f(x) = 2x^3 + ax^2 - bx + 3$. When $f(x)$ is divided by $x - 2$ the remainder is 15 and $f(1) = 0$.

a Calculate the values of a and b. **b** Find the other two factors of $f(x)$.

11 For the function f with rule $f(x) = x^3$, find the equation for the graph of the image under each of the following transformations:

a a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

b a dilation of factor 2 from the x-axis

c a reflection in the x-axis

d a reflection in the y-axis

e a dilation of factor 3 from the y-axis

12 For the function f with rule $f(x) = x^4$, find the equation for the graph of the image under each of the following transformations:

a a reflection in the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

b a dilation of factor 2 from the x-axis followed by a reflection in the y-axis

c a reflection in the x-axis followed by a translation of 2 units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis

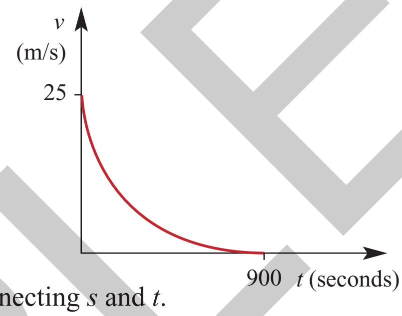
13 For each of the following find a sequence of transformations that take the graph of $y = x^3$ to the graph of:

a $y = 2(x - 1)^3 + 3$ **b** $y = -(x + 1)^3 + 2$ **c** $y = (2x + 1)^3 - 2$

Extended-response questions

1 There is a proposal to provide a quicker, more efficient and more environmentally 'friendly' system of inner-city public transport by using electric taxis. The proposal necessitates the installation of power sources at various locations as the taxis can only be driven for a limited time before requiring recharging.

The graph shows the speed, v m/s, which the taxi will maintain if it is driven at constant speed in such a way that it uses all its energy up in t seconds. The curve is a section of a parabola which touches the t-axis at $t = 900$. When $t = 0$, $v = 25$.



a Construct a rule for v in terms of t.

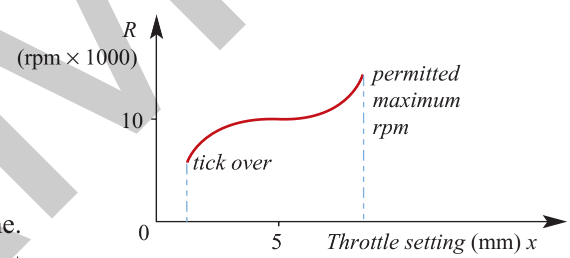
b If s metres is the distance that a taxi can travel before running out of electrical energy, write down a rule connecting s and t.

c Use a CAS calculator to help draw the graph of s against t.

d Originally the power sources were to be located at 2 km intervals. However there is a further proposal to place them at 3.5 km intervals. Is this new distance feasible?

e With the power sources at 2 km intervals, use your graph to determine approximately both the maximum and minimum speeds recommended for drivers. Explain your answer.

2 The figure shows part of a cubic graph that represents the relationship between the engine speed, R rpm, and the throttle setting, x mm from the closed position, for a new engine. It can be seen from the graph that the engine has a 'flat spot' where an increase in x has very little effect on R.



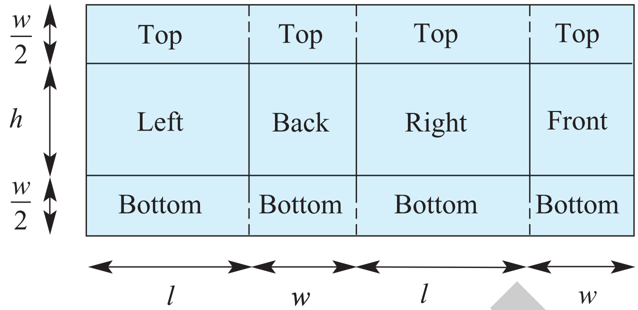
a Develop a cubic expression for R in terms of x of the form $R - k = a(x - h)^3$.

b Find a, if when the graph is extended it passes through the origin.

c In a proposed modification to the design, the 'flat spot' will occur when $x = 7$ mm. The speed of the engine in this case will be 12 000 rpm when $x = 7$ mm. Assuming that a cubic model still applies and that $R = 0$ when $x = 0$, write down an expression for R as a function of x.

Review

3 A net for making a cardboard box with overlapping flaps is shown in the figure. The dotted lines represent cuts and the solid lines represent lines along which the cardboard is folded.



a If $l = 35$ cm, $w = 20$ cm and $h = 23$ cm, calculate the area of the net.

b If the area of the net is to remain constant at the value calculated in **a** and $l = h$, write down an expression for V, the volume of the box in cm^3, as a function of l. (The maximum volume of the box will occur when $l = h$.)

CAS CALCULATOR

c Use a CAS calculator to help draw the graph of V against l.

d Find the value of l when the volume of the box is:

i 14 000 cm^3 **ii** 1 litre = 10 000 cm^3

e Find the maximum volume of the box and the value of l for which this occurs.

4 It is found that the shape of a branch of a eucalyptus tree can be modelled by a cubic function. The coordinates of several points on the branch are (0, 15.8), (10, 14.5), (15, 15.6), (20, 15).

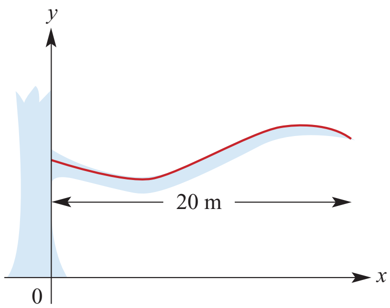
CAS CALCULATOR

a The rule for the function is of the form $y = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

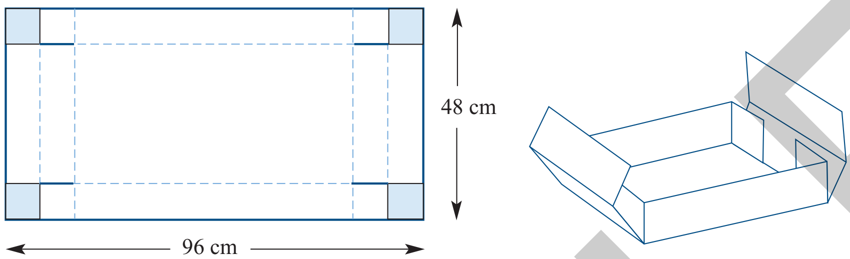
b Find the coordinates of the point on the branch that is:

i closest to the ground

ii furthest from the ground



5 A reinforced box is made by cutting congruent squares of side length x cm from the four corners of a rectangular piece of cardboard that measures 48 cm by 96 cm. The flaps are folded up.



a Find an expression for V, the volume of the box formed.

b Plot a graph of V against x on your CAS calculator:

  i What is the domain of the function V?

  ii Using your CAS calculator, find the maximum volume of the box and the value of x for which this occurs (approximate values required).

c Find the volume of the box when $x = 10$.

d It is decided that $0 \leq x \leq 5$. Find the maximum volume possible.

e If $5 \leq x \leq 15$, what is the minimum volume of the box?