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CHAPTER

20

Applications of Differentiation of Polynomials

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Objectives

- To be able to find the equation of the **tangent** and the **normal** at a given point of a polynomial curve.
- To use the derivative of a polynomial in **rates of change** problems.
- To be able to find the **stationary points** on the curves of certain polynomial functions and to state the nature of such points.
- To use differential calculus for **sketching the graphs** of polynomial functions.
- To be able to apply differential calculus to the solution of **maximum and minimum problems**.

20.1 Tangents and normals

The derivative of a function is a new function which gives the measure of the gradient at each point of the curve. If the gradient is known, it is possible to find the equation of the tangent for a given point on the curve.

Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then if f is differentiable for $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.

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Example 1

Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$ so $\left(1, \frac{3}{2}\right)$ is a point on the tangent.

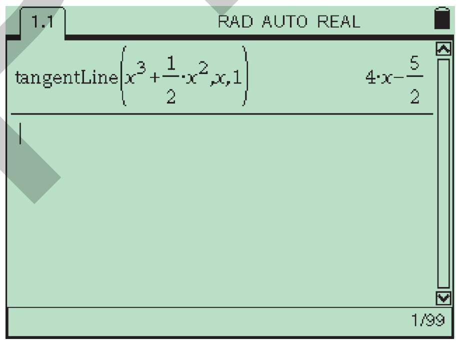
Further $\frac{dy}{dx} = 3x^2 + x$.

Thus the gradient of the tangent to the curve at $x = 1$ is 4 and the equation of the tangent is $y - \frac{3}{2} = 4(x - 1)$

which becomes $y = 4x - \frac{5}{2}$.

Using the TI-Nspire

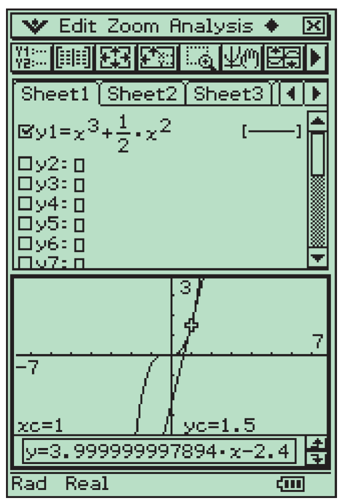
Use **Tangent Line** (menu 4 8) as shown.

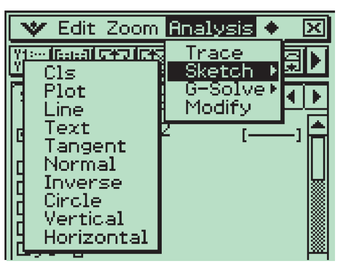


Using the Casio ClassPad

The tangent to the graph $y = x^3 + \frac{1}{2}x^2$ at $x = 1$ is found by first graphing the curve in Graph&Tab. With the graph window selected (bold box) select **Analysis—Sketch—Tangent**.

Now tap 1 on the calculator keyboard, **OK**, then **EXE**. To sketch others, use the scroll keys to move the + cursor to the point of interest and press **EXE** again.





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The **normal** to a curve at a point on the curve is the line which passes through the point and is perpendicular to the tangent at that point.

Recall from Chapter 2 that lines with gradients m_1 and m_2 are perpendicular if, and only if, $m_1 m_2 = -1$.

Thus if a tangent has a gradient m, the normal has gradient $-\frac{1}{m}$.

Example 2

Find the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ is on the normal.

Further $\frac{dy}{dx} = 3x^2 - 4x$.

Gradient of tangent $= -1$

Thus the gradient of the normal at $x = 1$ is $\frac{-1}{-1} = +1$

Hence the equation of the normal is $y + 1 = 1(x - 1)$

i.e. the equation of the normal is $y = x - 2$.

Exercise 20A

Examples 1, 2

1 Find the equation of the tangent and the normal at the given point for:

a $f(x) = x^2$, $(2, 4)$ **b** $f(x) = (2x - 1)^2$, $(2, 9)$

c $f(x) = 3x - x^2$, $(2, 2)$ **d** $f(x) = 9x - x^3$, $(1, 8)$

2 Find the equation of the tangent to the curve with equation $y = 3x^3 - 4x^2 + 2x - 10$ at the point of intersection with the y-axis.

3 Find the equation of the tangent to $y = x^2$ at the point $(1, 1)$ and the equation of the tangent to $y = \frac{1}{6}x^3$ at the point $\left(2, \frac{4}{3}\right)$.

Show that these tangents are parallel and find the perpendicular distance between them.

4 Find the equations of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$.

5 The curve with the equation $y = (x - 2)(x - 3)(x - 4)$ cuts the x-axis at the points $P = (2, 0)$, $Q = (3, 0)$, $R = (4, 0)$.

a Prove that the tangents at P and R are parallel.

b At what point does the normal to the curve at Q cut the y-axis?

6 For the curve with equation $y = x^2 + 3$ show that $y = 2ax - a^2 + 3$ is the equation of the tangent at the point $(a, a^2 + 3)$.

Hence find the coordinates of the two points on the curve, the tangents of which pass through the point $(2, 6)$.

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7 **a** Find the equation of the tangent at the point (2, 4) to the curve $y = x^3 - 2x$.
b Find the coordinates of the point where the tangent meets the curve again.

8 **a** Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point (1, 4).
b At what points of the curve is the tangent parallel to the line $4x + y - 3 = 0$?

20.2 Rates of change and kinematics

We have used the derivative of a function to find the gradient of the corresponding curve. It is clear that the process of differentiation may be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$ the average rate of change for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$. The instantaneous rate of change of f with respect to x when $x = a$ is $f'(a)$. Average rate of change has been discussed in earlier chapters.

The derivative of y with respect to x, $\dfrac{dy}{dx}$, gives the instantaneous rate of change of y with respect to x.

If $\dfrac{dy}{dx} > 0$ the change is an increase in the value of y corresponding to an increase in x and if $\dfrac{dy}{dx} < 0$ the change is a decrease in the value of y corresponding to an increase in x.

Example 3

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$
b the average rate of change for the interval $[2, 2 + h]$
c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a The average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b For the interval the average rate of change
$$= \frac{f(2+h) - f(2)}{2 + h - 2}$$
$$= \frac{(2+h)^2 + 2(2+h) - 8}{h}$$
$$= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h}$$
$$= \frac{6h + h^2}{h}$$
$$= 6 + h$$

c When $x = 2$, the instantaneous rate of change $= f'(2) = 6$. This can also be seen from the result of part b.

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Example 4

A balloon develops a microscopic leak and decreases in volume. Its volume $V(\text{cm}^3)$ at time t (seconds) is $V = 600 - 10t - \frac{1}{100}t^2$, $t > 0$.

a Find the rate of change of volume after

i 10 seconds **ii** 20 seconds

b For how long could the model be valid?

Solution

a $\frac{dV}{dt} = -10 - \frac{t}{50}$

i When $t = 10$

$$\frac{dV}{dt} = -10 - \frac{1}{5}$$
$$= -10\frac{1}{5}$$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$

$$\frac{dV}{dt} = -10 - \frac{2}{5}$$
$$= -10\frac{2}{5}$$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$. Consider $V = 0$.

$$600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore \quad t = \frac{10 \pm \sqrt{100 + \frac{1}{100} \times 600 \times 4}}{-0.02}$$

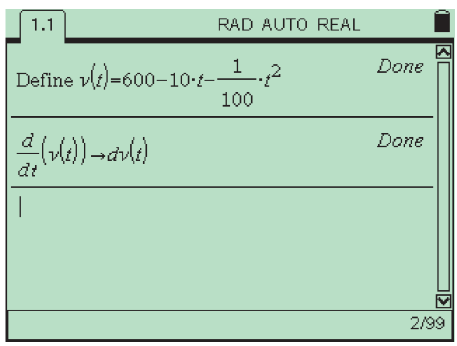
$$\therefore \quad t = -1056.78 \text{ or } t = 56.78 \text{ (to 2 decimal places)}$$

$\therefore$ the model may be suitable for $0 < t < 56.78$

Using the TI-Nspire

Define $v(t) = 600 - 10t - (1/100)t^2$.

Take the **Derivative** and store as $dv(t)$.



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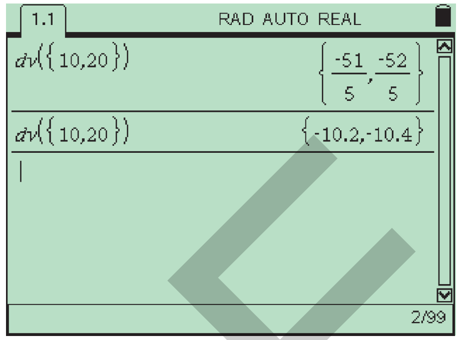
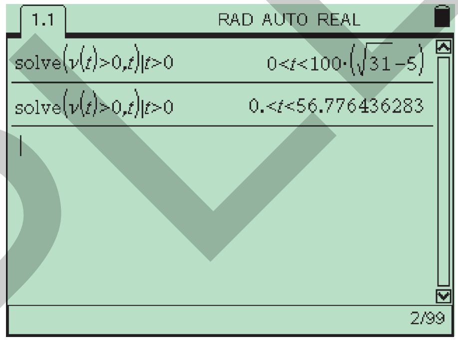
a Evaluate $dv(10)$ and $dv(20)$.
Press ctrl enter to obtain the answer as a decimal number.



b To find the domain, use
solve$(v(t) > t) \mid t > 0$.
Press ctrl enter to obtain the answer as a decimal number.



Using the Casio ClassPad

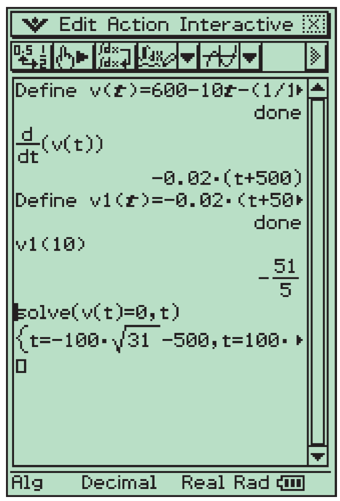
Use **Interactive—Define**
$v(t) = 600 - 10t - (1/100)t^2$.

Interactive—Calculation—diff and enter $v(t)$ for the expression, t for the variable and 1 for first derivative.

Define the derivative with a new name by using **Interactive—Define** and enter the derivative function name $v1(t)$, the variable t and the derivative function.

Evaluate $v1(10)$.

To find the domain, enter and highlight $v(t) = 0$ and use **Interactive—Equation/inequality—solve**, setting the variable to t.



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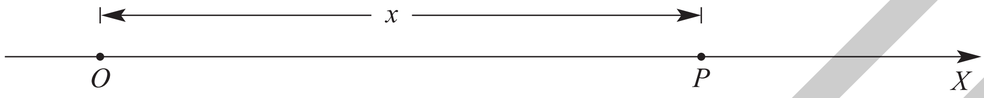
Applications of differentiation to kinematics

The **position coordinate** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.



Consider a particle which starts at O and begins to move. The position of the particle is determined by a number, x, called the position coordinate. If the unit is metres and if $x = -3$, the position is 3 m to the left of O, while if $x = 3$, its position is 3 m to the right of O.

The **displacement** is defined as the change in position of the particle relative to O. Sometimes there is a rule that enables the position coordinate, at any instant, to be calculated. In this case x is redefined as a function of t. Hence $x(t)$ is the displacement at time t. Specification of a displacement function together with the physical idealisation of a real situation constitute a **mathematical model** of the situation.

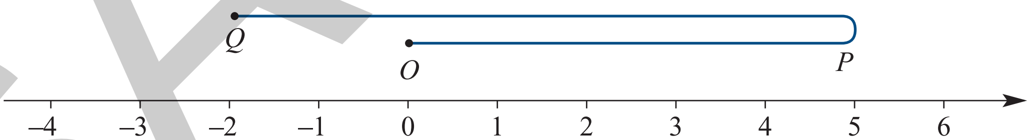
An example of a mathematical model follows.

A stone is dropped from the top of a vertical cliff 45 metres high. Assume that the stone is a particle travelling in a straight line. Let $x(t)$ metres be the downwards position of the particle from O, the top of the cliff, t seconds after the particle is dropped. If air resistance is neglected, an approximate model for the displacement is

$$x(t) = 5t^2 \text{ for } 0 \le t \le 3$$

It is important to distinguish between the scalar quantity **distance** and the vector quantity **displacement**.

Consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The final position of the particle is $x = -2$. However the **distance** it has moved is 12 units.

Example 5

A particle moves in a straight line so that its position x cm relative to O at time t seconds is given by $x = t^2 - 7t + 6, t \ge 0$.

a Find its initial position. **b** Find its position at $t = 4$.

Solution

a At $t = 0$, $x = +6$, i.e. the particle is 6 cm to the right of O.

b At $t = 4$, $x = (4)^2 - 7(4) + 6 = -6$, i.e. the particle is 6 cm to the left of O.

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Velocity

You should already be familiar with the concept of a rate of change through your studies in Chapter 18.

The **velocity** of a particle is defined as the rate of change of its position with respect to time. We can consider the **average rate of change**, i.e. the change in position over a period of time, or we can consider the **instantaneous rate of change**, which specifies the rate of change at a given instant in time.

If a particle moves from x_1 at time t_1 to x_2 at time t_2, its

$$\textbf{average velocity} = \frac{x_2 - x_1}{t_2 - t_1}$$

Velocity may be positive, negative or zero. If the velocity is positive, the particle is moving to the right, if it is negative the direction of motion is to the left. A velocity of zero means the particle is instantaneously at rest.

The instantaneous rate of change of position with respect to time is the instantaneous velocity. If the position, x, of the particle at time t is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

Common units of velocity (and speed) are:

$$\begin{aligned} &\text{1 metre per second} &&= 1\text{ m/s} = 1\text{ m s}^{-1}\\ &\text{1 centimetre per second} &&= 1\text{ cm/s} = 1\text{ cm s}^{-1}\\ &\text{1 kilometre per hour} &&= 1\text{ km/h} = 1\text{ km h}^{-1}\end{aligned}$$

The first and third units are connected in the following way:

$$\begin{aligned} 1\text{ km/h} &= 1000\text{ m/h}\\ &= \frac{1000}{60 \times 60}\text{ m/s}\\ &= \frac{5}{18}\text{ m/s}\\ \therefore \quad 1\text{ m/s} &= \frac{18}{5}\text{ km/h}\end{aligned}$$

Note the distinction between velocity and speed. **Speed** is the magnitude of the velocity.

Average speed for a time interval $[t_1, t_2]$ is equal to $\dfrac{\text{distance travelled}}{t_2 - t_1}$.

Instantaneous velocity $v = \dfrac{dx}{dt}$, where x is a function of time.

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Example 6

A particle moves in a straight line so that its position x cm relative to O at time t seconds is given by $x = t^2 - 7t + 6, t \geq 0$.

a Find its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 4 seconds?

d Determine its average speed for the first 4 seconds.

Solution

a

$$x = t^2 - 7t + 6$$

$$v = \frac{dx}{dt} = 2t - 7$$

at $t = 0$, $\quad v = -7$

The particle is moving to the left at 7 cm/s.

b

$$2t - 7 = 0$$

implies $\quad t = 3.5$

When $t = 3.5 \quad x = (3.5)^2 - 7(3.5) + 6$

$$= -6.25$$

So, at $t = 3.5$ seconds the particle is 6.25 cm to the left of O.

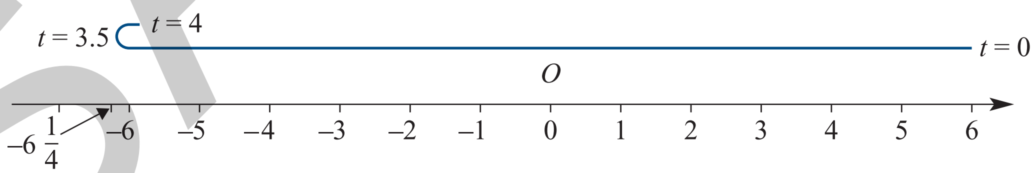
c Average velocity $= \dfrac{\text{change in position}}{\text{change in time}}$

At $t = 4$, $\quad x = -6$

$\therefore$ average velocity $= \dfrac{-6 - +6}{4} \qquad$ (as $x = 6$ when $t = 0$)

$= -3$ cm/s

d Average speed $= \dfrac{\text{distance travelled}}{\text{change in time}}$



The particle stopped at $t = 3.5$ and began to move in the opposite direction, so we must consider the distance travelled in the first 3.5 seconds (from $x = 6$ to $x = -6.25$) and then the distance travelled in the final 0.5 seconds (from $x = -6.25$ to $x = -6$).

Total distance travelled $= 12.25 + 0.25 = 12.5$

$\therefore$ average speed $= \dfrac{12.5}{4} = 3.125$ cm/s.

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Acceleration

The acceleration of a particle is defined as the rate of change of its velocity with respect to time.

Average acceleration for the time interval $[t_1, t_2]$ is defined by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

Instantaneous acceleration $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$.

For kinematics, the second derivative $\dfrac{d^2x}{dt^2}$ is denoted by $x''(t)$.

Acceleration may be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity. Note that the direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down. Also, although a particle may be instantaneously at rest its acceleration at that instant need not be zero. If acceleration has the same sign as velocity then the particle is 'speeding up'. If the sign is opposite, the particle is 'slowing down'.

The most commonly used units for acceleration are cm/s^2 and m/s^2.

Example 7

A particle moves in a straight line so that its position x cm relative to O at time t seconds is given by $x = t^3 - 6t^2 + 5$, $t \geq 0$.

a Find its initial position, velocity and acceleration, and hence describe its motion.

b Find the times when it is instantaneously at rest and determine its position and acceleration at those times.

Solution

a For $\quad x = t^3 - 6t^2 + 5 \quad v = 3t^2 - 12t \quad$ and $\quad a = 6t - 12.$

When $t = 0$, $\quad x = 5 \quad v = 0 \quad$ and $\quad a = -12$

The particle is instantaneously at rest 5 cm to the right of O, with an acceleration of -12 cm/s^2.

b
$$v = 3t^2 - 12t = 0$$
$$3t(t - 4) = 0$$
$$t = 0 \text{ or } t = 4$$

The particle is initially at rest and stops again after 4 seconds.

At $t = 0$, $x = 5$ and $a = -12$.
At $t = 4$, $x = (4)^3 - 6(4)^2 + 5 = -27$ and $a = 6(4) - 12 = 12$.

After 4 seconds the position of the particle is 27 cm to the left of O and its acceleration is 12 cm/s.

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Example 8

A car starts from rest and moves a distance s metres in t seconds, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$. What is the initial acceleration and the acceleration when $t = 2$?

Solution

$$s = \frac{1}{6}t^3 + \frac{1}{4}t^2$$

$$\text{Velocity } = \frac{ds}{dt} = \frac{1}{2}t^2 + \frac{1}{2}t$$

Velocity is given in metres per second or m/s.

Let $v = \frac{ds}{dt}$

Then $\frac{dv}{dt} = t + \frac{1}{2}$

Let $a = \frac{dv}{dt}$

When $t = 0, a = \frac{1}{2}$

When $t = 2, a = 2\frac{1}{2}$

Hence the required accelerations are $\frac{1}{2}$ m/s^2 and $2\frac{1}{2}$ m/s^2.

Example 9

A point moves along a straight line so that its distance x cm from a point O at time t seconds is given by the formula $x = t^3 - 6t^2 + 9t$.

a Find at what times and in what positions the point will have zero velocity.
b Find its acceleration at those instants.
c Find its velocity when its acceleration is zero.

Solution

$$\text{Velocity } = v = \frac{dx}{dt} = 3t^2 - 12t + 9$$

a When $v = 0$

$$3(t^2 - 4t + 3) = 0$$
$$(t - 1)(t - 3) = 0$$
$$t = 1 \quad \text{or} \quad t = 3$$

i.e. the velocity is zero when $t = 1$ and $t = 3$ and where $x = 4$ and $x = 0$.

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b Acceleration $= \dfrac{dv}{dt}$

$= 6t - 12$

$\therefore$ acceleration $= -6$ m/s^2 when $t = 1$ and acceleration $= 6$ m/s^2 when $t = 3$.

c Acceleration is zero when $6t - 12 = 0$, i.e. when $t = 2$

When $t = 2$

velocity $v = 3 \times 4 - 24 + 9$

$= -3$ m/s

SAMPLE

Skillsheet TEACHER

Exercise 20B

Example 3

1 Let $y = 35 + 12x^2$.

a Find the change in y as x changes from 1 to 2. What is the average rate of change of y with respect to x in this interval?

b Find the change in y as x changes from $2 - h$ to 2. What is the average rate of change of y with respect to x in this interval?

c Find the rate of change of y with respect to x when $x = 2$.

Example 4

2 According to a business magazine the expected assets, \$$M$, of a proposed new company will be given by $M = 200\,000 + 600t^2 - \dfrac{200}{3}t^3$, where t is the number of months after the business is set up.

a Find the rate of growth of assets at time t months.

b Find the rate of growth of assets at time $t = 3$ months.

c Will the rate of growth of assets be 0 at any time?

Examples 5, 6

3 The position of a body moving in a straight line, x cm from the origin, at time t seconds ($t \geq 0$) is given by $x = \dfrac{1}{3}t^3 - 12t + 6$.

a Find the rate of change of position with respect to time at $t = 3$.

b Find the time at which the velocity is zero.

4 Let $s = 10 + 15t - 4.9t^2$ be the height (in metres) of an object at time t (in seconds).

a Find the velocity at time t.

b Find the acceleration at time t.

5 As a result of a survey, the marketing director of a company found that the revenue, \$$R$, from selling n produced items at \$$P$ is given by the rule $R = 30P - 2P^2$.

a Find $\dfrac{dR}{dP}$ and explain what it means.

b Calculate $\dfrac{dR}{dP}$ when $P = 5$ and $P = 10$.

c For what selling prices is revenue rising?

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6 The population, P, of a new housing estate t years after 30 January 1997 is given by the rule $P = 100(5 + t - 0.25t^2)$.

a Find the rate of change of the population after:

i 1 year **ii** 2 years **iii** 3 years

7 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = \frac{5}{8}\left(10t^2 - \frac{t^3}{3}\right)$, $0 \le t \le 20$.

a Find the volume of water in the flask at time:

i $t = 0$ **ii** $t = 20$

b Find the rate of flow of water into the flask.

c Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

8 A model aeroplane flying level at 250 m above the ground suddenly dives. Its height h (m) above the ground at time (t seconds) after beginning to dive is given by

$$h(t) = 8t^2 - 80t + 250 \qquad t \in [0, 10].$$

Find the rate at which the plane is losing height at:

a $t = 1$ **b** $t = 3$ **c** $t = 5$

9 A particle moves along a straight line so that after t seconds its distance from O, a fixed point on the line, is s m, where $s = t^3 - 3t^2 + 2t$.

a When is the particle at O?

b What is its velocity and acceleration at these times?

c What is the average velocity during the first second?

10 A car starts from rest and moves a distance s (m) in t (s), where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$.

a What is the acceleration when $t = 0$?

b What is the acceleration when $t = 2$?

11 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \ge 0$) is given by $x = t^2 - 7t + 12$.

a Find its initial position. **b** What is its position at $t = 5$?

c Find its initial velocity.

d When does its velocity equal zero, and what is its position at this time?

e What is its average velocity in the first 5 seconds?

f What is its average speed in the first 5 seconds?

12 The position, x metres, at time t seconds ($t \ge 0$) of a particle moving in a straight line is given by $x = t^2 - 7t + 10$.

a When does its velocity equal zero? **b** Find its acceleration at this time.

c Find the distance travelled in the first 5 seconds.

d When does its velocity equal -2 m/s, and what is its position at this time?

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13 A particle moving in a straight line is x cm from the point O at time t seconds ($t \geq 0$), where $x = t^3 - 11t^2 + 24t - 3$.

a Find its initial position and velocity. b Find its velocity at any time.

c At what times is the particle stationary?

d What is the position of the particle when it is stationary?

e For how long is the particle's velocity negative?

f Find its acceleration at any time.

g When is the particle's acceleration zero, and what is its velocity and position at that time?

14 A particle moves in a straight line so that its position x cm relative to O at time t seconds ($t \geq 0$) is given by $x = 2t^3 - 5t^2 + 4t - 5$.

a When is its velocity zero, and what is its acceleration at that time?

b When is its acceleration zero, and what is its velocity at that time?

15 A particle moving in a straight line is x cm from the point O at time t seconds ($t \geq 0$), where $x = t^3 - 13t^2 + 46t - 48$.
When does the particle pass through O, and what is its velocity and acceleration at those times?

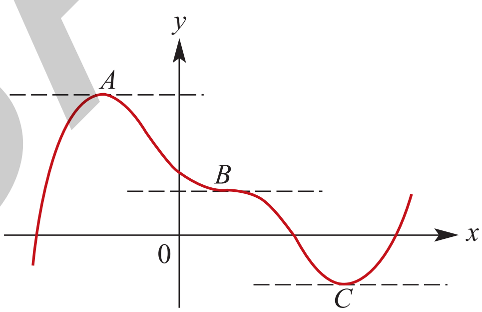
16 Two particles are moving along a straight path so that their displacements, x cm, from a fixed point P at any time t seconds are given by $x = t + 2$ and $x = t^2 - 2t - 2$.

a Find the time when the particles are at the same position.

b Find the time when the particles are moving with the same velocity.

20.3 Stationary points

In the previous chapter we have seen that the gradient at a point $(a, g(a))$ of the curve with rule $y = g(x)$ is given by $g'(a)$. A point $(a, g(a))$ on a curve $y = g(x)$ is said to be a stationary point if $g'(a) = 0$.



(Equivalently: for $y = g(x)$, $\frac{dy}{dx} = 0$ when $x = a$ implies that $(a, g(a))$ is a stationary point.)

For example, in the graph above there are stationary points at A, B and C. At such points the tangents are parallel to the x-axis (illustrated as dotted lines).

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The reason for the name **stationary points** becomes clear if we look at an application to motion of a particle.

Example 10

The displacement x metres of a particle moving in a straight line is given by $x = 9t - \frac{1}{3}t^3$ for $0 \le t \le 3\sqrt{3}$, where t seconds is the time taken.
Find the maximum displacement.

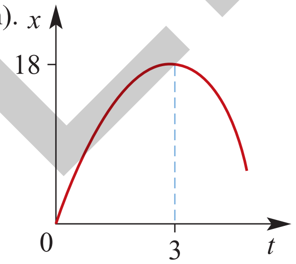
Solution

$\frac{dx}{dt} = 9 - t^2$, and maximum displacement occurs when $\frac{dx}{dt} = 0$.

So $t = 3$ or $t = -3$ (but $t = -3$ lies outside the domain).
At $t = 3$, $x = 18$.

Thus the stationary point is (3, 18) and the maximum displacement is 18 metres.

Note that the stationary point occurs when the rate of change of displacement with respect to time (i.e. velocity) is zero. The particle stopped moving forward at $t = 3$.



Example 11

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $p = 2t^3 - 5t^2 - 4t + 13$ for $t > 0$ **c** $y = 4 + 3x - x^3$

Solution

a $y = 9 + 12x - 2x^2$

$\frac{dy}{dx} = 12 - 4x$

Stationary point occurs when $\frac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$

i.e. at $x = 3$

When $x = 3$, $y = 9 + 12 \times 3 - 2 \times 3^2 = 27$

Thus the stationary point is at (3, 27).

SAMPLE

b $p = 2t^3 - 5t^2 - 4t + 13$

$\frac{dp}{dt} = 6t^2 - 10t - 4, t > 0$

$\frac{dp}{dt} = 0$ implies $2(3t^2 - 5t - 2) = 0$

$\therefore \quad (3t + 1)(t - 2) = 0$

$\therefore t = -\frac{1}{3}$ or $t = 2$

But $t > 0$, therefore the only acceptable solution is $t = 2$.

When $t = 2, p = 16 - 20 - 8 + 13 = 1$

So the corresponding stationary point is (2, 1).

c $y = 4 + 3x - x^3$

$\frac{dy}{dx} = 3 - 3x^2$

$\frac{dy}{dx} = 0$ implies $3(1 - x^2) = 0$

$\therefore \quad x = \pm 1$

$\therefore$ stationary points occur at (1, 6) and (−1, 2).

Example 12

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through (0, 5) with stationary point (2, 7). Find a, b, c.

Solution

When $x = 0, y = 5$

Thus $5 = c$

$\frac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2, \frac{dy}{dx} = 0$

Therefore $0 = 12 + 4a + b \qquad (1)$

The point (2, 7) is on the curve and therefore

$$7 = 2^3 + 2^2a + 2b + 5$$

$$\therefore \quad 2 = 8 + 4a + 2b$$

$$\therefore \quad 4a + 2b + 6 = 0 \qquad (2)$$

Subtract (2) from (1)

$$-b + 6 = 0$$

$$\therefore \quad b = 6$$

SAMPLE

Substitute in (1)

$$\begin{aligned} 0 &= 12 + 4a + 6 \\ -18 &= 4a \\ -\frac{9}{2} &= a \end{aligned}$$

$$\therefore \quad a = -\frac{9}{2}, b = 6, c = 5$$

Exercise 20C

Example 11 **1** Find the coordinates of the stationary points of each of the following functions:

a $f(x) = x^2 - 6x + 3$ **b** $y = x^3 - 4x^2 - 3x + 20, x > 0$

c $z = x^4 - 32x + 50$ **d** $q = 8t + 5t^2 - t^3$ for $t > 0$

e $y = 2x^2(x - 3)$ **f** $y = 3x^4 - 16x^3 + 24x^2 - 10$

Example 12 **2** The curve with equation $y = ax^2 + bx + c$ passes through $(0, -1)$ and has a stationary point at $(2, -9)$. Find a, b, c.

3 The curve with equation $y = ax^2 + bx + c$ has a stationary point at $(1, 2)$. When $x = 0$, the slope of the curve is $45°$. Find a, b, c.

4 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Find the values of a and b. **b** Find the coordinates of the turning point.

5 The curve of the equation $y = x^2 + ax + 3$ has a stationary point when $x = 4$. Find a.

6 The curve with equation $y = x^2 - ax + 4$ has a stationary point when $x = 3$. Find a.

7 Find the coordinates of the stationary points of each of the following:

a $y = x^2 - 5x - 6$ **b** $y = (3x - 2)(8x + 3)$

c $y = 2x^3 - 9x^2 + 27$ **d** $y = x^3 - 3x^2 - 24x + 20$

e $y = (x + 1)^2(x + 4)$ **f** $y = (x + 1)^2 + (x + 2)^2$

8 The curve with equation $y = ax^2 + bx + 12$ has a stationary point at $(1, 13)$. Find a and b.

9 The curve with equation $y = ax^3 + bx^2 + cx + d$ has a gradient of -3 at $\left(0, 7\frac{1}{2}\right)$ and a turning point at $(3, 6)$. Find a, b, c and d.

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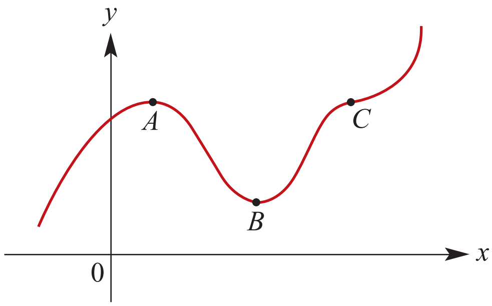
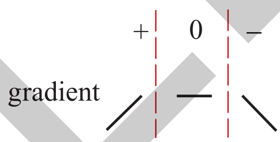
20.4 Types of stationary points

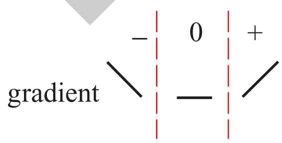
The graph below has three stationary points A, B, C.



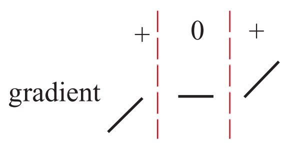
A Point A is called a **local maximum** point. Notice that immediately to the left of A the gradient is positive and immediately to the right the gradient is negative, as shown in the diagram on the right.



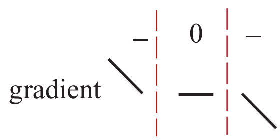
B Point B is called a **local minimum** point. Notice that immediately to the left of B the gradient is negative and immediately to the right the gradient is positive, as shown in the diagram on the right.



C The point C is called a **stationary point of inflexion**, as shown in the diagram on the right.



Clearly it is also possible to have stationary points of inflexion for which the diagram would be like this:



Stationary points of type A and B are referred to as **turning points**.

Example 13

Let the function $f: R \to R$, $f(x) = 3x^3 - 4x + 1$.

a Find the stationary points and state their nature. **b** Sketch the graph.

SAMPLE

Solution

a $y = f(x)$ has stationary points where $f'(x) = 0$.

$$f'(x) = 9x^2 - 4 = 0$$
$$\therefore\ x = \pm\frac{2}{3}$$

There are stationary points at $\left(-\frac{2}{3}, f\left(-\frac{2}{3}\right)\right)$, and $\left(\frac{2}{3}, f\left(\frac{2}{3}\right)\right)$,

that is at $\left(-\frac{2}{3}, 2\frac{7}{9}\right)$, and $\left(\frac{2}{3}, -\frac{7}{9}\right)$.

$f'(x)$ is of constant sign for each of

$$\left\{x\colon x < -\frac{2}{3}\right\}, \left\{x\colon -\frac{2}{3} < x < \frac{2}{3}\right\} \quad\text{and}\quad \left\{x\colon x > \frac{2}{3}\right\}$$

To calculate the sign of $f'(x)$ for each of these sets, simply choose a representative number in the set.

Thus $\quad f'(-1) = 9 - 4 = 5 > 0$
$\qquad\quad f'(0) = 0 - 4 = -4 < 0$
$\qquad\quad f'(1) = 9 - 4 = 5 > 0$

We can thus put together the following table:

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	−	0	+
shape of f	/	—	\	—	/

$\therefore$ there is a local maximum at $\left(-\frac{2}{3}, 2\frac{7}{9}\right)$ and a local minimum at $\left(\frac{2}{3}, -\frac{7}{9}\right)$.

b To sketch the graph of this function we need to find the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.
First, $f(0) = 1$ $\therefore$ the y-intercept is 1.
Consider $f(x) = 0$ which implies $3x^3 - 4x + 1 = 0$.
By inspection (factor theorem), $(x - 1)$ is a factor and by division

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

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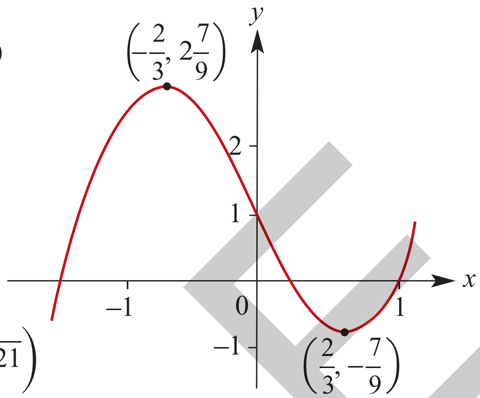
Now $(x-1)(3x^2+3x-1)=0$
implies that $x=1$ or $3x^2+3x-1=0$

$$\begin{aligned}&3x^2+3x-1\\&=3\left[\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-\frac{1}{3}\right]\\&=3\left[\left(x+\frac{1}{2}\right)^2-\frac{21}{36}\right]\\&=3\left(x+\frac{1}{2}-\frac{1}{6}\sqrt{21}\right)\left(x+\frac{1}{2}+\frac{1}{6}\sqrt{21}\right)\end{aligned}$$



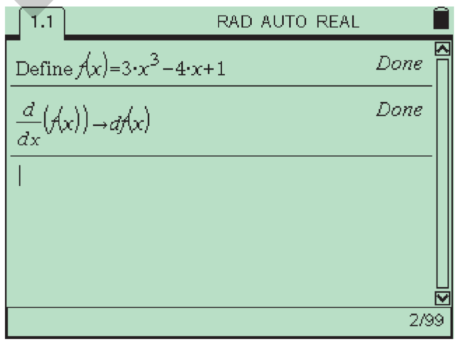
Thus the x-intercepts are at

$$x=-\frac{1}{2}+\frac{1}{6}\sqrt{21},\ x=-\frac{1}{2}-\frac{1}{6}\sqrt{21},\ x=1$$

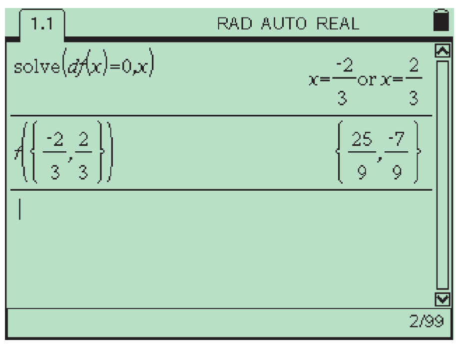
For $x>\frac{2}{3}$, $f(x)$ becomes larger.
For $x<-\frac{2}{3}$, $f(x)$ becomes smaller.

Using the TI-Nspire

Define the function, find its derivative and store it in $df(x)$.



Solve the equation $df(x)=0$ and determine the coordinates of the stationary points.



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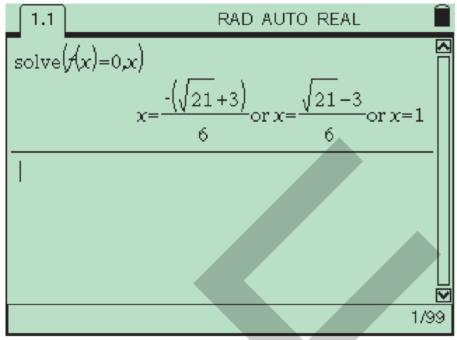
Then find the x-axis intercepts by solving the equation $f(x) = 0$.

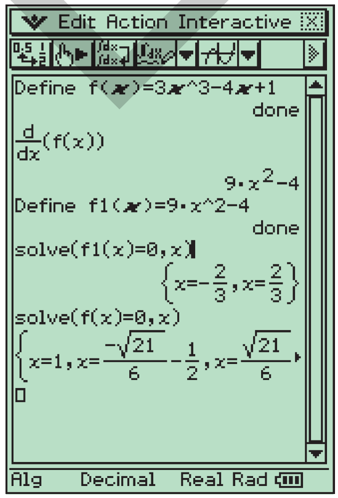


Using the Casio ClassPad

Define the function, find its derivative and store it as $\mathrm{f1}(x)$.

Solve the equation $\mathrm{f1}(x) = 0$ and determine the coordinates of the stationary points.

Then find the x-axis intercepts by solving the equation $\mathrm{f}(x) = 0$.



Exercise 20D

Example 13

1 For each of the following find all stationary points and state their nature. Sketch the graph of each function.

a $y = 9x^2 - x^3$ **b** $y = x^3 - 3x^2 - 9x$ **c** $y = x^4 - 4x^3$

2 Find the stationary points (and state their type) for each of the following functions:

a $y = x^2(x - 4)$ **b** $y = x^2(3 - x)$ **c** $y = x^4$

d $y = x^5(x - 4)$ **e** $y = x^3 - 5x^2 + 3x + 2$ **f** $y = x(x - 8)(x - 3)$

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Example 13

3 Sketch the graphs of each of the following functions:

a $y = 2 + 3x - x^3$ **b** $y = 2x^2(x - 3)$ **c** $y = x^3 - 3x^2 - 9x + 11$

4 The graphs corresponding to each of the following equations have a stationary point at $(-2, 10)$. For each graph find the nature of the stationary point at $(-2, 10)$.

a $y = 2x^3 + 3x^2 - 12x - 10$ **b** $y = 3x^4 + 16x^3 + 24x^2 - 6$

5 For the function $y = x^3 - 6x^2 + 9x + 10$:

a find the intervals where y is increasing, i.e. $\left\{x\colon \frac{dy}{dx} > 0\right\}$

b find the stationary points on the curve corresponding to $y = x^3 - 6x^2 + 9x + 10$

c sketch the curve carefully between $x = 0$ and $x = 4$.

6 Find the maximum value of the product of two numbers x and $10 - x$ which add up to 10.

7 For the function $f\colon R \to R,\ f(x) = 1 + 12x - x^3$, determine the values of x for which the function is increasing.

8 Let $f\colon R \to R$, where $f(x) = 3 + 6x - 2x^3$.

a Determine the values of x for which the graph of f has positive gradient.

b Find the values of x for which the graph of f has increasing gradient.

9 Let $f(x) = x(x + 3)(x - 5)$.

a Find the values of x for which $f'(x) = 0$.

b Sketch the graph of $f(x)$ for $-5 \le x \le 6$, giving the coordinates of the intersections with the axes and the coordinates of the turning points.

10 Sketch the graph of $y = x^3 - 6x^2 + 9x - 4$. State the coordinates of the axes intercepts and of the turning points.

11 Find the coordinates of the points on the curve $y = x^3 - 3x^2 - 45x + 2$ where the tangent is parallel to the x-axis.

12 Let $f(x) = x^3 - 3x^2$.

a Find:

i $\{x\colon f'(x) < 0\}$ **ii** $\{x\colon f'(x) > 0\}$ **iii** $\{x\colon f'(x) = 0\}$

b Sketch the graph of $y = f(x)$.

13 Sketch the graph of $y = x^3 - 9x^2 + 27x - 19$ and state the coordinates of stationary points.

14 Sketch the graph of $y = x^4 - 8x^2 + 7$.
All axis intercepts and all turning points should be identified and their coordinates given.

SAMPLE

20.5 Families of functions and transformations

It is assumed that a CAS calculator will be used throughout this section.

Example 14

Let the function $f(x) = (x - a)^2(x - b)$, where a and b are positive constants with $b > a$.

a Find the derivative of $f(x)$ with respect to x.
b Find the coordinates of the stationary points of the graph of $y = f(x)$.
c Show that the stationary point at $(a, 0)$ is always a local maximum.
d Find the values of a and b if the stationary points occur where $x = 3$ and $x = 4$.

Solution

a Use a CAS calculator to find that $f'(x) = (x - a)(3x - a - 2b)$.
b The coordinates of the stationary points are $(a, 0)$ and $\left(\frac{a+2b}{3}, \frac{4(a-b)^3}{27}\right)$.
c If $x < a$ then $f'(x) > 0$ and if $x > a$ and $x < \frac{a+2b}{3}$ then $f'(x) < 0$.
Therefore the stationary point is local maximum.
d $a = 3$, as $a < b$ and $\frac{a+2b}{3} = 4$ implies $b = \frac{9}{2}$.

Example 15

The graph of the function $y = x^3 - 3x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis. (a and b are positive constants.)

a Find the coordinates of the turning points of the graph of $y = x^3 - 3x^2$.
b Find the coordinates of the turning points of its image.

Solution

a The turning points have coordinates $(0, 0)$ and $(2, -4)$.
b The turning points of the image are (a, b) and $(2 + a, -4 + b)$.

Example 16

A cubic function has rule $y = ax^3 + bx^2 + cx + d$. It passes through the points $(1, 6)$ and $(10, 8)$ and has turning points where $x = -1$ and $x = 2$.

a Using matrix methods, find the values of a, b, c and d.
b Find the equation of the image of the curve under the transformation defined by the matrix equation $\mathbf{T}(\mathbf{X} + \mathbf{B}) = \mathbf{X}'$, where $\mathbf{T} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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Solution

a A CAS calculator has been used for matrix calculations. The equations obtained are:

$$6 = a + b + c + d$$
$$8 = 1000a + 100b + 10c + d$$

And as $\frac{dy}{dx} = 3ax^2 + 2bx + c$

$$0 = 3a - 2b + c$$

and $$0 = 12a + 4b + c$$

These can be written as a matrix equation.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1000 & 100 & 10 & 1 \\ 3 & -2 & 1 & 0 \\ 12 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

Therefore $$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1000 & 100 & 10 & 1 \\ 3 & -2 & 1 & 0 \\ 12 & 4 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{1593} \\ \frac{-2}{531} \\ \frac{-8}{531} \\ \frac{9584}{1593} \end{bmatrix}$$

Therefore $a = \frac{4}{1593}$, $b = \frac{-2}{531}$, $c = \frac{-8}{531}$, $d = \frac{9584}{1593}$.

b First solve the matrix equation for $\mathbf{X}$.

$\mathbf{T}^{-1}\mathbf{T}(\mathbf{X} + \mathbf{B}) = \mathbf{T}^{-1}\mathbf{X}'$

$\mathbf{X} + \mathbf{B} = \mathbf{T}^{-1}\mathbf{X}'$

and $\mathbf{X} = \mathbf{T}^{-1}\mathbf{X}' - \mathbf{B}$

Therefore $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{y'}{2} \\ -\frac{x'}{3} \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and $x = \frac{y'}{2} - 1$ and $y = -\frac{x'}{3} - 2$.

The image curve has equation

$$-\frac{x'}{3} = a\left(\frac{y'}{2} - 1\right)^3 + b\left(\frac{y'}{2} - 1\right)^2 + c\left(\frac{y'}{2} - 1\right) + d + 2$$

where a, b, c and d have the values given above.

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Exercise 20E

1 Let the function $f(x) = (x - 2)^2(x - b)$, where b is a positive constant with $b > 2$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(2, 0)$ is always a local maximum.

d Find the value of b, if the stationary points occur where $x = 2$ and $x = 4$.

2 Consider the function $f: R \to R$, defined by $f(x) = x - ax^2$, where a is a real number and $a > 0$.

a Determine the intervals on which f is:

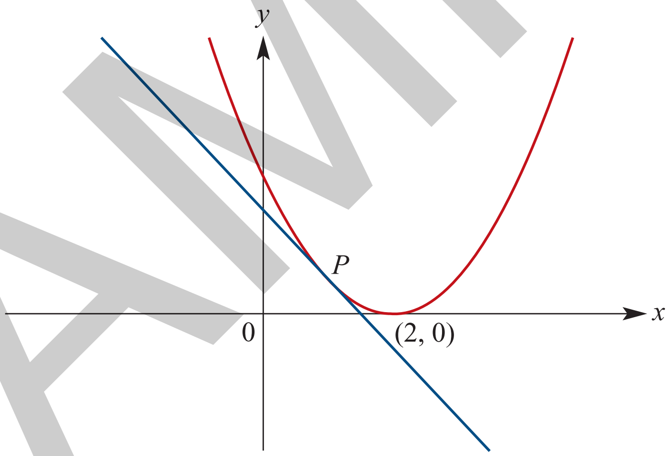
i a decreasing function **ii** an increasing function

b Find the equation of the tangent to the graphs of f at the point $\left(\frac{1}{a}, 0\right)$.

c Find the equation of the normal to the graphs of f at the point $\left(\frac{1}{a}, 0\right)$.

d What is the range of f?

3 A line with equation $y = mx + c$ is a tangent to the curve $y = (x - 2)^2$ at a point P, where $x = a$, such that $0 < a < 2$.



a **i** Find the gradient of the curve where $x = a$, for $0 < a < 2$.

ii Hence express m in terms of a.

b State the coordinates of the point P, expressing your answer in terms of a.

c Find the equation of the tangent where $x = a$.

d Find the x-axis intercept of the tangent.

4 **a** The graph of $f(x) = x^3$ is translated to the graph of $y = f(x + h)$. Find the possible value of h if $f(1 + h) = 27$.

b The graph of $f(x) = x^3$ is transformed to the graph of $y = f(ax)$. Find the possible values of a, if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 27)$.

c The cubic with equation $y = ax^3 - bx^2$ has turning point with coordinates $(1, 8)$. Find the values of a and b.

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SAMPLE

5 The graph of the function $y = x^4 + 4x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis. (a and b are positive constants.)

a Find the coordinates of the turning points of the graph of $y = x^4 + 4x^2$.

b Find the coordinates of the turning points of its image.

6 Consider the cubic function with rule $f(x) = (x - a)^2(x - 1)$, where $a > 1$.

a Find the coordinates of the turning points of the graph of $y = f(x)$.

b State the nature of each of the turning points.

c Find the equation of the tangent at which:

i $x = 1$ ii $x = a$ iii $x = \frac{a+1}{2}$

7 Consider the quartic function with rule $f(x) = (x - 1)^2(x - b)^2$, $b > 1$.

a Find the derivative of f.

b Find the coordinates of the turning points of f.

c Find the value of b such that the graph of $y = f(x)$ has a turning point at (2, 1).

8 A cubic function has rule $y = ax^3 + bx^2 + cx + d$. It passes through the points (1, 6) and (10, 8) and has turning points where $x = -1$ and $x = 1$.

a Using matrix methods, find the values of a, b, c and d.

b Find the equation of the image of the curve under the transformation defined by the matrix equation $\mathbf{T}(\mathbf{X} + \mathbf{B}) = \mathbf{X}'$, where $\mathbf{T} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

20.6 Applications to maximum and minimum and rate problems

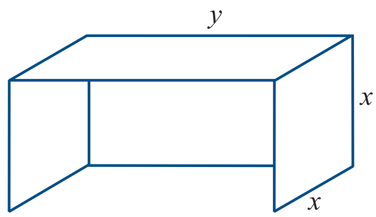
Example 17

A canvas shelter is made up with a back, two square sides and a top. The area of canvas available is 24 m^2.

a Find the dimensions of the shelter that will create the largest possible enclosed volume.

b Sketch the graph of V against x for a suitable domain.

c Find the values of x and y for which $V = 10$ m^3.



Solution

a The volume $V = x^2y$. One of the variables must be eliminated.
We know that area = 24 m^2.

$$\therefore \quad 2x^2 + 2xy = 24$$

Rearranging gives $y = \dfrac{24 - 2x^2}{2x}$

i.e. $y = \dfrac{12}{x} - x$

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Substituting in the formula for volume gives

$$V = 12x - x^3$$

Differentiation now gives

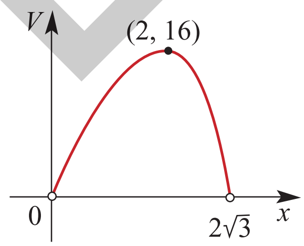
$$\frac{dV}{dx} = 12 - 3x^2$$

Stationary points occur when $\frac{dV}{dx} = 0$, which implies $12 - 3x^2 = 0$.

So stationary points occur when $x^2 = 4$,
i.e. when $x = \pm 2$, but negative values have no meaning in this problem, so the only solution is $x = 2$.
$\therefore$ maximum at $x = 2$
Dimensions are 2 m, 2 m, 4 m.

x		2	
$\frac{dV}{dx}$	+	0	−
shape of V	/	—	\

b Note that $V > 0$, $x > 0$ and $y > 0$.
This implies $12 - x^2 > 0$ and $x > 0$,
i.e. $0 < x < 2\sqrt{3}$



c Using a CAS calculator, numerically solve the equation $12x - x^3 = 10$.
The solutions are

$$x = 2.9304\ldots \quad \text{and} \quad x = 0.8925\ldots$$

Possible dimensions to the nearest centimetre are
2.93 m, 2.93 m, 1.16 m and 0.89 m, 0.89 m, 12.55 m.

Example 18

Given that $x + 2y = 4$, calculate the minimum value of $x^2 + xy - y^2$.

Solution

Rearranging $x + 2y = 4$ we have $x = 4 - 2y$.
Let $P = x^2 + xy - y^2$.

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Substituting for x: $P = (4-2y)^2 + (4-2y)y - y^2$

$$= 16 - 16y + 4y^2 + 4y - 2y^2 - y^2$$
$$= 16 - 12y + y^2$$
$$\frac{dP}{dy} = -12 + 2y$$

Stationary values occur when $\frac{dP}{dy} = 0$

i.e. when $\quad -12 + 2y = 0$

which implies $\quad y = 6$

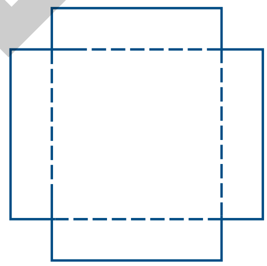
$\therefore$ a minimum when $y = 6$

y		6	
$\frac{dP}{dy}$	$-$	0	$+$
shape of P	$\backslash$	—	/

When $y = 6$, $x = -8$, the minimum value of $x^2 + xy - y^2$ is -20.

Example 19

From a square piece of metal of side length 2 m, four squares are removed as shown in the figure opposite.
The metal is then folded along the dotted lines to give an open box with sides of height x m.



a Show that the volume, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d Find the value(s) of x for which $V = 0.5$ m^3.

Solution

a Length of box $= (2 - 2x)$ metres, height $= x$ metres

$$\therefore \text{Volume} = (2-2x)^2 x$$
$$= (4 - 8x + 4x^2)x$$
$$= 4x^3 - 8x^2 + 4x \text{ m}^3$$

b Let $V = 4x^3 - 8x^2 + 4x$.
Maximum point will occur when $\frac{dV}{dx} = 0$.

$\frac{dV}{dx} = 12x^2 - 16x + 4$ and $\frac{dV}{dx} = 0$ implies that

$$12x^2 - 16x + 4 = 0$$
$$\therefore \quad 3x^2 - 4x + 1 = 0$$
$$\therefore \quad (3x-1)(x-1) = 0$$
$$\therefore x = \frac{1}{3} \text{ or } x = 1$$

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Note: When $x = 1$, the length of box $= 2 - 2x$, which is zero, $\therefore$ the only value to be considered is $x = \frac{1}{3}$.

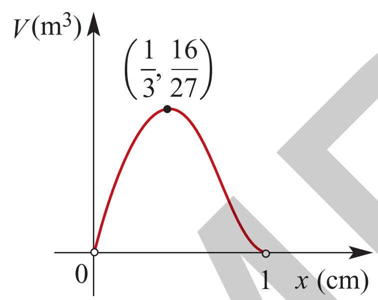
We show the entire gradient chart for completeness.

x		$\frac{1}{3}$		1
$\frac{dV}{dx}$	$+$	0	$-$	0
shape	/	—	\	—

$\therefore$ a maximum occurs when $x = \frac{1}{3}$

$$\text{and maximum volume} = \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3}$$
$$= \frac{16}{9} \times \frac{1}{3}$$
$$= \frac{16}{27}\ \text{m}^3$$

c



d This is achieved by solving the equation $V = 0.5$, i.e. $4x^3 - 8x^2 + 4x = 0.5$.

Using a CAS calculator gives $x = \frac{1}{2}$ or $\frac{3 \pm \sqrt{5}}{4}$

The domain for V is $(0, 1)$

$\therefore x = \frac{1}{2}$ or $\frac{3 - \sqrt{5}}{4}$

Endpoint maxima and minima

Calculus can be used to find a local maximum or minimum, but these are often not the actual maximum or minimum values of the function. The actual maximum value for a function defined on an interval is called the **absolute maximum**. The corresponding point on the graph of the function is not necessarily a stationary point. The actual minimum value for a function defined on an interval is called the **absolute minimum**. The corresponding point on the graph of the function is not necessarily a stationary point.

Example 20

Let $f : [-2, 4] \to R,\ f(x) = x^2 + 2$. Find the absolute maximum and the absolute minimum value of the function.

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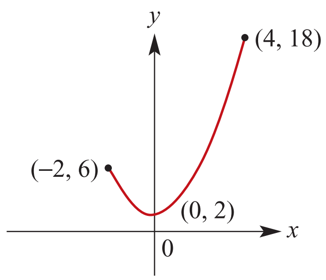
Solution

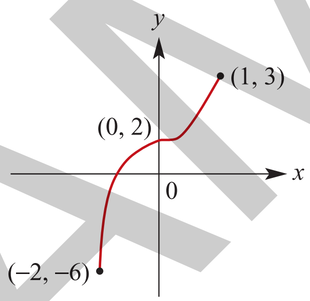


The minimum value occurs when $x = 0$ and is 2.
The maximum occurs when $x = 4$ and is 18.
The minimum value occurs at a stationary point of the graph, but the endpoint (4, 18) is not a stationary point.
The absolute maximum value is 18 and the absolute minimum value is 2.

Example 21

Let $f : [-2, 1] \to R$, $f(x) = x^3 + 2$. Find the maximum and the minimum value of the function.

Solution



The minimum value of -6 occurs when $x = -2$.
The maximum of 3 occurs when $x = 1$.
The absolute minimum and the absolute maximum do not occur at a stationary point.

Example 22

In Example 19, the maximum volume of a box was found. The maximum value corresponded to a local maximum of the graph of $V = 4x^3 - 8x^2 + 4x$. This was also the absolute maximum value.

If the height of the box must be less than 0.3 m, i.e. $x \leq 0.3$, what will be the maximum volume of the box?

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Solution

The local maximum of $V(x)$ defined on $[0, 1]$ was at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

But for the new problem $V(x) > 0$ for all $x \in [0, 0.3]$ and $\frac{1}{3}$ is not in this interval.

Therefore the maximum volume occurs when $x = 0.3$ and is 0.588.

Exercise 20F

Skillsheet TEACHER

1 The area, A km^2, of an oil slick is growing so that t hours after a leak $A = \frac{t}{2} + \frac{1}{10}t^2$.

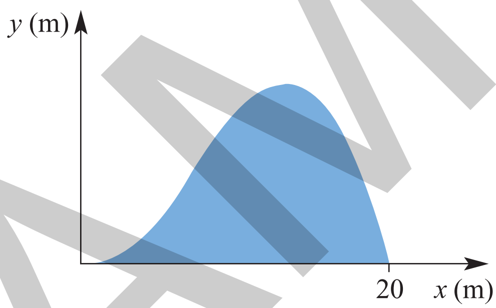
- **a** Find the area covered at the end of 1 hour.
- **b** Find the rate of increase of the area after 1 hour.

2 A particle moves in a straight line so that after t seconds it is s metres from a fixed point O on the line, and $s = t^4 + 3t^2$.

- **a** Find the acceleration when $t = 1$, $t = 2$, $t = 3$.
- **b** Find the average acceleration between $t = 1$ and $t = 3$.

3 A bank of earth has cross-section as shown in the diagram. All dimensions are in metres.

The curve defining the bank has equation $y = \frac{x^2}{400}(20 - x)$ for $x \in [0, 20]$.



- **a** Find the height of the bank where:
  i $x = 5$ **ii** $x = 10$ **iii** $x = 15$
- **b** Find the value of x for which the height is a maximum and state the maximum height of the mound.
- **c** Find the values of x for which:
  i $\frac{dy}{dx} = \frac{1}{8}$ **ii** $\frac{dy}{dx} = -\frac{1}{8}$

4 A cuboid has a total surface area of 150 cm^2 and a square base of side x cm.

- **a** Show that the height, h cm, of the cuboid is given by $h = \frac{75 - x^2}{2x}$.
- **b** Express the volume of the cuboid in terms of x.
- **c** Hence determine its maximum volume as x varies.
- **d** If the maximum side length of the square base of the cuboid is 4 cm, what is the maximum volume possible?

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Example 20

5 Let $f: [-2, 1] \to R$, $f(x) = x^3 + 2x + 3$. Find the absolute maximum and the absolute minimum value of the function for its domain.

6 Let $f: [0, 4] \to R$, $f(x) = 2x^3 - 6x^2$. Find the absolute maximum and the absolute minimum value of the function.

7 Let $f: [-2, 5] \to R$, $f(x) = 2x^4 - 8x^2$. Find the absolute maximum and the absolute minimum value of the function.

8 Let $f: [-2, 2] \to R$, $f(x) = 2 - 8x^2$. Find the absolute maximum and the absolute minimum value of the function.

Example 22

9 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Show that the volume, V cm^3, is given by $V = 15x^2 - 12x^3$.
b Find the $\frac{dV}{dx}$.
c Find the local maximum value of the graph of V against x for $x \in [0, 1.25]$.
d If $x \in [0, 0.8]$, find the absolute maximum value and the value of x for which this occurs.
e If $x \in [0, 1]$, find the absolute maximum value and the value of x for which this occurs.

10 For the variables x, y and z it is known that $x + y = 20$ and $z = xy$.

a If $x \in [2, 5]$, find the possible values of y.
b Find the maximum and minimum values of z.

11 For the variables x, y and z, it is known that $z = x^2y$ and $2x + y = 50$. Find the maximum value of z if:

a $x \in [0, 25]$
b $x \in [0, 10]$
c $x \in [5, 20]$

12 A piece of string 10 metres long is cut into two pieces to form two squares.

a If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}\left(x^2 - 10x + 50\right)$.
b Find $\frac{dA}{dx}$.
c Find the value of x that makes A a minimum.
d If two squares are formed but $x \in [0, 11]$, find the maximum possible area of the two squares.

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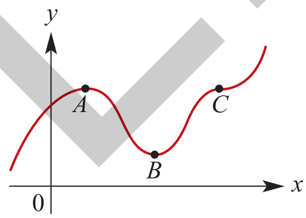
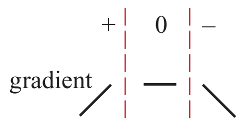
Chapter summary

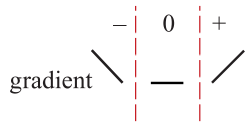
- Let P be the point on the curve $y = f(x)$ with coordinates (x_1, y_1). If f is differentiable for $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$.
  The equation of the normal at (x_1, y_1) is $y - y_1 = -\dfrac{1}{f'(x_1)}(x - x_1)$.
- For a body moving in a straight line with displacement s metres from a point O at time t seconds:
  velocity $\quad v = \dfrac{ds}{dt} \qquad$ acceleration $\quad a = \dfrac{dv}{dt}$
- A point with coordinates $(a, g(a))$ on a curve $y = g(x)$ is said to be a stationary point if $g'(a) = 0$.
- **Types of stationary points**
  The graph shown here has three stationary points, A, B, C.



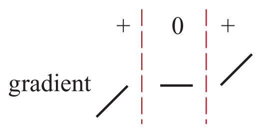
A The point A is called **a local maximum** point. Notice that immediately to the left of A the gradient is positive and immediately to the right the gradient is negative. A diagram to represent this is shown.



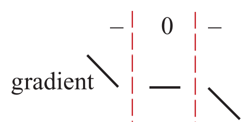
B The point B is called **a local minimum** point. Notice that immediately to the left of B the gradient is negative and immediately to the right the gradient is positive. A diagram to represent this is shown.



C The point C is called a **stationary point of inflexion**. A diagram for this is shown.



Clearly it is also possible to have stationary points of inflexion for which our diagram would be as shown.



Stationary points of type A and B are referred to as **turning points**.

- For a continuous function f defined on an interval, M is the absolute maximum value of the function if $f(x) \leq M$ for all x in the interval.
- For a continuous function f defined on an interval, N is the absolute minimum value of the function if $f(x) \geq N$ for all x in the interval.

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Multiple-choice questions

1 The equation of the tangent to the curve $y = x^3 + 2x$ at the point (1, 3) is

A $y = x$ **B** $y = 5x$ **C** $y = 5x + 2$
D $y = 5x - 2$ **E** $y = x - 2$

2 The equation of the normal to the curve $y = x^3 + 2x$ at the point (1, 3) is

A $y = -5x$ **B** $y = -5x + 2$ **C** $y = \frac{1}{5}x + 2\frac{2}{5}$
D $y = -\frac{1}{5}x + 2\frac{2}{5}$ **E** $y = -\frac{1}{5}x + 3\frac{1}{5}$

3 The equation of the tangent to the curve $y = 2x - 3x^3$ at the origin is

A $y = 2$ **B** $y = -2x$ **C** $y = x$ **D** $y = -x$ **E** $y = 2x$

4 The average rate of change of the function $f(x) = 4x - x^2$ between $x = 0$ and $x = 1$ is

A 3 **B** −3 **C** 4 **D** −4 **E** 0

5 A particle moves in a straight line so that its position S m relative to O at a time t seconds ($t > 0$) is given by $S = 4t^3 + 3t - 7$. The initial velocity of the particle is

A 0 m/s **B** −7 m/s **C** 3 m/s **D** −4 m/s **E** 15 m/s

6 The function $y = x^3 - 12x$ has stationary points at $x =$

A 0 and 12 **B** −4 and 4 **C** −2 and 4 **D** −2 and 2 **E** 2 only

7 The curve $y = 2x^3 - 6x$ has a gradient of 6 at $x =$

A 2 **B** $\sqrt{2}$ **C** −2 and 2 **D** $-\sqrt{2}$ and $\sqrt{2}$ **E** 0, $\sqrt{2}$

8 The rate of change of the curve $f(x) = 2x^3 - 5x^2 + x$ at $x = 2$ is

A 5 **B** −2 **C** 2 **D** −5 **E** 6

9 The average rate of change of the function $y = \frac{1}{2}x^4 + 2x^2 - 5$ between $x = -2$ and $x = 2$ is

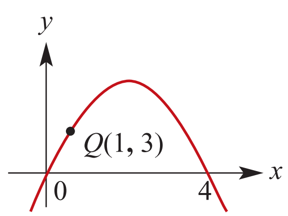
A 0 **B** 5.5 **C** 11 **D** 22 **E** 2.75

10 The minimum value of the function $y = x^2 - 8x + 1$ is

A 1 **B** 4 **C** −15 **D** 0 **E** −11

Short-answer questions (technology-free)

1 The graph of $y = 4x - x^2$ is shown.



a Find $\frac{dy}{dx}$.
b Find the gradient of the curve at $Q(1, 3)$.
c Find the equation of the tangent at Q.

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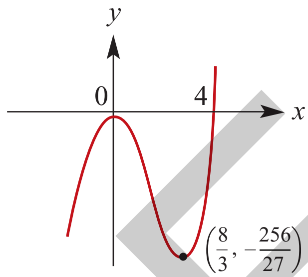
2 The graph of $y = x^3 - 4x^2$ is as shown.



a Find $\frac{dy}{dx}$.

b Find the gradient of the tangent of the curve at the point $(2, -8)$.

c Find the equation of the tangent at the point $(2, -8)$.

d Find the coordinates of the point Q where the tangent crosses the curve again.

3 Let $y = x^3 - 12x + 2$.

a Find $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$.

b State the nature of each of these stationary points.

c Find the corresponding y-value for each of these.

4 Write down the values of x for which each of the following derived functions are zero, and in each case prepare a gradient chart to determine whether the corresponding points are maxima, minima or stationary points of inflexion.

a $\frac{dy}{dx} = 3x^2$ **b** $\frac{dy}{dx} = -3x^3$

c $f'(x) = (x - 2)(x - 3)$ **d** $f'(x) = (x - 2)(x + 2)$

e $f'(x) = (2 - x)(x + 2)$ **f** $f'(x) = -(x - 1)(x - 3)$

g $\frac{dy}{dx} = -x^2 + x + 12$ **h** $\frac{dy}{dx} = 15 - 2x - x^2$

5 For each of the following find all stationary points and state the nature of each.

a $y = 4x - 3x^3$ **b** $y = 2x^3 - 3x^2 - 12x - 7$ **c** $y = x(2x - 3)(x - 4)$

6 Sketch the graphs of each of the following. Give the coordinates of their stationary points and of their axes intercepts.

a $y = 3x^2 - x^3$ **b** $y = x^3 - 6x^2$

c $y = (x + 1)^2(2 - x)$ **d** $y = 4x^3 - 3x$

e $y = x^3 - 12x^2$

Extended-response questions

Questions 1–8 involve rates of change.

1 The height, in metres, of a stone thrown vertically upwards from the surface of a planet is $2 + 10t - 4t^2$ after t seconds.

a Calculate the velocity of the stone after 3 seconds.

b Find the acceleration due to gravity.

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Review

2 A dam is being emptied. The quantity of water, V litres, remaining in the dam at any time t minutes after it starts to empty is given by $V(t) = 1000(30 - t)^3$, $t \geq 0$.

CAS CALCULATOR

a Sketch the graph of V against t.
b Find the time at which there are:
   i 2 000 000 litres of water in the dam **ii** 20 000 000 litres of water in the dam.
c At what rate is the dam being emptied at any time t?
d How long does it take to empty the dam?
e At what time is the water flowing out at 8000 litres per minute?
f Sketch the graphs of $y = V(t)$ and $y = V'(t)$ on the one set of axes.

3 In a certain area of Victoria the quantity of blackberries (W tonnes) ready for picking x days after 1 September is given by $W = \dfrac{x}{4000}\left(48\,000 - 2600x + 60x^2 - \dfrac{x^3}{2}\right)$, $0 \leq x \leq 60$.

CAS CALCULATOR

a Sketch the graph of W against x for $0 \leq x \leq 60$.
b After how many days will there be 50 tonnes of blackberries ready for picking?
c Find the rate of increase of W, in tonnes per day, when $x = 20$, 40 and 60.
d Find the value of W when $x = 30$.

4 A newly installed central heating system has a thermometer which shows the water temperature as it leaves the boiler (y °C). It also has a thermostat which switches off the system when $y = 65$.
The relationship between y and t, the time in minutes, is given by $y = 15 + \dfrac{1}{80}t^2(30 - t)$.

a Find the temperature at $t = 0$.
b Find the rate of increase of y with respect to t, when $t = 0$, 5, 10, 15 and 20.
c Sketch the graph of y against t for $0 \leq t \leq 20$.

5 The sweetness, S, of a pineapple t days after it begins to ripen is found to be given by $S = 4000 + (t - 16)^3$ units.

a At what rate is S increasing when $t = 0$?
b Find $\dfrac{dS}{dt}$ when $t = 4$, 8, 12, and 16.
c The pineapple is said to be unsatisfactory when our model indicates that the rate of increase of sweetness is zero. When does this happen?
d Sketch the graph of S against t up to the moment when the pineapple is unsatisfactory.

6 A slow train which stops at every station passes a certain signal box at noon. The motion of the train between the two adjacent stations is such that it is s km past the signal box at t minutes past noon, where $s = \dfrac{1}{3}t + \dfrac{1}{9}t^2 - \dfrac{1}{27}t^3$.

CAS CALCULATOR

a Use a calculator to help sketch the graphs of s against t and $\dfrac{ds}{dt}$ against t on the one set of axes. Sketch for $t \in [-2, 5]$.

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b Find the time of departure from the first station and the time of arrival at the second.

c Find the distance of each station from the signal box.

d Find the average velocity between the stations.

e Find the velocity with which the train passes the signal box.

7 Water is draining from a tank. The volume, V L, of water at time t (hours) is given by $V(t) = 1000 + (2 - t)^3$, $t \geq 0$ and $V(t) \geq 0$.

a What are the possible values of t?

b Find the rate of draining when:

i $t = 5$ **ii** $t = 10$

8 A mountain path can be approximately described by the following rule, where y is the elevation, in metres, above the sea level and x is the horizontal distance travelled, in kilometres.

$$y = \frac{1}{5}(4x^3 - 8x^2 + 192x + 144) \quad \text{for} \quad 0 \leq x \leq 7$$

a How high above sea level is the start of the track, i.e. $x = 0$?

b When $x = 6$, what is the value of y?

CAS CALCULATOR

c Use a calculator to draw a graph of the path. Sketch this graph.

d Does this model for the path make sense for $x > 7$?

e Find the gradient of the graph for the following distances (be careful of units):

i $x = 0$ **ii** $x = 3$ **iii** $x = 7$

Questions 9–32 are maxima and minima problems.

9 **a** On the one set of axes sketch the graphs of $y = x^3$ and $y = 2 + x - x^2$.

b For $x \leq 0$, $2 + x - x^2 \geq x^3$. Find the value of x, $x < 0$, for which the vertical distance between the two curves is a minimum and find the minimum distance. (Hint: Consider the function with rule $y = 2 + x - x^2 - x^3$ for $x \leq 0$.)

10 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the September rainfall, x, measured in mm, and is approximated by:

$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3),\ 0 \leq x \leq 10$$

CAS CALCULATOR

Find the rainfall that will produce the maximum and the minimum number of mosquitos. (First plot the graph of $y = M(x)$ using a calculator.)

11 Given that $x + y = 5$ and $P = xy$ find:

a y in terms of x **b** P in terms of x

c the maximum value of P and the corresponding values of x and y.

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Review

12 Given that $2x + y = 10$ and $A = x^2 y$, where $0 \le x \le 5$, find:

a y in terms of x **b** A in terms of x

c the maximum value of A and the corresponding values of x and y.

13 Given that $xy = 10$ and $T = 3x^2 y - x^3$, find the maximum value of T for $0 < x < \sqrt{30}$.

14 The sum of two numbers x and y is 8.

a Write down an expression for y in terms of x.

b Write down an expression for s, the sum of the squares of these two numbers, in terms of x.

c Find the least value of the sum of their squares.

15 Find two positive numbers whose sum is 4, and the sum of the cube of the first and the square of the second is as small as possible.

16 A rectangular patch of ground is to be enclosed with 100 metres of fencing wire. Find the dimensions of the rectangle so that the area enclosed will be a maximum.

17 The sum of two numbers is 24. If one number is x, find the value of x such that the product of the two numbers is a maximum.

18 A factory which produces n items per hour is found to have overhead costs of $\$\left(400 - 16n + \frac{1}{4}n^2\right)$ per hour. How many items should be produced every hour to keep the overhead costs to a minimum?

19 For $x + y = 100$ prove that the product $P = xy$ is a maximum when $x = y$, and find the maximum value of P.

20 A farmer has 4 km of fencing wire and wishes to fence in a rectangular piece of land through which a straight river flows. The river is to form one side of the enclosure. How can this be done to enclose as much land as possible?

21 Two positive quantities p and q vary in such a way that $p^3 q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum.

22 A beam has a rectangular cross-section of depth x cm and width y cm. The strength, S, of the beam is given by $S = 5x^2 y$. The perimeter of a cross-section of the beam is 120 cm.

a Find y in terms of x. **b** Express S in terms of x.

c What are the possible values for x? **d** Sketch the graph of S against x.

e Find the values of x and y which give the strongest beam.

f If the cross-sectional depth of the beam must be less than 19 cm, find the maximum strength of the beam.

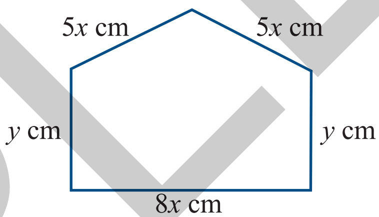
23 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$, with x representing the temperature of the water in degrees (°C). (This function is valid only if $6 \le x \le 20$.) Find the water temperature that results in the maximum number of salmon swimming upstream.

SAMPLE

24 A piece of wire 360 cm long is used to make the twelve edges of a rectangular box for which the length is twice the breadth.

- **a** Denoting the breadth of the box by x cm, show that the volume of the box, V cm^3, is given by $V = 180x^2 - 6x^3$.
- **b** Find the domain, S, of the function $V : S \rightarrow R$, $V(x) = 180x^2 - 6x^3$ which describes the situation.
- **c** Sketch the graph of the function with rule $y = V(x)$.
- **d** Find the dimensions of the box that has the greatest volume.
- **e** Find the values of x for which $V = 20\,000$. Give values correct to 2 decimal places.

25 A piece of wire of length 90 cm is bent into the shape shown in the diagram.

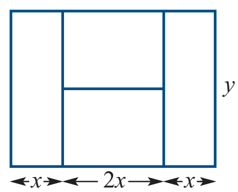


- **a** Show that the area, A cm^2, enclosed by the wire is given by $A = 360x - 60x^2$.
- **b** Find the values of x and y for which A is a maximum.

26 A piece of wire 100 cm in length is to be cut into two pieces, one piece of which is to be shaped into a circle and the other into a square.

- **a** How should the wire be cut if the sum of the enclosed areas is to be a minimum?
- **b** How should the wire be used to obtain a maximum area?

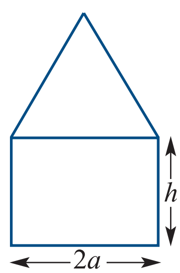
27 A roll of tape 36 metres long is to be used to mark out the edges and internal lines of a rectangular court of length $4x$ metres and width y metres, as shown in the diagram. Find the length and width of the court for which the area is a maximum.



28 A rectangular chicken run is to be built on flat ground. A 16-metre length of chicken wire will be used to form three of the sides; the fourth side of length x metres, will be part of a straight wooden fence.

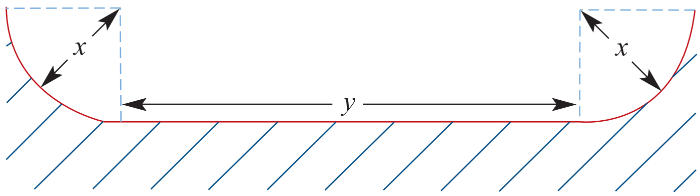
- **a** Let y be the width of the rectangle. Find an expression for A, the area of the chicken run in terms of x and y.
- **b** Find an expression for A in terms of x.
- **c** Find the possible values of x.
- **d** Sketch the graph of A against x for these values of x.
- **e** What is the largest area of ground the chicken run can cover?

29 The diagram illustrates a window that consists of an equilateral triangle and a rectangle. The amount of light that comes through the window is directly proportional to the area of the window. If the perimeter of such a window must be 8000 mm, find the values of h and a (correct to the nearest mm) which allow the maximum amount of light to pass.



SAMPLE

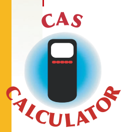
30 The diagram shows a cross-section of an open drainage channel. The flat bottom of the channel is y metres across and the sides are quarter circles of radius x metres. The total length of the bottom plus the two curved sides is 10 metres.



a Express y in terms of x.
b State the possible values that x can take.
c Find an expression for A, the area of the cross-section, in terms of x.
d Sketch the graph of $y = A(x)$, for possible values of x.
e Find the value of x which maximises A.
f Comment on the cross-sectional shape of the drain.

31 A cylinder closed at both ends has a total surface area of 1000 cm^2. The radius of the cylinder is x cm and the height h cm. Let V cm^3 be the volume of the cylinder.

a Find h in terms of x.
b Find V in terms of x.
c Find $\dfrac{dV}{dx}$.
d Find $\{x: \dfrac{dV}{dx} = 0\}$.



e Sketch the graph of V against x for a suitable domain.
f Find the maximum volume of the cylinder.
g Find the value(s) of x and h for which $V = 1000$, correct to 2 decimal places.

32 A cylindrical aluminium can able to contain half a litre of drink is to be manufactured. The volume of the can must therefore be 500 cm^3.

a Find the radius and height of the can which will use the least aluminium and, therefore, be the cheapest to manufacture.
b If the radius of the can must be no greater than 5 cm, find the radius and height of the can that will use the least aluminium.

SAMPLE

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