CHAPTER

15

Exponential Functions and Logarithms

Objectives

- To define and understand **exponential functions**.
- To **sketch graphs** of the various types of exponential functions.
- To understand the rules for manipulating **exponential and logarithmic expressions**.
- To solve **exponential equations**.
- To evaluate **logarithmic expressions**.
- To solve equations using **logarithmic methods**.
- To sketch graphs of functions of the form $\boldsymbol{y = \log_a x}$ and simple transformations of this.
- To understand and use a range of **exponential models**.
- To **sketch graphs** of exponential functions.
- To **apply** exponential functions to solving problems.

The function $f(x) = ka^x$, where k is a non zero constant and the base a is a positive real number other than 1, is called an **exponential function** (or index function).

Consider the following example of an exponential function. Assume that a particular biological organism reproduces by dividing every minute. The following table shows the population, P, after n one-minute intervals (assuming that all organisms are still alive).

n	0	1	2	3	4	5	6	n
P	1	2	4	8	16	32	64	2^n

Thus P defines a function which has the rule $P = 2^n$, an exponential (or index) function.

15.1 Graphs of exponential functions

Two types of graphs will be examined.

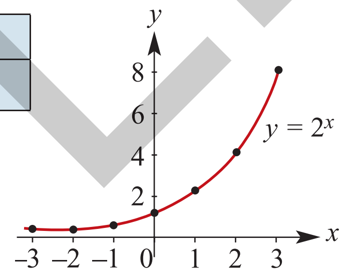
1 Graphs of $y = a^x$, $a > 1$

Example 1

Plot the graph of $y = 2^x$, and examine the table of values for $-3 \le x \le 3$. A calculator can be used.

Solution

x	-3	-2	-1	0	1	2	3
$y = 2^x$	0.125	0.25	0.5	1	2	4	8



It can be observed that, as negative values of increasing magnitude are considered, the graph approaches the x-axis from above. The x-axis is said to be an **asymptote**. As the magnitude of negative x-values becomes larger, 2^x takes values closer and closer to zero, but never reaches zero, i.e. the graph gets closer and closer to the x-axis. This is written as $x \to -\infty$, $y \to 0$ from the positive side, or as $x \to -\infty$, $y \to 0^+$.

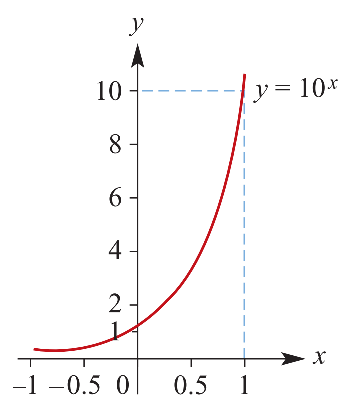
The y-axis intercept for the graph is (0, 1). The range of the function is R^+.

Example 2

Plot the graph of $y = 10^x$ and examine the table of values for $-1 \le x \le 1$. A calculator can be used.

Solution

x	-1	-0.5	0	0.5	1
$y = 10^x$	0.1	0.316	1	3.16	10



The x-axis is an asymptote, and the y-axis intercept is (0, 1). The y-values increase as the x-values increase. This rate of increase for $y = 10^x$ is greater than that for $y = 2^x$ for a given value of x.

It is worth noting at this stage that, for a and b positive numbers greater than 1, there is a positive number k such that $a^k = b$. This can be seen from the graphs of $y = 2^x$ and $y = 10^x$ above. Using a calculator to solve $2^k = 10$ graphically gives $k = 3.321928\ldots$ Hence $10^x = (2^{3.321928\ldots})^x = 2^{3.321928\ldots x}$. This means that the graph of $y = 10^x$ can be obtained from the graph of $y = 2^x$ by a dilation of factor $\frac{1}{k} = \frac{1}{3.321928\ldots}$ from the y-axis.

This shows that all graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations from the y-axis.

This will be discussed again later in the chapter.

2 Graphs of $y = a^x$, $0 < a < 1$

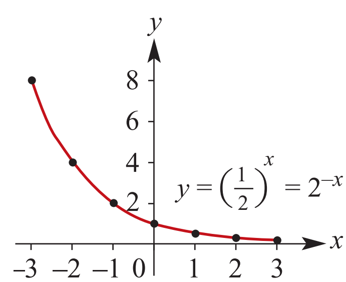
Example 3

Plot the graph of $y = \left(\frac{1}{2}\right)^x$ and examine the table of values for $-3 \le x \le 3$. A calculator can be used.

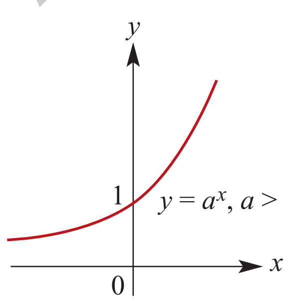
Solution

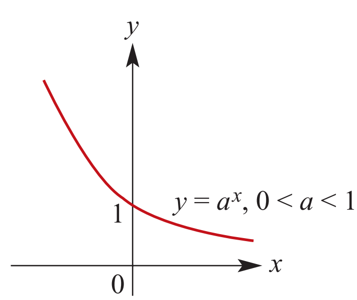
x	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{2}\right)^x = 2^{-x}$	8	4	2	1	0.5	0.25	0.125

The x-axis is an asymptote, and the y-axis intercept is (0, 1). For this graph the y-values decrease as the x-values increase. This is written as $x \to \infty$, $y \to 0$ from the positive side, or as $x \to \infty, y \to 0^+$. The range of the function is R^+.



In general:





For $0 < a < 1$, $y = a^x$ is equivalent to $y = b^{-x}$, where $b = \frac{1}{a}$.

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The graph of $y = a^{-x}$ is obtained from the graph of $y = a^x$ by a reflection in the y-axis. Thus, for example, the graph of $y = \left(\frac{1}{2}\right)^x$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis, and vice versa.

If $a = 1$, the resulting graph is a horizontal line with equation $y = 1$.

Example 4

Plot the graph of $y = 2^x$ on a CAS calculator and hence find:

a the value of y when $x = 2.1$, correct to 3 decimal places

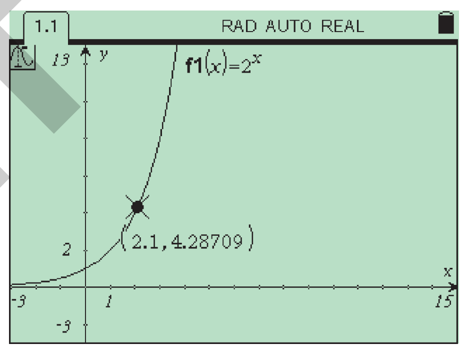
b the value of x when $y = 9$.

Solution

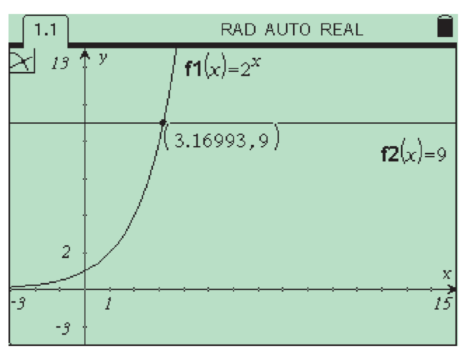
Using the TI-Nspire

Plot the graph.

a Use **Graph Trace** (menu 5 1) and type 2.1 enter to go straight to the point with x-coordinate 2.1.
When $x = 2.1$, $y = 4.287$, correct to 3 decimal places.



b The values of x for which $y = 9$ can be found by plotting the graph of $y = 9$ on the same screen and finding the **Intersection Point(s)** (menu 6 3).
When $y = 9$, $x = 3.170$, correct to 3 decimal places.



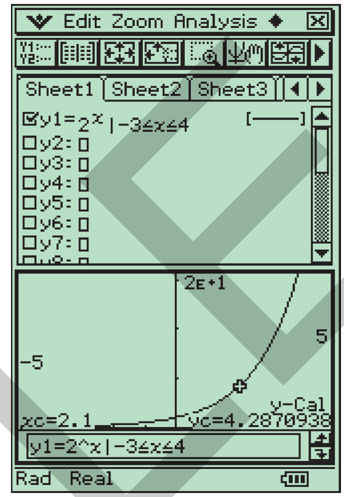
Using the Casio ClassPad

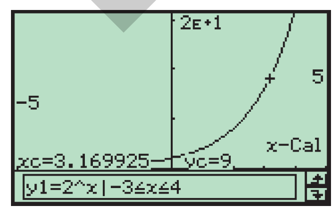
Plot the graph for $-3 \le x \le 4$.

a Click in the graph window and select **Analysis—G-solve—y-calc** and enter $x = 2.1$. $y = 4.287$ when $x = 2.1$ (correct to 3 decimal places).

b Select **Analysis—G-solve—x-calc** and enter $y = 9$. When $y = 9$, $x = 3.170$ (correct to 3 decimal places).

Note: An alternative method is to enter $y2 = 9$ and use **G-solve—Intersect**.





Transformations of exponential functions

Example 5

Sketch the graphs of each of the following functions. Give equations of asymptotes and the y-axis intercepts, and state the range of each of the functions. (x-axis intercepts need not be given.)

a $f: R \to R,\ f(x) = 2^x + 3$

b $f: R \to R,\ f(x) = 2 \times 3^x + 1$

c $f: R \to R,\ f(x) = -3^x + 2$

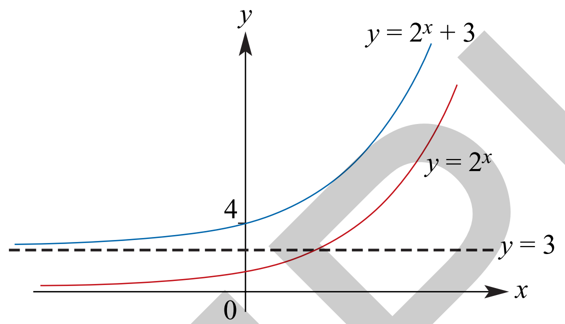
Solution

a For the function $f: R \to R$, $f(x) = 2^x + 3$, the corresponding graph is obtained by transforming the graph of $y = 2^x$ by a translation of 3 units in the positive direction of the y-axis.

The asymptote of $y = 2^x$, with equation $y = 0$, is transformed to the asymptote with equation $y = 3$ for the graph of $f(x) = 2^x + 3$. The asymptotic behaviour can be described as $x \to -\infty$, $y \to 3$ from the positive side, or as $x \to -\infty$, $y \to 3^+$.

As $f(0) = 2^0 + 3 = 4$, the y-axis intercept is 4.

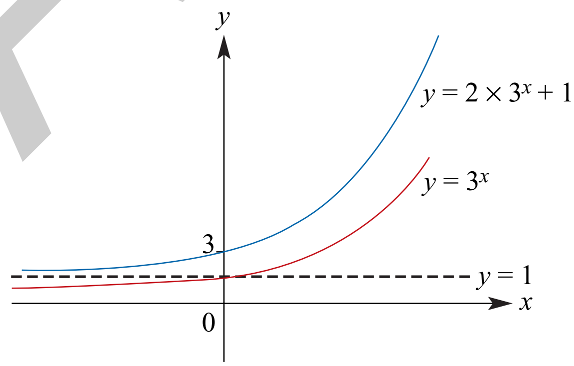
The range of the function $f: R \to R$, $f(x) = 2^x + 3$ is $(3, \infty)$.



b For the function $f: R \to R$, $f(x) = 2 \times 3^x + 1$, the corresponding graph is obtained by transforming the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis, followed by a translation of 1 unit in the positive direction of the y-axis.

The asymptote of $y = 3^x$, with equation $y = 0$, is transformed to the asymptote $y = 1$ for the graph of $f(x) = 2 \times 3^x + 1$.

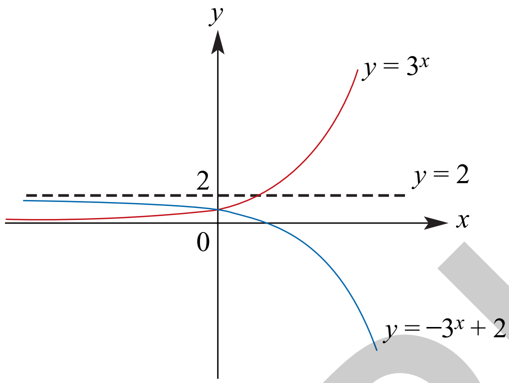
The asymptotic behaviour is described as $x \to -\infty$, $y \to 1$ from the positive side, or as $x \to -\infty$, $y \to 1^+$. The y-axis intercept is given by $f(0) = 2 \times 3^0 + 1 = 3$. The range of the function $f: R \to R$, $f(x) = 2 \times 3^x + 1$ is $(1, \infty)$.



c For the function $f: R \to R$, $f(x) = -3^x + 2$, the corresponding graph is obtained by transforming the graph of $y = 3^x$ by reflection in the x-axis followed by a translation of 2 units in the positive direction of the y-axis.

The asymptote of $y = 3^x$, with equation $y = 0$, is transformed to the asymptote $y = 2$ for the graph of $f(x) = -3^x + 2$.

The asymptotic behaviour is described by as $x \to -\infty$, $y \to 2$ from the negative side, or as $x \to -\infty$, $y \to 2^-$. The y-axis intercept is given by $f(0) = -3^0 + 2 = 1$. The range of the function $f: R \to R$, $f(x) = -3^x + 2$ is $(-\infty, 2)$.



Example 6

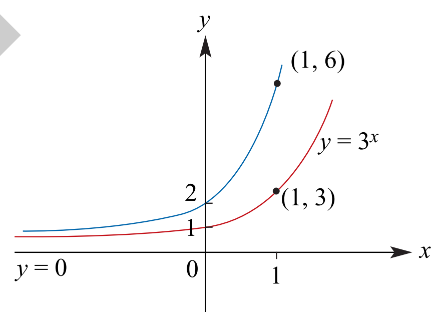
Sketch the graphs of each of the following:

a $y = 2 \times 3^x$ **b** $y = 3^{2x}$ **c** $y = 3^{\frac{x}{2}}$ **d** $y = -3^{2x} + 4$

Solution

a The graph of $y = 2 \times 3^x$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis.

The horizontal asymptote for both graphs has equation $y = 0$.



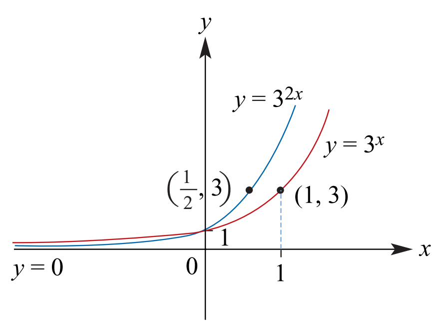
b The graph of $y = 3^{2x}$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

(In the notation introduced in Chapter 6, write it as $(x, y) \to \left(\frac{1}{2}x, y\right)$.

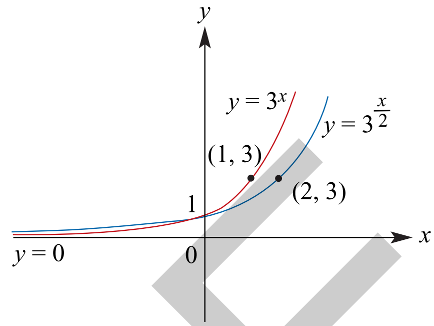
Then describe the transformation as $x' = \frac{1}{2}x$ and $y' = y$, and hence $x = 2x'$ and $y = y'$.

The graph of $y = 3^x$ is mapped to the graph of $y' = 3^{2x'}$)

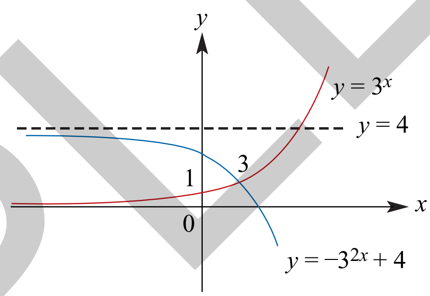
The horizontal asymptote for both graphs has equation $y = 0$.



c The graph of $y = 3^{\frac{x}{2}}$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 from the y-axis. (In the notation introduced in Chapter 6, write it as $(x, y) \to (2x, y)$.) Then describe the transformation as $x' = 2x$ and $y' = y$, and hence $x = \frac{x'}{2}$ and $y = y'$. The graph of $y = 3^x$ is mapped to the graph of $y' = 3^{\frac{x'}{2}}$).



d The graph of $y = -3^{2x} + 4$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the x-axis then by a translation of 4 units in the positive direction of the y-axis.



Exercise 15A

1 Using a calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 1.8^x$ **b** $y = 2.4^x$ **c** $y = 0.9^x$ **d** $y = 0.5^x$

2 Using a calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 2 \times 3^x$ **b** $y = 5 \times 3^x$ **c** $y = -2 \times 3^x$ **d** $y = -5 \times 3^x$

Example 4 **3** Using a calculator plot the graph of $y = 2^x$ for $x \in [-4, 4]$ and hence find the solution of the equation $2^x = 14$.

4 Using a calculator plot the graph of $y = 10^x$ for $x \in [-0.4, 0.8]$ and hence find the solution of the equation $10^x = 6$.

Example 5 **5** Sketch the graphs of each of the following functions. Give equations of asymptotes and y-axis intercepts, and state the range of each of the functions. (x-axis intercepts need not be given.)

a $f: R \to R,\ f(x) = 3 \times 2^x + 2$ **b** $f: R \to R,\ f(x) = 3 \times 2^x - 3$
c $f: R \to R,\ f(x) = -3^x - 2$ **d** $f: R \to R,\ f(x) = -2 \times 3^x + 2$
e $f: R \to R,\ f(x) = \left(\frac{1}{2}\right)^x + 2$ **f** $f: R \to R,\ f(x) = -2 \times 3^x - 2$

Example 6 **6** Sketch the graphs of each of the following:

a $y = 2 \times 5^x$ **b** $y = 3^{3x}$ **c** $y = 5^{\frac{x}{2}}$ **d** $y = -3^{2x} + 2$

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15.2 Reviewing rules for exponents (indices)

We shall review the rules for manipulating the exponential expression a^x, where a is a real number called the **base**, and x is a real number called the **exponent** or **index**. Other words which are synonyms for *index* are **power** and **logarithm**.

Multiplication: $a^m \times a^n$

If m and n are positive integers

then $$a^m = \underbrace{a \times a \times a \cdots \times a}_{\leftarrow m \text{ terms} \rightarrow}$$

and $$a^n = \underbrace{a \times a \cdots \times a}_{\leftarrow n \text{ terms} \rightarrow}$$

$$\begin{aligned}\therefore \quad a^m \times a^n &= \underbrace{(a \times a \times a \cdots \times a)}_{\leftarrow m \text{ terms} \rightarrow} \times \underbrace{(a \times a \cdots \times a)}_{\leftarrow n \text{ terms} \rightarrow} \\ &= \underbrace{(a \times a \times a \cdots \times a)}_{\leftarrow m+n \text{ terms} \rightarrow} \\ &= a^{m+n}\end{aligned}$$

Rule 1

To multiply two numbers in exponent form with the **same base**, **add** the exponents.

$$a^m \times a^n = a^{m+n}$$

Example 7

Simplify each of the following:

a $2^3 \times 2^{12}$ **b** $x^2y^3 \times x^4y$ **c** $2^x \times 2^{x+2}$ **d** $3a^2b^3 \times 4a^3b^2$

Solution

a $$\begin{aligned}2^3 \times 2^{12} &= 2^{3+12} \\ &= 2^{15}\end{aligned}$$

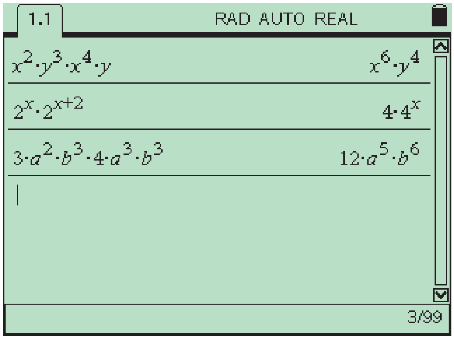
b $$\begin{aligned}x^2y^3 \times x^4y &= x^2 \times x^4 \times y^3 \times y \\ &= x^6y^4\end{aligned}$$

c $$\begin{aligned}2^x \times 2^{x+2} &= 2^{x+x+2} \\ &= 2^{2x+2}\end{aligned}$$

d $$\begin{aligned}3a^2b^3 \times 4a^3b^2 &= 3 \times 4a^2 \times a^3 \times b^3 \times b^2 \\ &= 12a^5b^5\end{aligned}$$

Using the TI-Nspire

b, c and **d** can be simplified as shown.



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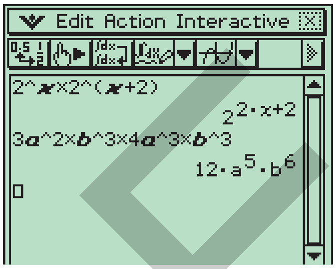
Using the Casio ClassPad

c and d

Enter $2\hat{}x*2\hat{}(x+2)$ and $3a\hat{}2*b\hat{}3*4a\hat{}3*b\hat{}3$ in the entry line.



Division: $a^m \div a^n$

If m and n are positive integers and $m > n$

$$\text{then} \qquad a^m \div a^n = \frac{\overbrace{a \times a \times a \ldots a}^{\leftarrow\, m \text{ terms} \,\rightarrow}}{\underbrace{a \times a \times a \ldots a}_{\leftarrow\, n \text{ terms} \,\rightarrow}}$$

$$= \underbrace{a \times a \times a \ldots a}_{\leftarrow\, (m-n) \text{ terms} \,\rightarrow} \qquad \text{(by cancelling)}$$

$$= a^{m-n}$$

Rule 2

To divide two numbers in exponent form with the **same base**, **subtract** the exponents.

$$a^m \div a^n = a^{m-n}$$

Define $a^0 = 1$ for $a \neq 0$ and $a^{-n} = \frac{1}{a^n}$ for $a \neq 0$.

Note: **Rule 1** and **Rule 2** also hold for negative indices m, n for $a \neq 0$.

For example:

$$2^4 \times 2^{-2} = \frac{2^4}{2^2} = \frac{2 \times 2 \times 2 \times 2}{2 \times 2} = 2^2 \qquad (\text{i.e. } 2^{4+(-2)})$$

$$2^{-4} \div 2^2 = \frac{1}{2^4} \times \frac{1}{2^2} = \frac{1}{2^4 \times 2^2} = 2^{-6} \qquad (\text{i.e. } 2^{-4-2})$$

$$2^3 \div 2^3 = 2^3 \times \frac{1}{2^3} = 1 = 2^0 \qquad (\text{i.e. } 2^{3-3})$$

Example 8

Simplify each of the following:

a $\dfrac{x^4y^3}{x^2y^2}$ **b** $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}}$ **c** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$

Solution

a $$\begin{aligned}\frac{x^4y^3}{x^2y^2} &= x^{4-2}y^{3-2}\\ &= x^2y\end{aligned}$$

b $$\begin{aligned}\frac{b^{4x} \times b^{x+1}}{b^{2x}} &= b^{4x+x+1-2x}\\ &= b^{3x+1}\end{aligned}$$

c $$\begin{aligned}\frac{16a^5b \times 4a^4b^3}{8ab} &= \frac{16 \times 4}{8} \times a^{5+4-1} \times b^{1+3-1}\\ &= 8a^8b^3\end{aligned}$$

Raising the power: $(a^m)^n$

Consider the following:

$$(2^3)^2 = 2^3 \times 2^3 = 2^{3+3} = 2^6 = 2^{3\times 2}$$
$$(4^3)^4 = 4^3 \times 4^3 \times 4^3 \times 4^3 = 4^{3+3+3+3} = 4^{12} = 4^{3\times 4}$$
$$(a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{2+2+2+2+2} = a^{10} = a^{2\times 5}$$

In general $(a^m)^n = a^{m\times n}$.

Rule 3

To raise the power of a to another power, **multiply** the indices.

$$(a^m)^n = a^{m\times n}$$

This rule holds for all integers m and n.

Products and quotients

Consider $(ab)^n$ and $\left(\dfrac{a}{b}\right)^n$.

$$\begin{aligned}(ab)^n &= \underbrace{(ab) \times (ab) \times \cdots \times (ab)}_{\leftarrow\ n \text{ terms}\ \rightarrow}\\ &= \underbrace{(a \times a \times \cdots a)}_{\leftarrow\ n \text{ terms}\ \rightarrow} \times \underbrace{(b \times b \times \cdots b)}_{\leftarrow\ n \text{ terms}\ \rightarrow}\\ &= a^nb^n\end{aligned}$$

Rule 4

$$(ab)^n = a^nb^n$$

$$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b} \times \frac{a}{b} \times \cdots \frac{a}{b}\\ &= \frac{a^n}{b^n}\end{aligned}$$

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Rule 5

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example 9

Simplify the following, expressing the answers in positive exponent form:

a 8^{-2} **b** $\left(\frac{1}{2}\right)^{-4}$ **c** $\frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}}$ **d** $\frac{3^{2n} \times 6^n}{8^n \times 3^n}$

Solution

a $8^{-2} = \frac{1}{8^2} = \frac{1}{(2^3)^2} = \frac{1}{2^6}$

b $\left(\frac{1}{2}\right)^{-4} = \frac{1}{2^{-4}} = 2^4$

c $\frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}} = \frac{3^{-3} \times 2^4 \times 3^4 \times 3^{-3} \times 2^{-3} \times 2^{-3}}{3^{-4} \times 3^{-4} \times 2^{-2}} = \frac{3^{-2} \times 2^{-2}}{3^{-8} \times 2^{-2}} = 3^6$

d $\frac{3^{2n} \times 6^n}{8^n \times 3^n} = \frac{(3^n \times 3^n) \times (3^n \times 2^n)}{2^{3n} \times 3^n} = \frac{3^n \times 3^n}{2^{2n}} = \left(\frac{3}{2}\right)^{2n}$

Exercise 15B

1 Use the stated rule for each of the following to give an equivalent expression in simplest form:

a $x^2 \times x^3$ **b** $2 \times x^3 \times x^4 \times 4$ Rule 1

c $\frac{x^5}{x^3}$ **d** $\frac{4x^6}{2x^3}$ Rule 2

e $(a^3)^2$ **f** $(2^3)^2$ Rule 3

g $(xy)^2$ **h** $(x^2y^3)^2$ Rule 4 (also use Rule 3 for h)

i $\left(\frac{x}{y}\right)^3$ **j** $\left(\frac{x^3}{y^2}\right)^2$ Rule 5 (also use Rule 3 for j)

2 Simplify the following:

Example 7

a $x^3 \times x^4 \times x^2$ **b** $2^4 \times 4^3 \times 8^2$ **c** $3^4 \times 9^2 \times 27^3$

d $(q^2p)^3 \times (qp^3)^2$ **e** $a^2b^{-3} \times (a^3b^2)^3$ **f** $(2x^3)^2 \times (4x^4)^3$

g $m^3p^2 \times (m^2n^3)^4 \times (p^{-2})^2$ **h** $2^3a^3b^2 \times (2a^{-1}b^2)^{-2}$

3 Simplify the following:

Example 8

a $\dfrac{x^3y^5}{xy^2}$ **b** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$

c $\dfrac{(-2xy)^2 \times 2(x^2y)^3}{8(xy)^3}$ **d** $\dfrac{(-3x^2y^3)^2}{(2xy)^3} \times \dfrac{4x^4y^3}{(xy)^3}$

Example 9

4 Simplify each of the following, expressing your answer in positive exponent form:

a $m^3n^2p^{-2} \times (mn^2p)^{-3}$ **b** $\dfrac{x^3yz^{-2} \times 2(x^3y^{-2}z)^2}{xyz^{-1}}$ **c** $\dfrac{a^2b \times (ab^{-2})^{-3}}{(a^{-2}b^{-1})^{-2}}$

d $\dfrac{a^2b^3c^{-4}}{a^{-1}b^2c^{-3}}$ **e** $\dfrac{a^{2n-1} \times b^3 \times c^{1-n}}{a^{n-3} \times b^{2-n} \times c^{2-2n}}$

5 Simplify each of the following:

a $3^{4n} \times 9^{2n} \times 27^{3n}$ **b** $\dfrac{2^n \times 8^{n+1}}{32^n}$ **c** $\dfrac{3^{n-1} \times 9^{2n-3}}{6^2 \times 3^{n+2}}$

d $\dfrac{2^{2n} \times 9^{2n-1}}{6^{n-1}}$ **e** $\dfrac{25^{2n} \times 5^{n-1}}{5^{2n+1}}$ **f** $\dfrac{6^{x-3} \times 4^x}{3^{x+1}}$

g $\dfrac{6^{2n} \times 9^3}{27^n \times 8^n \times 16^n}$ **h** $\dfrac{3^{n-2} \times 9^{n+1}}{27^{n-1}}$ **i** $\dfrac{8 \times 2^5 \times 3^7}{9 \times 2^7 \times 81}$

6 Simplify and evaluate:

a $\dfrac{(8^3)^4}{(2^{12})^2}$ **b** $\dfrac{(125)^3}{(25)^2}$ **c** $\dfrac{(81)^4 \div 27^3}{9^2}$

15.3 Rational exponents

$a^{\frac{1}{n}}$, where n is a natural number, is defined to be the n^{th} root of a, and denoted $\sqrt[n]{a}$. If $a \geq 0$ then $a^{\frac{1}{n}}$ is defined for all $n \in N$. If $a < 0$ then $\sqrt[n]{a}$ is only defined for n odd. (Remember that only real numbers are being considered.)

$$a^{\frac{1}{n}} = \sqrt[n]{a}, \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

Using this notation for square roots:

$$\sqrt{a} = \sqrt[2]{a} = a^{\frac{1}{2}}$$

Further, the expression a^x can be defined for rational exponents, i.e. when $x = \dfrac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

Example 10

Evaluate:

a $9^{-\frac{1}{2}}$ **b** $16^{\frac{5}{2}}$ **c** $64^{-\frac{2}{3}}$

Solution

a $9^{-\frac{1}{2}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

b $16^{\frac{5}{2}} = \left(16^{\frac{1}{2}}\right)^5 = \left(\sqrt[2]{16}\right)^5 = 4^5 = 1024$

c $64^{-\frac{2}{3}} = \dfrac{1}{64^{\frac{2}{3}}} = \dfrac{1}{\left(64^{\frac{1}{3}}\right)^2} = \dfrac{1}{\left(\sqrt[3]{64}\right)^2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

Earlier we stated the rules:

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(a^m)^n = a^{m \times n}$$

where m and n are integers.

Note: These rules are applicable for all rational exponents.

$$a^{\frac{m}{q}} \times a^{\frac{n}{p}} = a^{\frac{m}{q}+\frac{n}{p}}$$
$$a^{\frac{m}{q}} \div a^{\frac{n}{p}} = a^{\frac{m}{q}-\frac{n}{p}}$$
$$\left(a^{\frac{m}{q}}\right)^{\frac{n}{p}} = a^{\frac{m}{q}\times\frac{n}{p}}$$

Example 11

Simplify:

a $\dfrac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}}$ **b** $(x^{-2}y)^{\frac{1}{2}} \times \left(\dfrac{x}{y^{-3}}\right)^4$

Solution

a
$$\frac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}} = \frac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{\left(16^{\frac{1}{4}}\right)^3}$$
$$= \frac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{2^3} = \frac{3^{\frac{3}{4}} \times 2^{\frac{3}{4}}}{2^3}$$
$$= \frac{3^{\frac{3}{4}}}{2^{\frac{12}{4}-\frac{3}{4}}} = \frac{3^{\frac{3}{4}}}{2^{\frac{9}{4}}}$$

b
$$(x^{-2}y)^{\frac{1}{2}} \times \left(\frac{x}{y^{-3}}\right)^4 = x^{-1}y^{\frac{1}{2}} \times \frac{x^4}{y^{-12}}$$
$$= x^3 \times y^{\frac{25}{2}}$$

Exercise 15C

Example 10

1 Evaluate each of the following:

a $125^{\frac{2}{3}}$ b $243^{\frac{3}{5}}$ c $81^{-\frac{1}{2}}$ d $64^{\frac{2}{3}}$

e $\left(\frac{1}{8}\right)^{\frac{1}{3}}$ f $32^{-\frac{2}{5}}$ g $125^{-\frac{2}{3}}$ h $32^{\frac{4}{5}}$

i $1000^{-\frac{4}{3}}$ j $10\,000^{\frac{3}{4}}$ k $81^{\frac{3}{4}}$ l $\left(\frac{27}{125}\right)^{\frac{1}{3}}$

Example 11

2 Simplify:

a $\sqrt[3]{a^2b} \div \sqrt{ab^3}$ b $(a^{-2}b)^3 \times \left(\frac{1}{b^{-3}}\right)^{\frac{1}{2}}$ c $\dfrac{45^{\frac{1}{3}}}{9^{\frac{3}{4}} \times 15^{\frac{3}{2}}}$

d $2^{\frac{3}{2}} \times 4^{-\frac{1}{4}} \times 16^{-\frac{3}{4}}$ e $\left(\dfrac{x^3y^{-2}}{3^{-3}y^{-3}}\right)^{-2} \div \left(\dfrac{3^{-3}x^{-2}y}{x^4y^{-2}}\right)^2$ f $\left(\sqrt[5]{a^2}\right)^{\frac{3}{2}} \times \left(\sqrt[3]{a^5}\right)^{\frac{1}{5}}$

3 Simplify each of the following:

a $(2x-1)\sqrt{2x-1}$ b $(x-1)^2\sqrt{x-1}$ c $(x^2+1)\sqrt{x^2+1}$

d $(x-1)\sqrt[3]{x-1}$ e $\dfrac{1}{\sqrt{x-1}} + \sqrt{x-1}$ f $(5x^2+1)\sqrt[3]{5x^2+1}$

15.4 Solving exponential equations and inequations

Method 1

Express both sides of the equation as exponents to the same base and then equate the exponents since, if $a^x = a^y$ then $x = y$.

Example 12

Find the value of x for which:

a $4^x = 256$ b $3^{x-1} = 81$ c $5^{2x-4} = 25^{-x+2}$

Solution

a
$$\begin{aligned} 4^x &= 256 \\ 4^x &= 4^4 \\ \therefore x &= 4 \end{aligned}$$

b
$$\begin{aligned} 3^{x-1} &= 81 \\ 3^{x-1} &= 3^4 \\ x-1 &= 4 \\ \therefore x &= 5 \end{aligned}$$

c
$$\begin{aligned} 5^{2x-4} &= 25^{-x+2} \\ &= (5^2)^{-x+2} \\ &= 5^{-2x+4} \\ \therefore 2x-4 &= -2x+4 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

Example 13

Solve $9^x = 12 \times 3^x - 27$.

Solution

$$(3^x)^2 = 12 \times 3^x - 27$$

Let $y = 3^x$

Then $y^2 = 12y - 27$

$$y^2 - 12y + 27 = 0$$

$$(y-3)(y-9) = 0$$

Therefore $y - 3 = 0$ or $y - 9 = 0$

$y = 3$ or $y = 9$

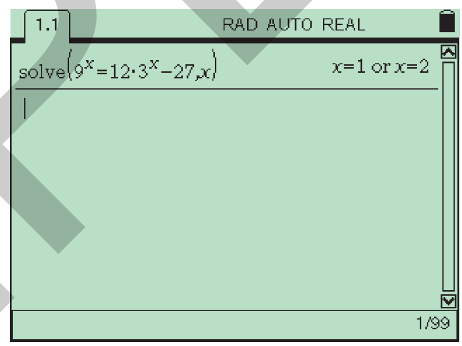
Hence $3^x = 3^1$ or $3^x = 3^2$

and $x = 1$ or $x = 2$

Using the TI-Nspire

Solution to Example 13

Use **solve( )** from the **Algebra** menu (menu 3 1).



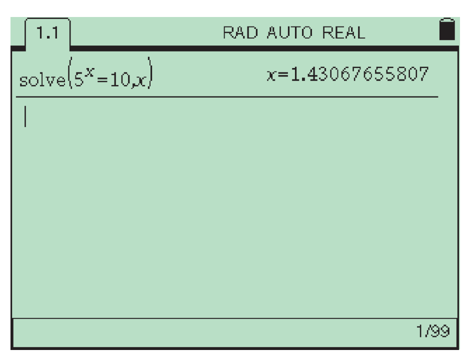
Example 14

Solve $5^x = 10$ correct to 2 decimal places.

Solution

Press ctrl enter to obtain the answer as a decimal number.

$x = 1.43$ (correct to 2 decimal places)



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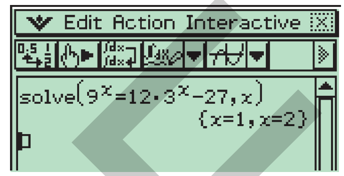
Using the Casio ClassPad

Solution to Example 13

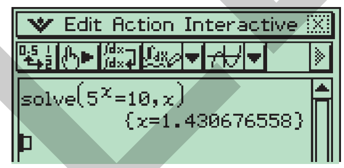
Enter and highlight the equation in Main, then select **Interactive—Equation/inequality—solve** and set the variable as x.



Solve $5^x = 10$ correct to 2 decimal places.

Solution

To answer the question as required, you may need to highlight the answer and click 0.5⇔½ to convert from exact to decimal approximation.



$x = 1.43$ (correct to 2 decimal places)

Solution of inequalities

The property that $a^x > a^y \Leftrightarrow x > y$, where $a \in (1, \infty)$, and $a^x > a^y \Leftrightarrow x < y$ when $a \in (0, 1)$ is used.

Example 15

Solve for x in each of the following:

a $16^x > 2$ **b** $2^{-3x+1} < \dfrac{1}{16}$

Solution

a
$$\begin{aligned} 2^{4x} &> 2^1 \\ \Leftrightarrow 4x &> 1 \\ \Leftrightarrow x &> \frac{1}{4} \end{aligned}$$

b
$$\begin{aligned} 2^{-3x+1} &< 2^{-4} \\ \Leftrightarrow -3x + 1 &< -4 \\ \Leftrightarrow -3x &< -5 \\ \Leftrightarrow x &> \frac{5}{3} \end{aligned}$$

Note: The CAS calculator can be used to help 'visualise' the inequality. Plot the graph of $y = 16^x$ and find x by using **intersect** to solve $16^x > 2$.

Exercise 15D

Example 12

1 Solve for x in each of the following:

a $3^x = 27$ **b** $4^x = 64$ **c** $49^x = 7$
d $16^x = 8$ **e** $125^x = 5$ **f** $5^x = 625$
g $16^x = 256$ **h** $4^{-x} = \dfrac{1}{64}$ **i** $5^{-x} = \dfrac{1}{125}$

2 Solve for n in each of the following:

a $5^n \times 25^{2n-1} = 125$ **b** $3^{2n-4} = 1$ **c** $3^{2n-1} = \frac{1}{81}$

d $\frac{3^{n-2}}{9^{1-n}} = 1$ **e** $3^{3n} \times 9^{-2n+1} = 27$ **f** $2^{-3n} \times 4^{2n-2} = 16$

g $2^{n-6} = 8^{2-n}$ **h** $9^{3n+3} = 27^{n-2}$ **i** $4^{n+1} = 8^{n-2}$

j $32^{2n+1} = 8^{4n-1}$ **k** $25^{n+1} = 5 \times 390\,625$

l $125^{4-n} = 5^{6-2n}$ **m** $4^{2-n} = \frac{1}{2048}$

3 Solve the following exponential equations:

a $2^{x-1} \times 4^{2x+1} = 32$ **b** $3^{2x-1} \times 9^x = 243$ **c** $(27 \times 3^x)^2 = 27^x \times 3^{\frac{1}{2}}$

Example 13 **4** Solve for x:

a $4(2^{2x}) = 8(2^x) - 4$ **b** $8(2^{2x}) - 10(2^x) + 2 = 0$

c $3 \times 2^{2x} - 18(2^x) + 24 = 0$ **d** $9^x - 4(3^x) + 3 = 0$

Example 14 **5** Use a calculator to solve each of the following, correct to 2 decimal places:

a $2^x = 5$ **b** $4^x = 6$ **c** $10^x = 18$ **d** $10^x = 56$

Example 15 **6** Solve for x in each of the following:

a $7^x > 49$ **b** $8^x > 2$ **c** $25^x \leq 5$ **d** $3^{x+1} < 81$

e $9^{2x+1} < 243$ **f** $4^{2x+1} > 64$ **g** $3^{2x-2} \leq 81$

15.5 Logarithms

Consider the statement $2^3 = 8$.

This may be written in an alternative form:

$$\log_2 8 = 3$$

which is read as 'the logarithm of 8 to the base 2 is equal to 3'.

In general, if $a \in R^+ \setminus \{1\}$ and $x \in R$ then the statements $a^x = n$ and $\log_a n = x$ are equivalent.

$$a^x = y \text{ is equivalent to } \log_a y = x$$

Further examples:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$

Example 16

Without the aid of a calculator evaluate the following:

a $\log_2 32$ **b** $\log_3 81$

Solution

a Let $\log_2 32 = x$

$$2^x = 32$$
$$2^x = 2^5$$

Therefore $x = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$

$$3^x = 81$$
$$3^x = 3^4$$

Therefore $x = 4$, giving $\log_3 81 = 4$.

Laws of logarithms

The index rules are used to establish other rules for computations with logarithms.

1 Let $a^x = m$ and $a^y = n$, where m, n and a are positive real numbers.

$$\begin{aligned} \therefore \quad mn &= a^x \times a^y \\ &= a^{x+y} \end{aligned}$$

$\log_a mn = x + y$ and, since $x = \log_a m$ and $y = \log_a n$, it follows that:

$$\log_a mn = \log_a m + \log_a n \qquad \textbf{Rule 1}$$

For example:

$$\begin{aligned} \log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5) \\ &= \log_{10} 1000 \\ &= 3 \end{aligned}$$

2 $\dfrac{m}{n} = \dfrac{a^x}{a^y} = a^{x-y}$

$\therefore \log_a\left(\dfrac{m}{n}\right) = x - y$ and so:

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n \qquad \textbf{Rule 2}$$

For example:

$$\begin{aligned} \log_2 32 - \log_2 8 &= \log_2\left(\frac{32}{8}\right) \\ &= \log_2 4 \\ &= 2 \end{aligned}$$

3 If $m = 1$, $\log_a\left(\dfrac{1}{n}\right) = \log_a 1 - \log_a n$

$$= -\log_a n$$

Therefore

$$\log_a\left(\frac{1}{n}\right) = -\log_a n \qquad \textbf{Rule 3}$$

4
$$\begin{aligned} m^p &= (a^x)^p \\ &= a^{xp} \end{aligned}$$

$\therefore \log_a(m^p) = xp$

and so

$$\log_a(m^p) = p\log_a m \qquad \textbf{Rule 4}$$

For example:

$$3\log_3 4 = \log_3(4^3) = \log_3 64$$

Example 17

Without using a calculator simplify the following:

$$2\log_{10} 3 + \log_{10} 16 - 2\log_{10}\left(\frac{6}{5}\right)$$

Solution

$$\begin{aligned}2\log_{10} 3 + \log_{10} 16 - 2\log_{10}\left(\frac{6}{5}\right) &= \log_{10} 3^2 + \log_{10} 16 - \log_{10}\left(\frac{6}{5}\right)^2\\ &= \log_{10} 9 + \log_{10} 16 - \log_{10}\left(\frac{36}{25}\right)\\ &= \log_{10}\left(9 \times 16 \times \frac{25}{36}\right)\\ &= \log_{10} 100\\ &= 2\end{aligned}$$

Example 18

Solve each of the following equations for x:

a $\log_5 x = 3$ **b** $\log_5 (2x + 1) = 2$

c $\log_2 (2x + 1) - \log_2 (x - 1) = 4$ **d** $\log_3 (x - 1) + \log_3 (x + 1) = 1$

Solution

a $\log_5 x = 3 \Leftrightarrow x = 5^3 = 125$

b $\log_5 (2x + 1) = 2 \Leftrightarrow 2x + 1 = 5^2$

$$\begin{aligned}\therefore \quad 2x + 1 &= 25\\ 2x &= 24\\ x &= 12\end{aligned}$$

c $\log_2(2x+1) - \log_2(x-1) = 4$

Then $\log_2\left(\dfrac{2x+1}{x-1}\right) = 4$

$\Leftrightarrow$ $\dfrac{2x+1}{x-1} = 2^4$

$2x+1 = 16(x-1)$

Then $17 = 14x$

Therefore $\dfrac{17}{14} = x$

d $\log_3(x-1) + \log_3(x+1) = 1$

Therefore $\log_3[(x-1)(x+1)] = 1$

which implies $x^2 - 1 = 3$ and $x = \pm 2$

But the expression is not defined for $x = -2$, therefore $x = 2$.

Skillsheet TEACHER

Exercise 15E

1 Use the stated rule for each of the following to give an equivalent expression in simplest form:

a $\log_2 10 + \log_2 a$	**b** $\log_{10} 5 + \log_{10} 2$	Rule 1
c $\log_2 9 - \log_2 4$	**d** $\log_2 10 - \log_2 5$	Rule 2
e $\log_5\left(\frac{1}{6}\right)$	**f** $\log_5\left(\frac{1}{25}\right)$	Rule 3
g $\log_2(a^3)$	**h** $\log_2(8^3)$	Rule 4

Example 16 **2** Without using a calculator evaluate each of the following:

a $\log_3 27$ **b** $\log_5 625$ **c** $\log_2\left(\frac{1}{128}\right)$

d $\log_4\left(\frac{1}{64}\right)$ **e** $\log_x x^4$ **f** $\log_2 0.125$

g $\log_{10} 10\,000$ **h** $\log_{10} 0.000\,001$ **i** $-3\log_5 125$

j $-4\log_{16} 2$ **k** $2\log_3 9$ **l** $-4\log_{16} 4$

Example 17 **3** Without using a calculator simplify each of the following:

a $\frac{1}{2}\log_{10} 16 + 2\log_{10} 5$ **b** $\log_2 16 + \log_2 8$

c $\log_2 128 + \log_3 45 - \log_3 5$ **d** $\log_4 32 - \log_9 27$

e $\log_b b^3 - \log_b \sqrt{b}$ **f** $2\log_x a + \log_x a^3$

g $x\log_2 8 + \log_2 8^{1-x}$ **h** $\frac{3}{2}\log_a a - \log_a \sqrt{a}$

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Example 18

4 Solve for x:

a $\log_3 9 = x$
b $\log_3 x = 3$
c $\log_5 x = -3$
d $\log_{10} x = \log_{10} 4 + \log_{10} 2$
e $\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 = 2$
f $\log_{10} x = \frac{1}{2}\log_{10} 36 - 2\log_{10} 3$
g $\log_x 64 = 2$
h $\log_5 (2x - 3) = 3$
i $\log_3 (x + 2) - \log_3 2 = 1$
j $\log_x 0.01 = -2$

5 Solve each of the following for x.

a $\log_x\left(\frac{1}{25}\right) = -2$
b $\log_4 (2x - 1) = 3$
c $\log_4 (x + 2) - \log_4 6 = 1$
d $\log_4 (3x + 4) + \log_4 16 = 5$
e $\log_3 (x^2 - 3x - 1) = 0$
f $\log_3 (x^2 - 3x + 1) = 0$

6 If $\log_{10} x = a$ and $\log_{10} y = c$, express $\log_{10}\left(\frac{100x^3y^{-\frac{1}{2}}}{y^2}\right)$ in terms of a and c.

7 Prove that $\log_{10}\left(\frac{ab^2}{c}\right) + \log_{10}\left(\frac{c^2}{ab}\right) - \log_{10}(bc) = 0$.

8 If $\log_a\left(\frac{11}{3}\right) + \log_a\left(\frac{490}{297}\right) - 2\log_a\left(\frac{7}{9}\right) = \log_a(k)$, find k.

9 Solve each of the following equations for x:

a $\log_{10} (x^2 - 2x + 8) = 2\log_{10} x$
b $\log_{10} (5x) - \log_{10} (3 - 2x) = 1$
c $3\log_{10} (x - 1) = \log_{10} 8$
d $\log_{10} (20x) - \log_{10} (x - 8) = 2$
e $2\log_{10} 5 + \log_{10} (x + 1) = 1 + \log_{10} (2x + 7)$
f $1 + 2\log_{10} (x + 1) = \log_{10} (2x + 1) + \log_{10} (5x + 8)$

15.6 Using logarithms to solve exponential equations and inequations

In Section 15.4 two methods were shown for solving exponential equations.

If $a \in R^+ \setminus \{1\}$ and $x \in R$ then the statements $a^x = n$ and $\log_a n = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations.

Method 3—Using logarithms

Example 19

Solve for x if $2^x = 11$.

Solution

Take either the $\log_{10}$ or $\log_e$ (since these are the only logarithmic functions available on your calculator) of both sides of the equation:

Then $\log_{10} 2^x = \log_{10} 11$

i.e. $x \log_{10} 2 = \log_{10} 11$

Therefore $x = \dfrac{\log_{10} 11}{\log_{10} 2}$

Use your calculator to evaluate both the $\log_{10} 11$ and $\log_{10} 2$.

Then $x = \dfrac{1.041}{0.301}$ (to 3 decimal places)

≈ 3.46 (to 2 decimal places)

Example 20

Solve $3^{2x-1} = 28$.

Solution

$$\begin{aligned}
\log_{10} 3^{2x-1} &= \log_{10} 28 \\
(2x-1)\log_{10} 3 &= \log_{10} 28 \\
2x - 1 &= \frac{\log_{10} 28}{\log_{10} 3} \\
&= \frac{1.4472}{0.4771} \\
& 3.0331 \\
\therefore 2x &= 4.0331 \\
x &= 2.017 \quad \text{(to 3 decimal places)}
\end{aligned}$$

Example 21

Solve $\{x: 0.7^x \geq 0.3\}$.

Solution

Taking $\log_{10}$ of both sides:

$$\begin{aligned}
\log_{10} 0.7^x &\geq \log_{10} 0.3 \\
x \log_{10} 0.7 &\geq \log_{10} 0.3 \\
x &\leq \frac{\log_{10} 0.3}{\log_{10} 0.7} \quad \text{Note change of direction of inequality as } \log_{10} 0.7 < 0. \\
x &\leq \frac{-0.5229}{-0.1549} \\
x &\leq 3.376
\end{aligned}$$

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Example 22

Sketch the graph of $f(x) = 2 \times 10^x - 4$, giving the equation of the asymptote and the axes intercepts.

Solution

$f(0) = 2 \times 10^0 - 4 = 2 - 4 = -2$

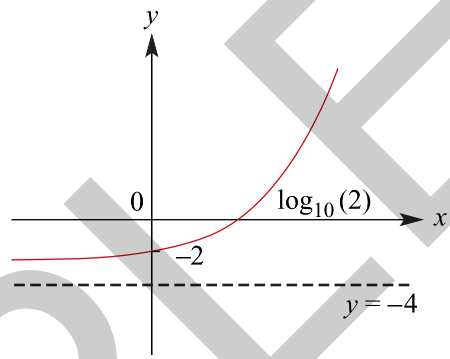
As $x \to -\infty$, $y \to -4$ from the positive side, or as $x \to -\infty$, $y \to -4^+$.

The equation of the horizontal asymptote is $y = -4$.

For the x-axis intercept consider $2 \times 10^x - 4 = 0$.

Hence $2 \times 10^x = 4$ and thus $10^x = 2$.

Therefore $x = \log_{10} 2 = 0.3010$ (correct to 4 decimal places)



Exercise 15F

1 Solve each of the following equations correct to 2 decimal places:

Example 20

a $2^x = 7$ **b** $2^x = 0.4$ **c** $3^x = 14$
d $4^x = 3$ **e** $2^{-x} = 6$ **f** $0.3^x = 2$
g $5^x = 3^{x-2}$ **h** $8^x = 2005^{x+1}$ **i** $3^{x-1} = 10$
j $0.2^{x+1} = 0.6$

Example 21

2 Solve for x. Give values correct to 2 decimal places if necessary:

a $2^x > 8$ **b** $3^x < 5$ **c** $0.3^x > 4$
d $3^{x-1} \leq 7$ **e** $0.4^x \leq 0.3$

Example 22

3 For each of the following sketch the graph of $y = f(x)$, giving the equation of the asymptote and the axes intercepts:

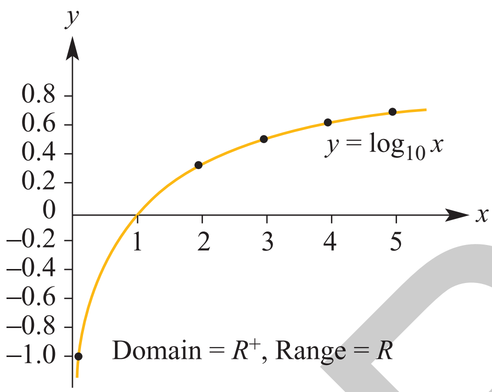
a $f(x) = 2^x - 4$ **b** $f(x) = 2 \times 3^x - 6$ **c** $f(x) = 3 \times 10^x - 5$
d $f(x) = -2 \times 10^x + 4$ **e** $f(x) = -3 \times 2^x + 6$ **f** $f(x) = 5 \times 2^x - 6$

4 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$, where d is the diameter of the tree, in cm, t years after the beginning of this period. The diameter is 52 cm after 1 year and 80 cm after 3 years. Calculate the values of the constants d_0 and m.

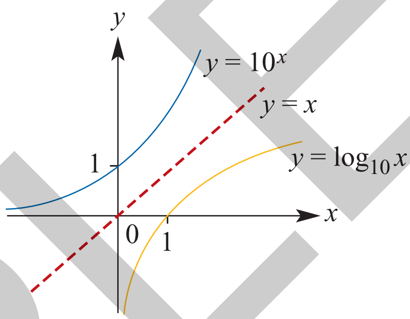
15.7 Graph of $y = \log_a x$, where $a > 1$

A table of values for $y = \log_{10} x$ is given below (the values are correct to 2 decimal places). Use your calculator to check these values.

x	0.1	1	2	3	4	5
$y = \log_{10} x$	-1	0	0.30	0.48	0.60	0.70



Domain = R^+, Range = R



The graph of $y = \log_{10} x$ is the reflection in the line $y = x$ of the graph of $y = 10^x$.

Note that $\log_{10} 1 = 0$ as $10^0 = 1$ and as $x \to 0^+$, $y \to -\infty$.

The inverse of a one-to-one function was introduced in Section 6.7.

The function $f^{-1}: (0, \infty) \to R,\ f^{-1}(x) = \log_{10} x$

is the inverse function of $f: R \to R,\quad f(x) = 10^x$

In general, $y = \log_a x$ is the rule for the inverse of the function with rule $y = a^x$, where $a > 0$ and $x > 0$.

Note that $\log_a (a^x) = x$ for all x and $a^{\log_a x} = x$ for positive values of x, as they are inverse functions.

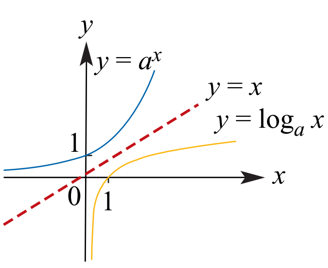
For $a > 1$, the graphs of logarithm functions all have this same shape.

Properties of $y = a^x$, $a > 1$

- domain $= R$
- range $= R^+$
- $a^0 = 1$
- $x \to -\infty$, $y \to 0^+$

Properties of $y = \log_a x$, $a > 1$

- domain $= R^+$
- range $= R$
- $\log_a 1 = 0$
- $x \to 0^+$, $y \to -\infty$



Example 23

Find the inverse of:

a $y = 10^{2x}$ **b** $y = \log_{10} (2x)$

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Solution

a $y = 10^{2x}$

Interchanging x and y gives

$$x = 10^{2y}$$

Therefore $2y = \log_{10} x$

and $y = \frac{1}{2}\log_{10} x$

b $y = \log_{10}(2x)$

Interchanging x and y gives

$$x = \log_{10}(2y)$$

Therfore $10^x = 2y$

and $y = \frac{1}{2} \times 10^x$

Example 24

Find the inverse of each of the following:

a $f(x) = 2^x + 3$ **b** $f(x) = \log_2(x - 2)$ **c** $f(x) = 5 \times 2^x + 3$

Solution

a Let $y = 2^x + 3$

Interchanging x and y gives

$$x = 2^y + 3$$

$\therefore$ $x - 3 = 2^y$

and $y = \log_2(x - 3)$

$\therefore$ $f^{-1}(x) = \log_2(x - 3)$

domain of $f^{-1} = (3, \infty)$

b Let $y = \log_2(x - 2)$

Interchanging x and y gives

$$x = \log_2(y - 2)$$

$\therefore$ $2^x = y - 2$

and $y = 2^x + 2$

$\therefore$ $f^{-1}(x) = 2^x + 2$

domain of $f^{-1} = R$

c Let $y = 5 \times 2^x + 3$

Interchanging x and y gives

$$x = 5 \times 2^y + 3$$

$\therefore$ $\frac{x-3}{5} = 2^y$

and $y = \log_2\left(\frac{x-3}{5}\right)$

$\therefore$ $f^{-1}(x) = \log_2\left(\frac{x-3}{5}\right)$

domain of $f^{-1} = (3, \infty)$

Transformations can be applied to the graphs of logarithm functions. This is shown in the following example.

Example 25

Sketch the graphs of each of the following. Give the maximal domain, the equation of the asymptote and the axes intercepts.

a $f(x) = \log_2(x - 3)$ **b** $f(x) = \log_2(x + 2)$ **c** $f(x) = \log_2(3x)$

Solution

a $f(x) = \log_2 (x - 3)$

For the function to be defined

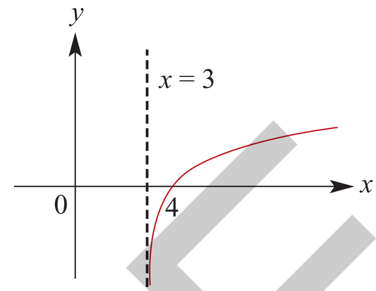
$x - 3 > 0$, i.e. $x > 3$

The maximal domain is $(3, \infty)$

x-axis intercept: $\log_2 (x - 3) = 0$

implies $x - 3 = 2^0$, i.e. $x = 4$

Asymptote: $x \to 3^+, y \to -\infty$



b $f(x) = \log_2 (x + 2)$

For the function to be defined

$x + 2 > 0$, i.e. $x > -2$

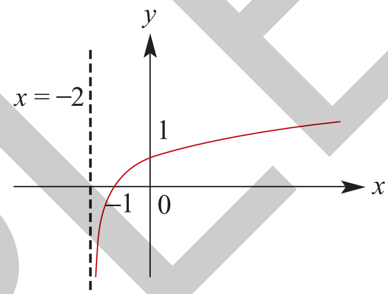
The maximal domain is $(-2, \infty)$

$f(0) = \log_2 (2) = 1$

x-axis intercept: $\log_2 (x + 2) = 0$

implies $x + 2 = 2^0$, i.e. $x = -1$

Asymptote: $x \to -2^+, y \to -\infty$



c $f(x) = \log_2 3x$

For the function to be defined

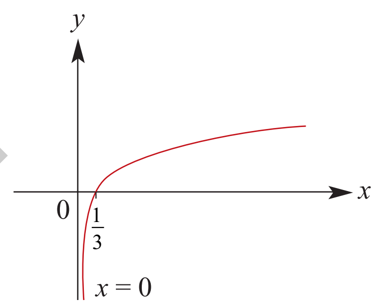
$3x > 0$, i.e. $x > 0$

The maximal domain is $(0, \infty)$

For the x-axis intercept $\log_2 (3x) = 0$

implies $3x = 2^0$, i.e. $x = \frac{1}{3}$

Asymptote: $x \to 0^+, y \to -\infty$



Exercise 15G

1 Sketch the graph of each of the following and state the domain and range for each:

a $y = \log_{10} (2x)$ **b** $y = 2 \log_{10} x$ **c** $y = \log_{10} \left(\frac{1}{2}x\right)$

d $y = 2 \log_{10} (3x)$ **e** $y = -\log_{10} x$ **f** $y = \log_{10} (-x)$

Example 23 2 Determine the inverse of each of the following:

a $y = 10^{0.5x}$ **b** $y = 3 \log_{10} x$ **c** $y = 10^{3x}$

d $y = 2 \log_{10} (3x)$

Example 24 3 Find the rule for the inverse function of each of the following:

a $f(x) = 3^x + 2$ **b** $f(x) = \log_2 (x - 3)$ **c** $f(x) = 4 \times 3^x + 2$

d $f(x) = 5^x - 2$ **e** $f(x) = \log_2 (3x)$ **f** $f(x) = \log_2 \left(\frac{x}{3}\right)$

g $f(x) = \log_2 (x + 3)$ **h** $f(x) = 5 \times 3^x - 2$

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Example 25

4 Sketch the graphs of each of the following. Give the maximal domain, the equation of the asymptote and the axes intercepts:

a $f(x) = \log_2 (x - 4)$ **b** $f(x) = \log_2 (x + 3)$ **c** $f(x) = \log_2 (2x)$

d $f(x) = \log_2 (x + 2)$ **e** $f(x) = \log_2 \left(\frac{x}{3}\right)$ **f** $f(x) = \log_2 (-2x)$

5 Use a calculator to solve each of the following equations correct to 2 decimal places:

a $2^{-x} = x$ **b** $\log_{10} (x) + x = 0$

6 Use a calculator to plot the graphs of $y = \log_{10} (x^2)$ and $y = 2 \log_{10} x$ for $x \in [-10, 10], x \neq 0$.

7 On the same set of axes plot the graph of $y = \log_{10} (\sqrt{x})$ and $y = \frac{1}{2} \log_{10} x$ for $x \in (0, 10]$.

8 Use a calculator to plot the graphs of $y = \log_{10} (2x) + \log_{10} (3x)$ and $y = \log_{10} (6x^2)$

9 Find a and k such that the graph of $y = a10^{kx}$ passes through the points (2, 6) and (5, 20).

15.8 Exponential models and applications

Fitting data

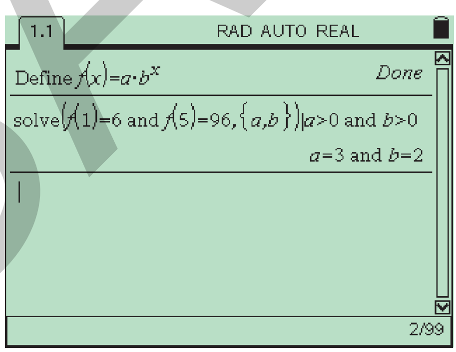
Using a TI-Nspire calculator

It is known that the points (1, 6) and (5, 96) lie on a curve with equation $y = a \times b^x$.

Define $f(x) = a \times b^x$.

Then use **solve(**$f(1) = 6$ and $f(5) = 96, \{a, b\})|a > 0$ and $b > 0$.

The result is as shown.



The equations can also be solved for a and b ‘by hand’:

$$a \times b^1 = 6 \quad \text{and} \quad a \times b^5 = 96$$

Divide the second equation by the first to obtain $b^4 = 16$. Hence $b = 2$. Substitute in the first equation to obtain $a = 3$.

There are many practical situations in which the relationship between variables is exponential.

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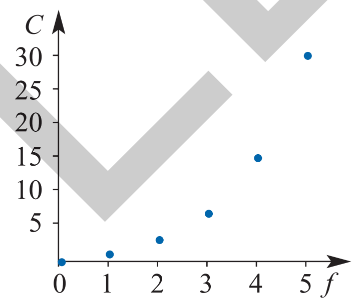
Example 26

Take a rectangular piece of paper approximately 30 cm × 6 cm. Fold the paper in half, successively, until you have folded it five times. Tabulate the times folded, f, and the number of creases in the paper, C.

Solution

Times folded, f	0	1	2	3	4	5
Creases, C	0	1	3	7	15	31

The rule connecting C and f is
$C = 2^f - 1,\ f \in N \cup \{0\}$.



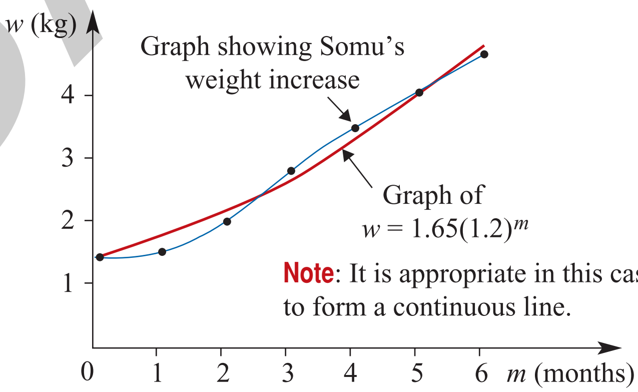
Example 27

The table below shows the increase in weight of Somu, an orang-utan born at the Eastern Plains Zoo. Draw a graph to show Somu's weight increase for the first six months.

Months, m	0	1	2	3	4	5	6
Weight, w kg	1.65	1.7	2.2	3.0	3.7	4.2	4.8

Solution

Plotting these values:



Note: It is appropriate in this case to form a continuous line.

We shall now plot, on the same set of axes, the graph of the exponential function $w = 1.65(1.2)^m$, $0 \le m \le 6$.

Table of values

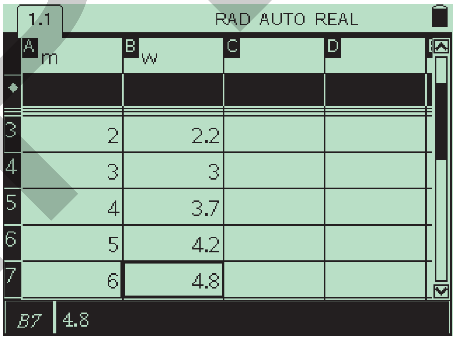
m	0	1	2	3	4	5	6
w	1.65	1.98	2.38	2.85	3.42	4.1	4.93

It can be seen from the graphs that the exponential model $w = 1.65(1.2)^m$ approximates to the actual weight gain and would be a useful model to predict weight gains for any future orang-utan births at the zoo. This model describes a growth rate for the first 6 months of 20% per month.

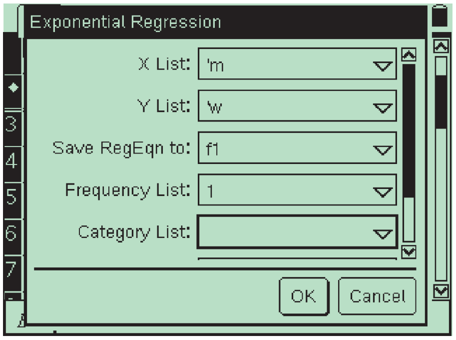
This problem can also be attempted with a CAS calculator.

Using the TI-Nspire

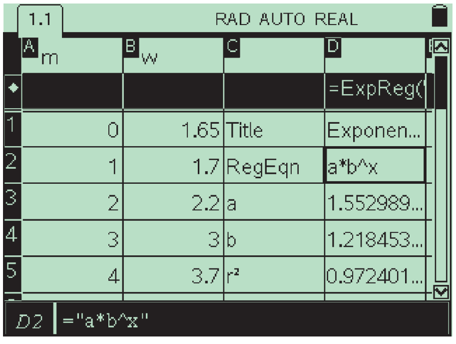
Enter the data in either a **Calculator** application as lists or in a **Lists & Spreadsheet** application as shown.



Choose **Exponential Regression** (menu 4 1 A) from the list of available regressions and complete as shown.
Use the **tab** key (tab) to move between the cells, use the **selection tool** ( ) to open a cell, and use the **up/down arrows** (▲ ▼) to move to the entry to be selected. Select this entry using the **selection tool** ( ).



This now gives the values of a and b, and the equation has been entered in $f_1(x)$.



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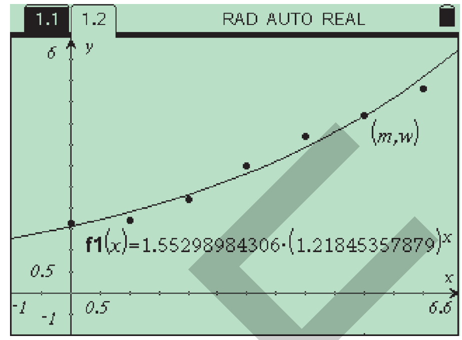
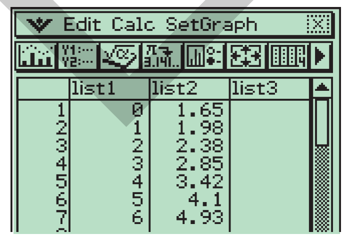
The curve can be shown in a **Graphs & Geometry** application together with the **Scatter Plot** (menu 3 4) using an appropriate **Window** (menu 4).



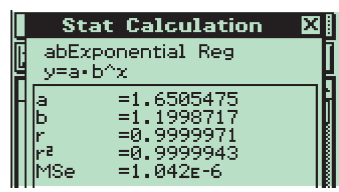
Using the Casio ClassPad

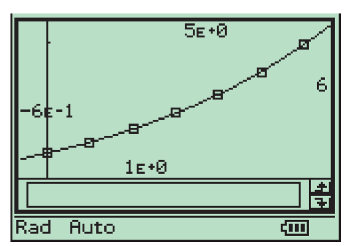
In Statistics enter the data in list 1 and list 2 as shown.

In **SetGraph—Setting** ensure that you set the graph to Scatter and the lists to list 1 and list 2.



Now select **Calc—abExponential Reg** and choose the settings shown.





Example 28

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let P = population of grey kangaroos at the start.

$\therefore$ number of grey kangaroos after n years $= P(1.11)^n$,

and number of red kangaroos after n years $= 10P(0.95)^n$.

When the proportions are reversed

$$\begin{aligned} P(1.11)^n &= 10 \times [10P(0.95)^n] \\ (1.11)^n &= 100(0.95)^n \end{aligned}$$

Taking $\log_{10}$ of both sides

$$\begin{aligned} \log_{10}(1.11)^n &= \log_{10} 100(0.95)^n \\ n \log_{10} 1.11 &= \log_{10} 100 + n \log_{10} 0.95 \\ n \times 0.04532 &= 2 + n(-0.0223) \\ n &= \frac{2}{0.0676} \\ &= 29.6 \end{aligned}$$

i.e. the proportions of kangaroo populations will be reversed by the 30th year.

Exponential growth

The above two examples are examples of exponential change. In the following, A is a variable that is subject to exponential change.

Let A be the quantity at time t. Then $A = A_0 a^t$, where A_0 is a positive constant and a is a real number.

If $a > 1$ the model represents **growth**. If $a < 1$ the model represents **decay**.

Physical situations in which this is applicable include:

- the growth of cells
- population growth
- continuously compounded interest
- radioactive decay
- cooling of materials.

Consider a sum of money, \$10 000, invested at a rate of 5% per annum but compounded continually. That is it is compounded at every instant.

If there are n compound periods in a year, the interest paid per period is $\frac{5}{n}$%. Therefore at the end of the year the amount of the investment, A, is

$$A = 10\,000\left(1 + \frac{5}{100n}\right)^n = 10\,000\left(1 + \frac{1}{20n}\right)^n$$

Enter the function Y1 = (1 + 1/(20X))ˆX. Look at the table of values with an increment of one. It is found that the value approaches 1.05127 (correct to 5 decimal places) for n large. Hence, for continuous compounding it can be written that $A = 10\,000 \times (1.05126\ldots)^x$, where x is the number of years of the investment.

Exercise 15H

Example 27

1 Find an exponential model of the form $y = ab^x$ to fit the following data.

x	0	2	4	5	10
y	1.5	0.5	0.17	0.09	0.006

2 Find an exponential model of the form $p = ab^t$ to fit the following data

t	0	2	4	6	8
p	2.5	4.56	8.3	15.12	27.56

3 A sheet of paper 0.2 mm thick is cut in half and one piece is stacked on top of the other.

a If this process is repeated complete the following table:

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	⋮
⋮	⋮	⋮
10		

b Write down a formula which shows the relationship between T and n.

c Draw a graph of T against n for $n \le 10$.

d What would be the total thickness, T, after 30 cuts?

CAS CALCULATOR

4 The populations (in millions), p and q, of two neighbouring American states, P and Q, over a period of 50 years from 1950 are modelled by functions $p = 1.2 \times 2^{0.08t}$ and $q = 1.7 \times 2^{0.04t}$, where t is the number of years since 1950.

a Plot the graphs of both functions using a calculator.

b Find when the population of state P is:

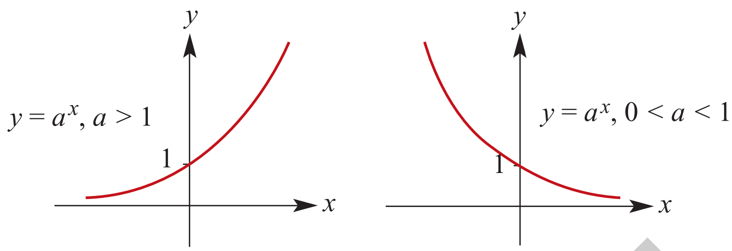
i equal to the population of state Q

ii twice the population of state Q.

MC TEST D&D TEST

Chapter summary

- Groups of exponential functions
  Examples:



- To **multiply** two numbers in exponent form with the same base, **add** the exponents:
  $a^m \times a^n = a^{m+n}$
- To **divide** two numbers in exponent form with the same base, **subtract** the exponents:
  $a^m \div a^n = a^{m-n}$
- To raise the power of a to another power, **multiply** the exponents:
  $(a^m)^n = a^{m \times n}$
- If $a^x = a^y$ then $x = y$.
- For rational exponents: $a^{\frac{1}{n}} = \sqrt[n]{a}$
  and $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$
- For $a \in R^+ \setminus \{1\}$ and $x \in R$: $a^x = y$ is equivalent to $\log_a y = x$
- **Laws of logarithms**
  1 $\log_a (mn) = \log_a m + \log_a n$
  2 $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$
  3 $\log_a \left(\frac{1}{n}\right) = -\log_a n$
  4 $\log_a (m^p) = p \log_a m$
- Exponential equations can be solved by taking logarithms of both sides.
  e.g. If $2^x = 11$
  then $\log_{10} 2^x = \log_{10} 11$ and hence x can be solved.

Multiple-choice questions

1 $8x^3 \div 4x^{-3} =$

A 2 B $2x^0$ C $2x^6$ D $2x^{-1}$ E $\dfrac{2}{x^9}$

2 The expression $\dfrac{a^2b}{(2ab^2)^3} \div \dfrac{ab}{16a^0} =$

A $\dfrac{2}{a^2b^6}$ B $\dfrac{2a^2}{b^6}$ C $2a^2b^6$ D $\dfrac{2}{ab^6}$ E $\dfrac{1}{128ab^5}$

3 The function $f: R \to R, f(x) = 3 \times 2^x - 1$ has range

A R B $R \setminus \{-1\}$ C $(-1, \infty)$ D $(1, \infty)$ E $[1, \infty)$

4 The function $f: R^+ \to R$, where $f(x) = \log_2 3x$, has an inverse function f^{-1}. The rule for f^{-1} is given by

A $f^{-1}(x) = 2^x$ **B** $f^{-1}(x) = 3^x$ **C** $f^{-1}(x) = \frac{1}{3}2^x$

D $f^{-1}(x) = 2^{\frac{x}{3}}$ **E** $f^{-1}(x) = \log_2\left(\frac{1}{3}x\right)$

5 If $\log_{10}(x-2) - 3\log_{10} 2x = 1 - \log_{10} y$, then y is equal to

A $\frac{80x^3}{x-2}$ **B** $1 + \frac{8x^3}{x-2}$ **C** $\frac{60x}{x-2}$

D $1 + \frac{6x}{x-2}$ **E** $1 - \frac{x-2}{8x^3}$

6 The solution of the equation $5 \times 2^{5x} = 10$ is x equals

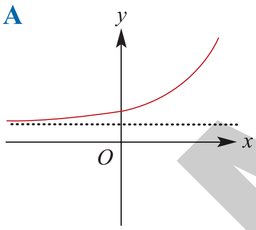
A $\frac{1}{2}$ **B** $\frac{1}{5}$ **C** $\frac{1}{5}\log_2 10$ **D** $\frac{1}{2}\log_2 5$ **E** $\frac{1}{5}2^5$

7 The equation of the asymptote of $y = 3\log_2(5x) + 2$ is

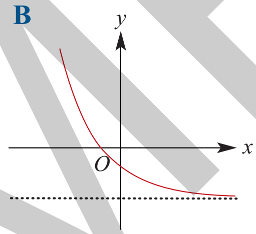
A $x = 0$ **B** $x = 2$ **C** $x = 3$ **D** $x = 5$ **E** $y = 2$

8 Which of the following graphs could be the graph of the function $f(x) = 2^{ax} + b$, where a and b are positive?

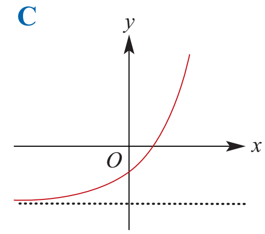
A



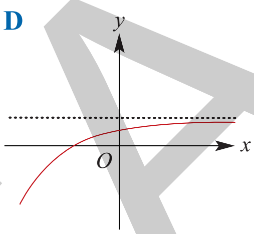
B



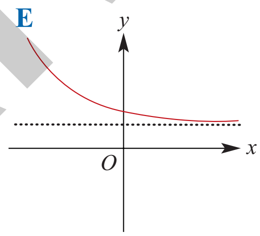
C



D



E



9 Which one of the following functions has a graph with a vertical asymptote with equation $x = b$?

A $y = \log_2(x - b)$ **B** $y = \frac{1}{x+b}$ **C** $y = \frac{1}{x+b} - b$

D $y = 2^x + b$ **E** $y = 2^{x-b}$

10 The expression $\frac{2mh}{(3mh^2)^3} \div \frac{mh}{81m^2}$ is equal to

A $\frac{6}{mh^6}$ **B** $\frac{6m^2}{h^6}$ **C** $6m^2h^6$ **D** $\frac{6}{m^2h^6}$ **E** $\frac{1}{128mh^5}$

Short-answer questions (technology-free)

1 Simplify each of the following, expressing your answer with positive index:

a $\dfrac{a^6}{a^2}$ **b** $\dfrac{b^8}{b^{10}}$ **c** $\dfrac{m^3n^4}{m^5n^6}$ **d** $\dfrac{a^3b^2}{(ab^2)^4}$

e $\dfrac{6a^8}{4a^2}$ **f** $\dfrac{10a^7}{6a^9}$ **g** $\dfrac{8(a^3)^2}{(2a)^3}$ **h** $\dfrac{m^{-1}n^2}{(mn^{-2})^3}$

i $(p^{-1}q^{-2})^2$ **j** $\dfrac{(2a^{-4})^3}{5a^{-1}}$ **k** $\dfrac{6a^{-1}}{3a^{-2}}$ **l** $\dfrac{a^4+a^8}{a^2}$

2 Use logarithms to solve each of the following equations:

a $2^x = 7$ **b** $2^{2x} = 7$ **c** $10^x = 2$ **d** $10^x = 3.6$

e $10^x = 110$ **f** $10^x = 1010$ **g** $2^{5x} = 100$ **h** $2^x = 0.1$

3 Evaluate each of the following:

a $\log_2 64$ **b** $\log_{10} 10^7$ **c** $\log_a a^2$ **d** $\log_4 1$

e $\log_3 27$ **f** $\log_2\left(\dfrac{1}{4}\right)$ **g** $\log_{10} 0.001$ **h** $\log_2 16$

4 Express each of the following as single logarithms:

a $\log_{10} 2 + \log_{10} 3$ **b** $\log_{10} 4 + 2\log_{10} 3 - \log_{10} 6$

c $2\log_{10} a - \log_{10} b$ **d** $2\log_{10} a - 3 - \log_{10} 25$

e $\log_{10} x + \log_{10} y - \log_{10} x$ **f** $2\log_{10} a + 3\log_{10} b - \log_{10} c$

5 Solve each of the following for x:

a $3^x(3^x - 27) = 0$ **b** $(2^x - 8)(2^x - 1) = 0$

c $2^{2x} - 2^{x+1} = 0$ **d** $2^{2x} - 12 \times 2^x + 32 = 0$

6 Sketch the graph of:

a $y = 2.2^x$ **b** $y = -3.2^x$ **c** $y = 5.2^{-x}$

d $y = 2^{-x} + 1$ **e** $y = 2^x - 1$ **f** $y = 2^x + 2$

7 Solve the equation $\log_{10} x + \log_{10} 2x - \log_{10}(x+1) = 0$

8 Given $3^x = 4^y = 12^z$, show that $z = \dfrac{xy}{x+y}$.

9 Evaluate $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$.

10 **a** Given that $\log_p 7 + \log_p k = 0$, find k.

b Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$, find q.

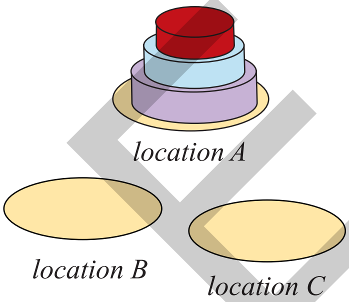
11 Solve:

a $2 \times 4^{a+1} = 16^{2a}$ (for a) **b** $\log_2 y^2 = 4 + \log_2(y+5)$ (for y)

Extended-response questions

1 This problem is based on the so-called 'Tower of Hanoi' puzzle. Given a number of different sized discs, the problem is to move a pile of discs (where n, the number of discs, is ≥ 1) to a second location (if starting at A then either to B or C) according to the following rules:

- Only one disc can be moved at a time.
- A total of only three locations can be used to 'rest' discs.
- A larger sized disc cannot be placed on top of a smaller disc.
- The task must be completed in the smallest possible number of moves.



a Using two coins complete the puzzle. Repeat first with three coins and then four coins and thus complete the table.

Number of discs, n	1	2	3	4
Minimum number of moves, M	1			

b Work out the formula which shows the relationship between M and n. Use your formula to extend the table of values for $n = 5$, 6 and 7.

c Plot the graph of M against n.

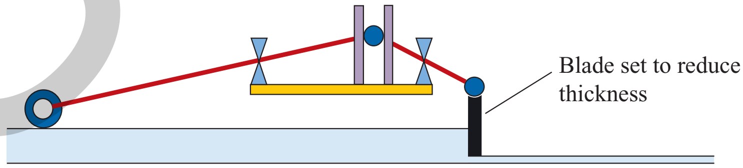
d Investigate, for both $n = 3$ and 4, to find whether there is a pattern for the number of times each particular disc is moved.

2 To control an advanced electronic machine, 2187 different switch positions are required. There are two kinds of switches available:

- Switch 1: These can be set in 9 different positions.
- Switch 2: These can be set to 3 different positions.

If n of switch type 1 and $n + 1$ of switch type 2 are used, calculate the value of n to give the required number of switch positions.

3 Research is being carried out to investigate the durability of paints of different thicknesses. The automatic machine shown in the diagram is proposed for producing a coat of paint of a particular thickness.



The paint is spread over a plate and a blade sweeps over the plate reducing the thickness of the paint. The process involves the blade moving at three different speeds.

a Operating at the initial setting the blade reduces the paint thickness to one-eighth of the original thickness. This happens n times. What fraction of the paint thickness remains? Express this as a power of $\frac{1}{2}$.

b The blade is then reset so that it removes three-quarters of the remaining paint. This happens $(n - 1)$ times. At the end of this second stage express the remaining thickness as a power of $\frac{1}{2}$.

c The third phase of the process involves the blade being reset to remove half of the remaining paint. This happens $(n - 3)$ times. At what value of n would the machine have to be set to reduce a film of paint 8192 units thick to 1 unit thick?

4 A hermit has little opportunity to replenish supplies of tea and so, to eke out supplies for as long as possible, he dries out the tea leaves after use and then stores the dried tea in an airtight box. He estimates that after each re-use of the leaves the amount of tannin in the used tea will be half the previous amount. He also estimates that the amount of caffeine in the used tea will be one-quarter of the previous amount.
The information on the label of the tea packet states that the tea contains 729 mg of caffeine and 128 mg of tannin.

a Write down expressions for the amount of caffeine when the tea leaves are re-used for the first, second, third and nth times.

b Do the same for the amount of tannin remaining.

c Find the number of times he can re-use the tea leaves if a ‘tea’ containing more than three times as much tannin as caffeine is undrinkable.

5 A new type of red synthetic carpet was produced in two batches. The first batch had a brightness of 15 units and the second batch 20 units. After a period of time it was discovered that the first batch was losing its brightness at the rate of 5% per year while the second lost its brightness at the rate of 6% per year.

a Write down expressions for the brightness of each batch after n years.

b A person bought some carpet from the first batch when it was a year old and some new carpet from the second batch. How long would it be before the brightness of the two carpets was the same?

CAS CALCULATOR

6 The value of shares in Company X increased linearly over a two-year period according to the model $x = 0.8 + 0.17t$, where t is the number of months from the beginning of January 1997 and \$$x$ is the value of the shares at time t.

The value of shares in Company Y increased over the same period of time according to the model $y = 10^{0.03t}$, where \$$y$ is the value of these shares at time t months.

The value of shares in Company Z increased over the same period according to the model $z = 1.7 \log_{10} (5(x + 1))$, where \$$z$ is the value of the shares at time t months.

Use a calculator to sketch the graphs of each of the functions on the one screen.

a Find the values of the shares in each of the three companies at the end of June 1997.

b Find the values of the shares in the three companies at the end of September 1998.

Review

c During which months were shares in Company X more valuable than shares in Company Y?

d For how long and during which months were the shares in Company X the most valuable?

7 In 2000 in a game park in Africa it was estimated that there were approximately 700 wildebeest and that their population was increasing at 3% per year. At the same time, in the park there were approximately 1850 zebras and their population was decreasing at the rate of 4% per year. Use a calculator to plot the graphs of both functions.

CAS CALCULATOR

a After how many years was the number of wildebeest greater than the number of zebras?

b It is also estimated that there were 1000 antelope and their numbers were increasing by 50 per year. After how many years was the number of antelope greater than the number of zebras?

8 Students conducting a science experiment on cooling rates measure the temperature of a beaker of liquid over a period of time. The following measurements were taken.

Time (min)	3	6	9	12	15	18	21
Temperature (°C)	71.5	59	49	45.5	34	28	23.5

a Find an exponential model to fit the data collected.

b Use this model to estimate:

i the initial temperature of the liquid

ii the temperature of the liquid after 25 minutes.

It is suspected that one of the temperature readings was incorrect.

c Re-calculate the model to fit the data, omitting the incorrect reading.

d Use the new model to estimate:

i the initial temperature of the liquid **ii** the temperature of the liquid at $t = 12$.

e If the room temperature is 15°C, find the approximate time at which the cooling of the liquid ceased.

9 The curve with equation $y = ab^x$ passes through the points (1, 1) and (2, 5)

a Find the values of a and b.

b Let $b^x = 10^z$.

i Take logarithms of both sides (base 10) to find z as an expression in x.

ii Find the value of k and a such that $y = a10^{kx}$ passes through the points (1, 1) and (2, 5).

10 **a** Find an exponential model of the form $y = a.b^x$ to fit the following data.

x	0	2	4	5	10
y	2	5	13	20	200

b Express the model you have found in **a** in the form $y = a10^{kx}$.

c Hence find an expression for x in terms of y.