CHAPTER 12

Counting Methods

Objectives

- To introduce addition and multiplication principles.
- To define and apply the concept of arrangements.
- To define and apply the concept of selections.
- To define and compute \( \binom{n}{r} \).
- To relate combinations to Pascal’s Triangle.
- To apply counting methods to probability.

In Chapter 10 it was found that often, when determining the probability of an event occurring, the number of outcomes contained in both the event and the sample space of interest need to be known. To do this, the sample space was listed and the outcomes counted. Sometimes a tree diagram was used or a table constructed to make sure that all the possibilities were accounted for.

When dealing with more complicated probability problems, listing the sample space and the event becomes too difficult. There may be hundreds of outcomes for a particular experiment, and even if they were comparatively easy to list we would soon tire of the task. In this chapter we will look at ways of counting the number of outcomes for various experiments and this will enable us to deal with more complicated probability problems.

12.1 Addition and multiplication principles

Some people find the decision about what to wear when they get up in the morning to be very difficult, and the more clothes they own, the more complex the decision becomes! Let us consider the number of choices they might have by looking at some examples.
Example 1

Sandi can’t decide whether to wear a windcheater or a jacket. She has four windcheaters and two jackets. How many choices does she have?

Solution

As Sandi is going to wear a windcheater or a jacket she has a total of six choices from among these two items.

Example 2

Sandi’s next choice is whether to wear jeans or a skirt. She has three pairs of jeans and four skirts. How many choices does she have?

Solution

Once again, as Sandi will wear jeans or a skirt, she has a total of seven choices from these two items.

Addition rule

In general, to choose between alternatives simply add up the available number for each alternative.

Example 3

At the library Alan is having trouble deciding which book to borrow. He has a choice of three mystery novels, three biographies or two science fiction books. How many choices of book does he have?

Solution

As he is choosing between alternatives (mystery novels or biographies or science fiction) then he has a total of \(3 + 3 + 2 = 8\) choices.

Sometimes the question arises of determining the number of possibilities when making successive choices.

Example 4

When travelling from home to school James takes a bus or walks to the main road, where he can catch a tram or a train or another bus to his destination. How many ways does James have for travelling to school?
Solution

A tree diagram may help to answer this question.

By counting the branches of the tree diagram it is found that there are six choices. This answer could be deduced by noting there are two choices for the first part of the journey and three choices for the second, and $2 \times 3 = 6$.

**Multiplication rule**

When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

**Example 5**

Consider Sandi’s situation. She has six choices of windcheater or jacket, and seven choices of jeans or skirts. How many choices does she have for a complete outfit?

**Solution**

As Sandi will wear either a windcheater or jacket and jeans or a skirt, then we cannot consider these to be alternative choices. We could draw a tree diagram to list the possibilities, but this would be arduous. Using the multiplication rule, however, we can quickly determine the number of choices to be $6 \times 7 = 42$.

**Exercise 12A**

1. Find how many choices of book are possible if one book is to be selected from the following:
   a. eight novels, three dictionaries
   b. three mysteries, two dramas, seven science fiction
   c. twenty-two romances, fourteen mysteries, one autobiography
   d. ten novels, three biographies, twelve encyclopedias, four atlases
2 Find how many different meals are possible if three courses (one entrée, one main course and one dessert) are to be selected from a menu that lists:

a three entrées, four main courses, five desserts
b ten entrées, ten main courses, five desserts
c five entrées, seven main courses, ten desserts
d eight entrées, eight main courses, eight desserts.

3 The menu in a restaurant lists four choices of entrée, eight of main course and four of dessert. Find the number of choices of meal possible:

a if one of each of the three courses must be chosen
b if you can choose to omit the entrée.

4 John cannot decide how to spend his evening. He can read a book, watch a video or go to the movies. If he can choose between three books, seven videos and ten movies, how many different choices does he have?

5 A student has to select a two-unit study for her course. She has to choose one unit in each semester. In semester one she has to choose one of two mathematics units, three language units and four science units. In semester two she has a choice of two history units, three geography units and two art units. How many choices does she have for the complete study?

6 Dominic is travelling from Melbourne to Brisbane and has the following choices. He can fly directly from Melbourne to Brisbane on one of three airlines, or he can fly from Melbourne to Sydney on one of four airlines and then travel from Sydney to Brisbane with one of five bus lines, or he can go on one of three bus lines directly from Melbourne to Brisbane. In how many ways could Dominic travel from Melbourne to Brisbane?

7 A particular new model of car is available in five choices of colour, three choices of transmission, four types of interior and two types of engine. Air conditioning is optional. How many different types of car are possible?

8 A company uses one letter followed by four digits for product codes. If any of the letters A–Z is allowed in the first position, and any of the digits 0–9 in the next four positions, how many different product codes are possible? (The letters and digits may be used more than once.)

9 In Victoria a licence plate generally consists of three letters followed by three numbers. If any of the letters A–Z is allowed in the first three positions, and any of the digits 0–9 in the second three positions, how many different number plates are possible? (The letters and digits may be used more than once.)

10 Morse code consists of a succession of dots and dashes. The symbols formed by the code may consist of one, two, three or four dots or dashes. How many different symbols may be represented by this code?
12.2 Arrangements

Arrangements are concerned with arranging objects in a definite order.

Example 6

How many ways are there of arranging three different books on a shelf?

Solution

Consider the bookshelf as having three possible positions in which books can be placed:

<table>
<thead>
<tr>
<th>position 1</th>
<th>position 2</th>
<th>position 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of choices</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

As we have three books, there are three choices of book to place in position 1. Having placed a book in position 1, there are only two choices of book for position 2, and similarly after two books are placed there is only one choice of book for position 3.

Using the multiplication principle stated in the previous section, we know that the total number of choices will be the product of these individual choices:

Number of arrangements of three books in a row is $3 \times 2 \times 1 = 6$

In general, if $n$ objects are arranged in a row there are $n$ choices for the first position, $n-1$ choices for the second position, $n-2$ choices for the third position and so on, until 1 is reached.

Thus:

The number of ways of arranging $n$ objects in a row is

$$n \times (n-1) \times (n-2) \times (n-3) \cdots \times 2 \times 1$$

Example 7

Twelve students are to have their photos taken. How many ways are there of arranging the group if they all sit in a row?

Solution

As there are twelve students the number of arrangements is

$$12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479 001 600$$

which is rather a large number of choices!
Continuing to write the expression \( n(n-1)(n-2)(n-3) \cdot \cdot \cdot 2 \times 1 \) can be rather cumbersome, so for convenience this is written as \( n! \) which is read as ‘\( n \) factorial’.

The notation \( n! \) (read \( n \) factorial) is an abbreviation for the product of all the integers from \( n \) down to 1:

\[
n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdot \cdot \cdot \times 2 \times 1
\]

Not all arrangements require the use of every object available, as is shown in the following example.

### Example 8

A painter is to paint the five circles of the Olympic flag. He cannot remember the colours to use for any of the circles, but he knows they should all be different. He has eight colours of paint available. In how many ways can he paint the circles on the flag?

#### Solution

In a similar manner to the previous example we list the painter’s choices:

<table>
<thead>
<tr>
<th>Circle</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of choices</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Thus the total number of arrangements possible is \( 8 \times 7 \times 6 \times 5 \times 4 = 6720 \).

Could the factorial notation be used to express the answer to Example 8? In that example the number of arrangements of eight objects in groups of five was to be counted. The answer is

\[
8 \times 7 \times 6 \times 5 \times 4 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{8!}{3!}
\]

This is the number of ways of choosing and arranging five objects from eight different ones, or the number of arrangements of eight objects taken five at a time.

In general, the number of arrangements of \( n \) objects in groups of size \( r \) is given by:

\[
\frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \cdot \cdot \cdot (n-r+1)
\]

Arrangements are also called permutations in mathematics, and this expression for the number of arrangements of \( n \) objects in groups of size \( r \) is often denoted by the symbol \( ^nP_r \).
Example 9

Find the number of different four-figure numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if each digit:

a can only be used once b can be used more than once

Solution

a As we are arranging nine objects (n) in groups of four (r):

\[ \frac{9!}{5!} = \frac{9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 9 \times 8 \times 7 \times 6 = 3024 \]

b There are nine choices for each of the four positions, so the total number of choices is \(9 \times 9 \times 9 \times 9 = 9^4 = 6561\).

Consider the number of arrangements of \(n\) objects in a group of size \(n\). From first principles we have found that this is equal to \(n!\). Using our rule for the number of arrangements of \(n\) objects in groups of size \(n\) gives us the answer of:

\[ \frac{n!}{(n-n)!} = \frac{n!}{0!} \]

Equating these expression gives an important definition for 0!:

\[ 0! = 1 \]

If a more complicated arrangement is required in general, the restriction should be dealt with first, as shown in the following example.

Example 10

How many different even four-figure numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, if each digit may be used only once?

Solution

As the number is to be even it must end in 2, 4, 6, or 8, and so we have four choices for the last digit. Having now selected the last digit we need to choose any three digits from the remaining seven to fill up the other places. Consider each of the four positions to be filled as a ‘box’, and write in each box the number of choices available for that position. The total number of arrangements is then found by multiplying together the numbers in the boxes. Remember, the restriction should be dealt with first. Thus:

\[
\begin{array}{cccc}
7 & 6 & 5 & 4 \\
\end{array}
\]

number of choices for the first three positions number of choices for the last digit if the number is to be even

Multiplying gives \(7 \times 6 \times 5 \times 4 = 840\).
Exercise 12B

1 Evaluate:
   a \(3!\)  b \(5!\)  c \(7!\)  d \(2!\)  e \(0!\)  f \(1!\)

2 Evaluate:
   a \(\frac{5!}{3!}\)  b \(\frac{9!}{7!}\)  c \(\frac{3!}{0!}\)  d \(\frac{8!}{6!}\)  e \(\frac{5!}{0!}\)  f \(\frac{10!}{7!}\)

Example 3
3 In how many ways can five books be arranged in a row on a shelf?

Example 4
4 In how many ways can seven students be seated on a row of seven chairs?

Example 5
5 In how many ways can four pictures be hung in a row on a wall?

Example 6
6 In how many ways can six cups be hung on six hooks?

Example 7
7 In how many ways can three pictures be hung along a wall if there are ten pictures available?

8 If there are eight swimmers in the final of a 1500 m freestyle event, in how many ways can the first three places be filled?

9 Find the number of ways in which the letters of the word TROUBLE can be arranged:
   a if they are all used  b if they are used three at a time

10 Find the number of ways in which the letters of the word PANIC can be arranged:
   a if they are all used  b if they are used four at a time

Example 8
11 Find the number of four-letter code words that can be made from the letters of the word COMPLEX:
   a if no letter can be used more than once  b if the letters can be re-used

12 Find how many code words of three or four letters can be made from the letters of the word NUMBER:
   a if no letter can be used more than once  b if the letters can be re-used

Example 9
13 If no digit can be used more than once, find how many numbers formed from the digits 3, 4, 5, 6, 7 are:
   a three-digit numbers  b three-digit numbers and even
c greater than 700

14 If no number can be used more than once, find how many numbers can be formed from the digits 3, 4, 5, 6, 7, 8, which are:
   a two- or three-digit numbers  b six-digit numbers and even
c greater than 7000

Example 10
15 Four boys and two girls sit in a line on stools in front of a counter. Find the number of ways in which they can arrange themselves:
   a if there are no restrictions  b if the two girls wish to be at the ends
12.3 Selections

In the previous section methods for counting the number of ways in which objects can be chosen and then arranged were discussed. Sometimes the interest is solely in the number of different groups of objects that can be selected.

Example 11

Four flavours of ice cream — vanilla, chocolate, strawberry and caramel — are available at the school canteen. How many different double-cone selections are possible if two different flavours must be used?

Solution

The possibilities are:
- vanilla and chocolate
- vanilla and strawberry
- vanilla and caramel
- chocolate and strawberry
- chocolate and caramel
- strawberry and caramel

giving a total of six different selections.

In this example the selection 'vanilla and chocolate' is considered to be the same as 'chocolate and vanilla', and so was counted only once. Such choices without regard to order are called selections or combinations and the symbol \( \binom{n}{r} \) is used to represent the number of ways in which groups of size \( r \) can be chosen from a total of \( n \) objects when order is unimportant.

When the total group size \( n \) is not large the combinations can be listed, but obviously a more efficient method is preferable. Consider again Example 8 concerning the colours on the Olympic flag. In this example the colours to be used are first chosen, and then they are arranged on the flag. This is shown as:

Choose the colours | Arrange them | Possible arrangements
--- | --- | ---
\( \binom{8}{5} \times 5! \) | \( \frac{8!}{3!} \) | \( \frac{8!}{3!5!} \)

So, if \( \binom{8}{5} \times 5! = \frac{8!}{3!} \), an expression for \( \binom{8}{5} \) can be found by dividing both sides by 5!. That is:

\[ \binom{8}{5} = \frac{8!}{3!5!} \]

Note that the two figures on the bottom line (3 and 5) add to 8.
In general, the number of combinations of \( n \) objects in groups of size \( r \) is
\[
{n \choose r} = \frac{n!}{r!(n-r)!}
\]
A commonly used alternative notation for \( n \choose r \) is \( \left( \begin{array}{c} n \\ r \end{array} \right) \).

A CAS calculator can be used to determine values of \( n \choose r \). From the MATH menu, select 7:Probability, then select 3:nCr( and complete nCr(4, 2) to evaluate \( \left( \begin{array}{c} 4 \\ 2 \end{array} \right) \).

### Example 12

Consider again Example 11. The number of combinations of four ice creams in groups of size two is
\[
\left( \begin{array}{c} 4 \\ 2 \end{array} \right) = \frac{4!}{2!2!} = \frac{4 \times 3}{2} = 6
\]
This is the same as the answer we had before.

Once again, not all combination problems are so straightforward, as shown by the following example.

### Example 13

A team of three boys and three girls is to be chosen from a group of eight boys and five girls. How many different teams are possible?

**Solution**

Three boys can be chosen from eight in \( \left( \begin{array}{c} 8 \\ 3 \end{array} \right) \) ways, and three girls from five in \( \left( \begin{array}{c} 5 \\ 3 \end{array} \right) \) ways. Thus the total number of possible teams is:
\[
\left( \begin{array}{c} 8 \\ 3 \end{array} \right) \times \left( \begin{array}{c} 5 \\ 3 \end{array} \right) = 56 \times 10 = 560
\]

**Pascal’s Triangle**

Consider the possibilities when choosing from small groups of objects. In the first instance, consider when the group has only one object, \( A \). One object can be chosen from the group in only one way: \( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = 1 \). Next, consider when the group has two objects, \( A \) and \( B \).

Both of them can be chosen in only 1 way: \( (AB) \)
\[
\left( \begin{array}{c} 2 \\ 2 \end{array} \right) = 1
\]
One of them can be chosen in 2 ways: \( (A) \) or \( (B) \)
\[
\left( \begin{array}{c} 2 \\ 1 \end{array} \right) = 2
\]
In order to build up the pattern, it can be said that in both of these cases there is only one way in which to choose no objects from a group: \( \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = 1 \) and \( \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = 1 \).
Suppose that the original group has three objects, $A$, $B$, and $C$.

Three objects can be chosen in 1 way: $(ABC)$ $^3C_3 = 1$

Two objects can be chosen in 3 ways: $(AB) (BC) (AC)$ $^3C_2 = 3$

One object can be chosen in 3 ways: $(A) (B) (C)$ $^3C_1 = 3$

No objects can be chosen in 1 way: $( )$ $^3C_0 = 1$

Choosing from four objects we have:

$^4C_4 = 1$ $^4C_3 = 4$ $^4C_2 = 6$ $^4C_1 = 4$ $^4C_0 = 1$

Rearranging these choices allows us to show a pattern. We will also add a top line for a group of 0 objects.

| 0 objects | 1 |
| 1 object  | 1 | 1 |
| 2 objects | 1 | 2 | 1 |
| 3 objects | 1 | 3 | 3 | 1 |
| 4 objects | 1 | 4 | 6 | 4 | 1 |
| 5 objects | ? | ? | ? | ? | ? |

The pattern can be continued indefinitely by noting:

- The first and last number in the row are always 1.
- The number in any position in a row is the sum of the two numbers in the row above which are to the left and right of it, as shown.

This triangle of numbers is usually called Pascal’s Triangle after the French mathematician Blaise Pascal who made ingenious use of it in his studies of probability.

An interesting application of Pascal’s Triangle can be discovered by adding each row:

1 1 2 4 8 16 32
Is there a pattern in the sequence?

<table>
<thead>
<tr>
<th>Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>?</td>
</tr>
</tbody>
</table>

It can be seen from the table that the sum of the entries in row $n$ appears to be $2^n$, and this is in fact true. That is:

$$nC_0 + nC_1 + nC_2 + \cdots + nC_{n-1} + nC_n = 2^n$$

This can be used to solve some problems concerned with combinations.

**Example 14**

Nick is making a list for his party. He has seven friends, and he can’t decide how many to invite. If he may choose any number from one to all seven to come to the party, how many possible party lists does he have? (Assume he will invite at least one person to his party.)

**Solution**

Nick may invite one person to the party and he has $\binom{7}{1} = 7$ ways of doing this. If he invites two people to the party he has $\binom{7}{2} = 21$ ways of doing this.

Continuing in this way we can see that Nick’s total number of choices is:

$$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$$

Since we know that:

$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 2^7$$

then the required answer is $2^7 - \binom{7}{0} = 128 - 1 = 127$

**Exercise 12C**

1. For each of the following examples determine the number of selections possible by listing the possibilities:
   a. An ice cream with two different scoops is selected from a choice of vanilla, chocolate and strawberry.
   b. Two students from the group of Jaime, Georgia and Wey are chosen to represent the class in debating.
   c. Two students from the group of Thomas, William, Jackson and Phillip are chosen for the tennis team.
   d. Three scarves are chosen from a blue scarf, a green scarf, a red scarf and a white scarf.

2. Evaluate:
   a. $\binom{5}{3}$
   b. $\binom{5}{2}$
   c. $\binom{7}{4}$
   d. $\binom{7}{3}$

   Compare your answers for parts a and b, and for parts c and d.
3 Evaluate:
\[
\begin{align*}
&\quad \binom{20}{18} \quad \binom{100}{99} \quad \binom{100}{2} \quad \binom{250}{248} \\
&\text{4 Evaluate:} \\
&\quad \binom{6}{3} \quad \binom{7}{1} \quad \binom{8}{2} \quad \binom{50}{48} \\
&\text{5 How many netball teams of seven can be chosen from 13 players?} \\
&\text{6 An ice cream parlour has 25 different flavours of ice cream available. How many different} \\
&\text{three-scoop ice cream sundaes are available if three different flavours are to be used and} \\
&\text{the order of the scoops does not matter?} \\
&\text{7 How many different hands of seven cards can be dealt from a normal pack of 52 cards?} \\
&\text{8 In Tattslotto six numbers are selected from 45. How many different possible selections are} \\
&\text{there? (Do not attempt to consider supplementary numbers.)} \\
&\text{9 A student has the choice of three mathematics subjects and four science subjects. In how} \\
&\text{many ways can they choose to study one mathematics and two science subjects?} \\
&\text{10 A survey is to be conducted, and eight people are to be chosen from a group of 30.} \\
&\quad \text{a In how many different ways could the eight be chosen?} \\
&\quad \text{b If the group contains 10 men and 20 women, how many groups containing exactly two} \\
&\quad \text{men are possible?} \\
&\text{11 From a standard 52-card deck, how many 7-card hands have exactly 5 spades and 2 hearts?} \\
&\text{12 In how many ways can a committee of five be selected from eight women and four men:} \\
&\quad \text{a without restriction?} \quad \text{b if there must be three women on the committee?} \\
&\text{13 Six females and five males are interviewed for five positions. If all are found to be} \\
&\text{acceptable for any position, in how many ways could the following combinations be} \\
&\text{selected?} \\
&\quad \text{a three males and two females} \quad \text{b four females and one male} \\
&\quad \text{c five female} \quad \text{d five people regardless of sex} \\
&\quad \text{e at least four females} \\
&\text{14 The selectors of a sporting team need to choose 10 athletes from the 15 track and 12 field} \\
&\text{athletes who have qualified to compete.} \\
&\quad \text{a How many groups are possible?} \\
&\quad \text{b How many groups would contain track athletes only?} \\
&\quad \text{c How many groups would contain field athletes only?} \\
&\quad \text{d How many groups would contain half track and half field athletes?}
15 A student representative committee of five is to be chosen from four male and six female students. How many committees could be selected which contain more female than male students?

Example 14

16 Joanne is offered a selection of five different sweets. She can choose to pass or to select any number of them. In total how many choices does she have?

17 Eight people have auditioned for roles in a play. The director can choose none, or any number of them for his production. In how many ways can selections be made from these eight people, any number at a time?

18 How many colours can be obtained by mixing a set volume of red, blue, green, orange and white paints if any number of paints can be used at a time?

19 How many varieties of fruit salad, using at least two fruits, can be obtained from apples, oranges, pears, passionfruit, kiwi fruit and nectarines, taken any number at a time?

20 In how many ways can a group of six people be divided into:

a two equal groups?

b two unequal groups, if there must be at least one person in each group?

12.4 Applications to probability

As already established in Chapter 10, the probability of an event occurring may be determined by dividing the number of favourable outcomes by the total number of possible outcomes. Establishing the number of favourable outcomes and the total number of outcomes is often achieved by using permutations and combinations.

Example 15

Four-letter ‘words’ are to be made by arranging letters of the word SPECIAL. What is the probability the ‘word’ will start with a vowel?

Solution

Arranging the seven letters in groups of four may be done in \(7 \times 6 \times 5 \times 4\) different ways, i.e. 840, and this is the number of possible outcomes. Consider the number of arrangements which start with a vowel. Since there are three vowels we have 3 choices for the first letter. Having done this, we have six letters remaining which are to be placed in the three remaining positions, and this can be done in \(6 \times 5 \times 4\) different ways. Thus the number of arrangements which start with a vowel is:

\[3 \times 6 \times 5 \times 4 = 360\]

Thus, the probability of the arrangement starting with a vowel is:

\[
\frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} = \frac{360}{840} = \frac{3}{7}
\]
Example 16

Three students are to be chosen to represent the class in a debate. If the class consists of six boys and four girls, what is the probability that the teams will contain:

i exactly one girl?  
ii at least two girls?

Solution

Since there is a total of 10 students, the number of possible teams is \(^{10}C_3 = 120\).

i If the team is to contain one girl then this can be done in \(\binom{4}{1} = 4\) different ways. Having placed one girl, the other two places must be filled by boys, and this can be done in \(\binom{6}{2} = 15\) different ways. Thus the total number of teams containing one girl and two boys is \(4 \times 15 = 60\) and the probability that the team contains exactly one girl is \(\frac{60}{120} = 0.5\).

ii If the team is to contain at least two girls then it may contain two or three girls. The number of teams containing:

- exactly two girls is \(\binom{6}{1} \times \binom{4}{2} = 36\)
- exactly three girls is \(\binom{6}{0} \times \binom{4}{3} = 4\)

Thus the total number of teams containing at least two girls is 40, and the probability of this is \(\frac{40}{120} = \frac{1}{3}\).

Exercise 12D

1 A four-digit number (with no repetitions) is to be formed from the set of numbers \{1, 2, 3, 4, 5, 6\}. Find the probability that the number:

   a is even  
   b is odd

Example 15

Two three-letter ‘words’ are to be made by arranging the letters of the word COMPUTER. What is the probability the ‘word’ will start with a vowel?

Example 16

Three letters are chosen at random from the word HEART and arranged in a row. Find the probability that:

   a the letter H is first  
   b the letter H is chosen  
   c both vowels are chosen

4 Three men and three women are to be randomly seated in a row. Find the probability that both the end places will be filled by women.

Example 16

A netball team of seven players is to be chosen from six men and seven women. Find the probability that the selected team contains more men than women.
6 Bill is making a sandwich. He may choose any combination of the following: lettuce, tomato, carrot, cheese, cucumber, beetroot, onion, ham. Find the probability that:
   a the sandwich contains ham
   b the sandwich contains three ingredients
   c the sandwich contains at least three ingredients

7 A bag contains five white, six red and seven blue balls. If three balls are selected at random, without replacement, find the probability they are:
   a all red
   b all different colours

8 A four-digit number (with no repetitions) is to be formed from the set of numbers \{0, 1, 2, 3, 4, 5, 6, 7\}. Find the probability that the number:
   a is even
   b is odd
   c is less than 4000
   d is less than 4000, given that it is greater than 3000

9 Susie chooses four pieces of bubble gum from a jar containing five red, two blue and three green pieces of bubble gum. Calculate the probability that Susie selects:
   a no green bubble gum
   b at least one green bubble gum
   c at least one green bubble gum and at least one red bubble gum
   d at least one red bubble gum given that there is at least one green bubble gum

10 A hand of five cards is dealt from a normal pack of 52 cards. Find the probability that the hand will contain:
   a no aces
   b at least one ace
   c the ace of spades
   d the ace of spades given that there is at least one ace

11 A committee of three is to be chosen from a group of four men and five women. Find the probability that the committee contains:
   a all women
   b at least one woman
   c exactly two men given that at least one man is chosen
Chapter summary

- \( n! \) (\( n \) factorial) = \( n(n-1)(n-2)\ldots 2 \times 1 \) and \( 0! = 1 \).
- Evaluation of \( ^nC_r \) gives the number of combinations or selections of \( n \) objects in subgroups of size \( r \). Order is not important.
- When the combination problem involves restrictions, deal with these first.
- Combinations may be used when determining probabilities. In the appropriate cases, the probability is given by dividing the number of favourable outcomes by the total number of outcomes.

Multiple-choice questions

1. For his holiday reading Geoff has selected eight detective novels, three biographies and four science fiction books, but he only has room in his case for three books. If he selects one book from each group, how many combinations of book are possible?
   - A 15  
   - B 28  
   - C 56  
   - D 20  
   - E 96

2. Georgia is choosing her five subjects for Year 12. She has already chosen three subjects. She will choose one of the three mathematics subjects, and either one of five languages or one of three science subjects. How many different subject combinations are possible?
   - A 11  
   - B 15  
   - C 9  
   - D 24  
   - E 45

3. In how many ways can 10 people be arranged in a queue at the bank?
   - A 10!  
   - B \( \frac{10!}{2!} \)  
   - C \( \frac{10!}{2!8!} \)  
   - D \( \frac{10!}{8!} \)  
   - E \( ^{10}C_1 \)

4. How many different numberplates can be made using two letters followed by four digits, if neither the letters nor the digits can be repeated?
   - A 8  
   - B 720  
   - C 5690  
   - D 3276000  
   - E 6760000

5. \( ^{21}C_3 \) is equal to:
   - A 21!  
   - B \( \frac{21!}{3!} \)  
   - C \( \frac{21!}{18!3!} \)  
   - D \( \frac{21!}{18!} \)  
   - E \( \frac{18!3!}{21!} \)

6. In how many ways can a hand of six cards be dealt from a pack of 52 cards?
   - A 6!  
   - B \( ^{52}C_6 \)  
   - C \( ^{46}C_6 \)  
   - D \( \frac{52!}{6!} \)  
   - E 52!

7. In how many ways can three DVDs be chosen from a group of 12 DVDs?
   - A 12  
   - B 36  
   - C 220  
   - D 1320  
   - E 7983600

8. A class consists of 10 girls and 14 boys. In how many ways could a committee of two girls and two boys be chosen?
   - A \( ^{10}C_2 \times ^{14}C_2 \)  
   - B \( ^{24}C_4 \)  
   - C \( 10! \times 14! \)  
   - D \( 10 \times 9 \times 14 \times 13 \)  
   - E \( \frac{10!14!}{4!} \)

9. Three-letter ‘words’ are to be made by arranging the letters of the word METHODS. What is the probability that the word begins with a vowel?
   - A \( \frac{1}{105} \)  
   - B \( \frac{1}{21} \)  
   - C \( \frac{1}{2} \)  
   - D \( \frac{5}{26} \)  
   - E \( \frac{2}{7} \)
10 What is the probability that a team of four chosen at random from a group of eight friends, four males and four females, would consist of three women and one man?

A $\frac{1}{2}$  
B $\frac{3}{4}$  
C $\frac{3}{8}$  
D $\frac{3}{64}$  
E $\frac{8}{35}$

Short-answer questions (technology-free)

1 Evaluate:
   a $1000 \text{C}_{998}$  
   b $1000 \text{,000C}_{999,999}$  
   c $1000 \text{,000C}_{1}$

2 How many integers from 100 to 999, inclusive, have three different digits?

3 How many different three-digit house numbers can be constructed from six brass numerals 1, 2, 3, 4, 5, 6?

4 A supermarket sells $n$ different brands of liquid dishwashing soap. Each brand offers four different sized bottles, small, medium, large and economy, and each is available as either lemon-scented or pine-scented. How many different types of liquid dishwashing soap bottles are available at this supermarket?

5 Of the integers from 1000 to 9999 how many have at least one digit a 5 or 7?

6 A bushwalking club has 80 members, 50 men and 30 women. A committee consisting of two men and one woman is to be selected. How many different committees are possible?

7 There are five vowels and 21 consonants in the English alphabet. How many different four-letter ‘words’ can be formed that contain two different vowels and two different consonants?

8 A pizza restaurant offers the following toppings: onions, green peppers, mushrooms, anchovies and pepperoni.
   a How many different kinds of pizza with three different toppings can be ordered?
   b How many different kinds, with any number of toppings (between none and all five) can be ordered?

9 Seven people are to be seated in a row. Calculate the number of ways in which this can be done so that two particular persons, $A$ and $B$, always have exactly one of the others between them.

10 Three letters are chosen at random from the word OLYMPICS and arranged in a row. What is the probability that:
   a the letter O is first?  
   b the letter Y is chosen?  
   c both vowels are chosen?

Extended-response questions

1 Permutations are formed using all of the digits 1, 2, 3, 4, . . . , 9 without repetition.
   Determine the number of possible permutations in each of the following cases:
   a Even and odd digits alternate.
   b The digits 1 and 2 are together but not necessarily in that order.
2 There are 10 chairs in a row.
   a In how many ways can three people be seated?
   b In how many of these will the two end chairs be occupied?
   c In how many of these will the two end chairs be empty?

3 All possible three-digit numbers are formed from the odd digits \{1, 3, 5, 7, 9\}.
   a How many such numbers are possible if each digit is used only once?
   b How many of the numbers from part a are larger than 350?

4 In how many ways can a committee of four be chosen from five married couples if:
   a all individuals are eligible for selection?
   b the committee must consist of two women and two men?
   c a husband and wife cannot both be selected?

5 Geoff has five flat batteries, and 10 charged batteries. Unfortunately his little brother mixes them up, and he can’t tell them apart. He selects four batteries at random for his calculator.
   a How many different combinations of the 15 batteries could Geoff select?
   b In how many of these are all four batteries charged?
   c In how many of these is at least one battery flat?

6 There are seven mints and 11 jubes in the lolly jar. Steve puts his hand in the jar and selects four lollies at random.
   a How many different combinations of the lollies are there?
   b In how many of these are there no mints?
   c In how many of these are there two mints and two jubes?

7 In Tattslotto, a player picks a selection of six numbers from the numbers 1 to 45. To determine the winners eight numbers are chosen at random – the first six are designated as the winning numbers, and the other two as the supplementary numbers. Prizes are as follows:
   Division 1: 6 winning numbers
   Division 2: 5 winning numbers and 1 supplementary
   Division 3: 5 winning numbers
   Division 4: 4 winning numbers
   Division 5: 3 winning numbers and 1 supplementary

   Find the number of combinations which satisfy each of the divisions, and hence the probabilities of winning each of the five divisions.

8 In Keno, a player selects between three and 10 numbers from 1 to 80. Each selection is called a ‘spot’. If you select five numbers, you are playing a ‘Spot 5’ game. To determine the winners, 20 numbers are drawn randomly from the 80 numbers. If your selected numbers are among the 20, you win. The amount you win depends on the ‘spot’ you are playing.

   Find the probability of winning a:
   a ‘Spot 6’ game
   b ‘Spot 5’ game