Back to Menu >>>

CHAPTER 9

MODULE 1

Arithmetic and geometric sequences

- What is an arithmetic sequence?
- What is a geometric sequence?
- How do we find the *n*th term of an arithmetic or geometric sequence?
- How do we find the sum of the first *n* terms of an arithmetic or geometric sequence?
- How do we find the sum to infinity of a geometric sequence?
- How can we use arithmetic and geometric sequences to model real-world situations?
- How do we distinguish graphically between an arithmetic and a geometric sequence?

9.1 Sequences

We call a list of numbers written down in succession a **sequence**; for example, the numbers drawn in a lottery:

$$12, 22, 5, 6, 16, 43, \ldots$$

For this sequence, there is no clear rule that will enable you to predict with certainty the next number in the sequence. Such sequences are called **random** sequences. Random sequences arise in situations where chance plays a role in determining outcomes, for example the number of people in successive cars arriving at a parking lot, or the number generated in a Tattslotto draw.

If, however, we list house numbers on the left-hand side of a suburban street:

$$1, 3, 5, 7, 9, 11, \ldots$$

we see they also form a sequence. This sequence differs from the random sequence above in that there is a clear pattern or rule that enables us to determine how the sequence will continue; each term in the sequence is two more than its predecessor. Such a sequence is called a **rule-based** sequence. In this module, we will concentrate on rule-based sequences.

Back to Menu >>>

Exercise 9A

1 Label each of the sequences as either rule based or probably random. If rule based, write down the next value in the sequence.

a 1, 2, 3, 4, . . . **b** 100, 99, 98, 97, . . . **c** 23, 9, 98, 1, . . .
d 1, 1, 1, 0, 1, 0, 0, 0, . . . **e** 1, 1, 1, 1, . . . **f** 1, 0, 1, 0, 1, 0, . . .
g 2, 4, 6, 8, . . . **h** 2, 4, 8, 16, . . . **i** 10, 20, 40, 80, . . .

2 For each of the following rule-based sequences, find the missing term:

a 1, 10, 100, □, . . . **b** 10, □, 6, 4, 2, . . . **c** 3, 9, □, 81, . . .
d $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \square, \ldots$ **e** $5^1, 5^2, \square, 5^8, \ldots$ **f** $-2, -6, \square, -14, \ldots$
g $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \square, \ldots$ **h** $10, -10, \square, -10, \ldots$ **i** $2, -2, \square, -10, \ldots$

9.2 Arithmetic sequences

A sequence in which each successive term can be found by **adding the same number** is called an **arithmetic** sequence.

For example, the sequence 2, 7, 12, 17, 22, . . . is arithmetic because each successive term can be found by adding 5.

term 1	term 2	term 3	term 4	term 5
2	7	12	17	22
+5	+5	+5	+5	+5

The sequence 19, 17, 15, 13, 11, . . . is also arithmetic because each successive term can be found by adding −2, or equivalently, subtracting 2.

term 1	term 2	term 3	term 4	term 5
19	17	15	13	11
–2	–2	–2	–2	–2

The common difference

Because of the way in which an arithmetic sequence is formed, the **difference** between successive terms is constant. In the language of arithmetic sequences, we call it the **common difference**. In the sequence

2, 7, 12, 17, 22, . . .

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

the common difference is +5, while in the sequence

19, 17, 15, 13, 11, . . .

the common difference is −2.

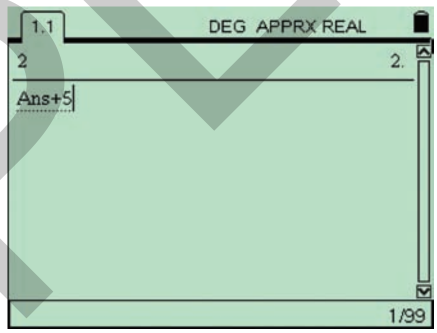
Once you know the first term in an arithmetic sequence and its common difference, the rest of the terms in the sequence can be readily generated. If you want to generate a large number of terms, your graphics calculator will do this with little effort.

How to generate the terms of an arithmetic sequence using the TI-Nspire CAS

Generate the first five terms of the arithmetic sequence: 2, 7, 12, 17, 22, . . .

Steps

1 Start a new document by pressing ctrl + N.

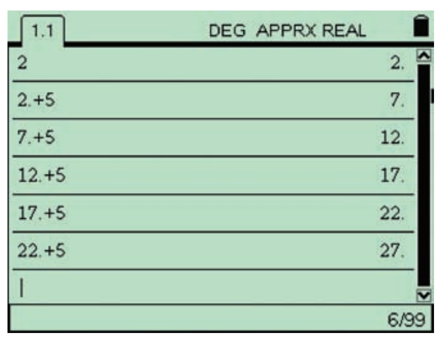


2 Select **1:Add Calculator**.
Enter the value of the first term, **2**. Press enter.

3 The common difference for the sequence is 5. So, type **+5**. Press enter. The second term in the sequence, **7**, is generated.

4 Pressing enter again generates the next term, **12**.

5 Keep pressing enter until the desired number of terms is generated.



Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Back to Menu >>>

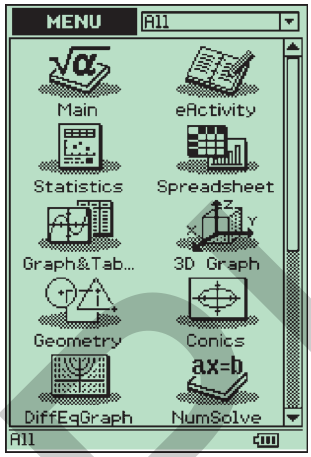
How to generate the terms of an arithmetic sequence in the main application using the ClassPad

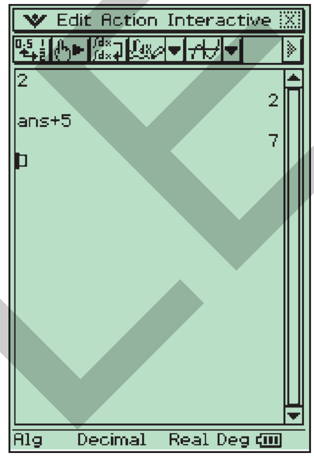
Generate the first five terms of the arithmetic sequence: 2, 7, 12, 17, 22, . . .

Steps

1 From the application menu screen, locate the built-in **Main** application. Tap Main to open, giving the screen shown opposite.

Note: Tapping Menu from the icon panel (just below the touch screen) will display the application menu, if it is not already visible.

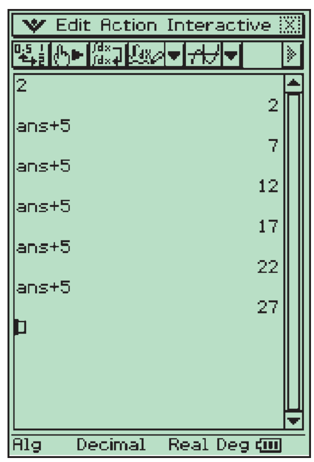




2 a Starting with a clean screen, enter the value of the first term, **2**.

b The common difference for this sequence is 5. So, type +**5**. Then press EXE. The second term in the sequence (7) is displayed.

3 Pressing EXE again generates the next term, **12**. Keep pressing EXE until the required number of terms is generated.



Being able to recognise an arithmetic sequence is another skill that you need to develop. The key idea here is that the successive terms in an arithmetic sequence differ by a constant amount (the common difference).

Example 1 Testing for an arithmetic sequence

a Is the sequence 20, 17, 14, 11, 8, . . . arithmetic?

Solution

Strategy: Subtract successive terms in the sequence to see whether they differ by a constant amount. If they do, the sequence is arithmetic.

1 Write down the terms of the sequence.	20, 17, 14, 11, 8, . . .
2 Subtract successive terms.	$17 - 20 = -3$ $14 - 17 = -3$ $11 - 14 = -3$ and so on
3 Write down your conclusion	Sequence is arithmetic as terms differ by a constant amount.

b Is the sequence 4, 8, 16, 32, . . . arithmetic?

Solution

1 Write down the terms of the sequence.	4, 8, 16, 32, . . .
2 Subtract successive terms.	$8 - 4 = 4$ $16 - 8 = 8$ $32 - 16 = 16$ and so on
3 Write down your conclusion	Sequence is not arithmetic as terms differ by different amounts.

Exercise 9B

1 Which of the following sequences are arithmetic?

a 1, 2, 3, 4, . . .	**b** 100, 99, 98, 97, . . .	**c** 23, 25, 29, 31, . . .
d 1, 1, 0, 1, . . .	**e** 3, 9, 27, . . .	**f** 56, 60, 64, . . .
g 6, 24, 36, 48, . . .	**h** 2, 4, 8, 16, . . .	**i** 10, 20, 40, 80, . . .
j −10, −6, −2, 2, . . .	**k** 100, 50, 0, −50, . . .	**l** −20, −15, −10, −5, . . .
m −100, −200, −400, . . .	**n** 1, −3, −7, −11, . . .	**o** 2, 4, 16, 256, . . .

2 Which of the following sequences are arithmetic?

a 1.1, 2.1, 3.1, 4.1, . . .	**b** 9.8, 8.8, 7.8, 6.8, . . .	**c** 1.01, 1.02, 2.02, . . .
d 1.1, 1.1, 0.1, 1.1, . . .	**e** 36.3, 37.0, 37.7, . . .	**f** 0.01, 0.02, 0.04, . . .
g 6000, 7450, 8990, . . .	**h** 2000, 4000, 8000, . . .	**i** 10.9, 20.8, 30.7, . . .
j −100, −60, −20, 20, . . .	**k** −10.2, −5.1, 0, 5.1, . . .	**l** −67.8, −154.1, −205.7, . . .

3 Consider the sequence 20, 24, 28, 32, 36, . . .

a Why is the sequence arithmetic?
b What is the common difference?
c What is the next term in the sequence?

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Back to Menu >>>

d Starting with 20, how many times do you have to add the common difference to get to the 8th term? What is the value of this term?

e Starting with 20, how many times do you have to add the common difference to get to term 13? What is the value of term 13?

4 Consider the sequence $5, 3, 1, -1, -3, \ldots$

a Why is the sequence arithmetic?

b What is the common difference?

c What is the next term in the sequence?

d Starting with 5, how many times do you have to add the common difference to get to the 7th term? What is the value of this term?

e What is the value of the 10th term? What is the value of the 50th term?

9.3 The *n*th term of an arithmetic sequence and its applications

Some notation

To find a general rule for finding the *n*th term of an arithmetic sequence, we first need to introduce some notation. With a rule, we can calculate the value of any term in the series without having to write out all the preceding terms first.

- To represent the successive terms in a sequence we will use the notation:

  $$t_1, t_2, t_3, t_4, \ldots t_n$$

  where t_n indicates the *n*th term.
  In the sequence 2, 7, 12, 17, 22, . . . $t_1 = 2$ and $t_4 = 17$.

- The *first term* in the sequence has its own symbol: we call it a.
  In the sequence 2, 7, 12, 17, 22, . . . $a = 2$.

- Finally, we use the symbol d to represent the *common difference*.
  In the sequence 2, 7, 12, 17, 22, . . . $d = 5$.

The rule

Consider an arithmetic sequence with first term a and common difference d.
Then:

$$\begin{aligned}
t_1 &= a\\
t_2 &= t_1 + d = a + d\\
t_3 &= t_2 + d = a + 2d\\
t_4 &= t_3 + d = a + 3d\\
t_5 &= t_4 + d = a + 4d
\end{aligned}$$

and so on. Thus, following the pattern, we can write:

$$t_n = a + (n - 1)d$$

This gives us the following rule for the nth term of an arithmetic sequence.

Rule for finding the nth term in an arithmetic sequence

The nth term of an arithmetic sequence is given by

$$t_n = a + (n - 1)d$$

where a (= t_1) is the value of the first term and d is the common difference.

Example 2 Identifying a and d in an arithmetic sequence

For the arithmetic sequence 30, 28, 26, 24, . . . , write down the values of a, d and t_3.

Solution

1 a is the first term — $a = 30$
2 d is the common difference. — $d = 28 - 30 = 26 - 28 = -2$
3 t_3 is the third term. — $t_3 = 26$

Example 3 Using the rule to find the nth term of an arithmetic sequence

The first term of an arithmetic sequence is $a = 6$ and the common difference is $d = 2$. Use a rule to determine the 11th term in the sequence.

Solution

1 Use the rule $t_n = a + (n - 1)d$ — $t_n = a + (n - 1)d$
2 In this example, $a = 6$, $d = 2$. For the 11th term, $n = 11$. — $a = 6$, $d = 2$, $n = 11$
3 Substitute these values in the rule and evaluate. — $\therefore t_{11} = 6 + (11 - 1)2 = 26$
4 Write down your answer. — The 11th term is 26.

Example 4 Using the rule to find the nth term of an arithmetic sequence

Use a rule to determine the 15th term of the arithmetic sequence 18, 15, 12, 9, . . .

Solution

1 Use the rule $t_n = a + (n - 1)d$ — $t_n = a + (n - 1)d$
2 For this sequence, the first term is: $a = 18$. Arithmetic sequence, so: $d = t_2 - t_1 = -3$. For the 15th term, $n = 15$. — $a = 18$, $d = 15 - 18 = -3$, $n = 15$
3 Substitute these values in the rule and evaluate. — $\therefore t_{15} = 18 + (15 - 1) \times (-3) = -24$
4 Write down your answer. — The 15th term is -24.

Example 5 **Finding an expression for the *n*th term of an arithmetic sequence**

For the arithmetic sequence 6, 10, 14, 18, . . . , find an expression for the *n*th term.

Solution

1 Use the rule $t_n = a + (n - 1)d$	$t_n = a + (n - 1)d$
2 For this sequence, $a = 6$ and $d = 4$.	$a = 6, d = 4$
3 Substitute these values to obtain an expression for the *n*th term. Simplify.	$\therefore t_n = 6 + (n - 1) \times 4$ $= 6 + 4n - 4$ $= 2 + 4n$
4 Write down your answer.	The *n*th term is $t_n = 2 + 4n$.

Example 6 **Using two terms to determine *a* and *d***

In an arithmetic sequence, the 5th term is 10 and the 9th term is 18. Write down the first three terms of the sequence.

Solution

Method 1 (the routine way)

Strategy: To answer this question we use the values of the 5th and 9th terms to set up two equations involving *a* and *d* which we solve. The values of *a* and *d* can then be used to generate the sequence.

1 As $t_5 = 10$ we can write:	$10 = a + (5 - 1)d$ $\therefore 10 = a + 4d$ (1)
2 As $t_9 = 18$ we can write:	$18 = a + (9 - 1)d$ $\therefore 18 = a + 8d$ (2)
3 Solve the two equations for *a* and *d*. Subtract (1) from (2) to solve for *d*. Substitute $d = 2$ in (1) to solve for *a*.	$\therefore 8 = 4d$ (2) − (1) $\therefore d = 2$ Substitute $d = 2$ in (1) $10 = a + 4 \times 2$ $\therefore a = 2$
4 Use the values of *a* and *d* to write down your answer.	The first three terms are: 2, 4, 6.

Method 2 (quicker than Method 1 but requires more thinking)

1 The difference between the 9th and 5th terms in the sequence is 18 − 10 = 8 so we can write:	$4d = 8$ $\therefore d = 2$
2 As $t_5 = 10$ we can write:	$t_5 = 10 = a + 4d$
3 Substitute $d = 2$ and solve for *a*.	$\therefore 10 = a + 4 \times 2$ $\therefore a = 2$
4 Write down your answer.	The first three terms are: 2, 4, 6.

A calculator note

Previous examples have not required the use of the graphics calculator for their solution (Examples 2, 5 and 6), or the calculator is not the most efficient way to solve the problem

Back to Menu >>>

(Examples 3 and 4). However, the next type of example is most efficiently solved using a graphics calculator.

As often happens, there are several graphics calculator methods that can be used. The method we have chosen uses sequence mode. This is perhaps not the quickest and easiest method. However, it has the advantage of being the only method that works for *all* the problems you will meet in this module. It is therefore worth your while learning it now.

How to generate the terms of a sequence using the TI-Nspire CAS

Generate the terms in an arithmetic sequence with $a = 10$ and $d = 4$.

Steps

Strategy: Find an expression for the nth term of the sequence as for Example 5. A graphics calculator can then be used to display the sequence in a table.

1 For this sequence, $a = 10$ and $d = 4$.

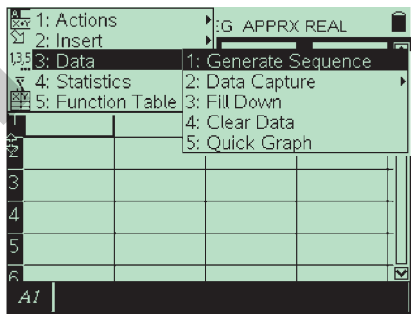
$a = 10, \ d = 4$

2 Use $t_n = a + (n - 1)d$ to write down an expression for the nth term, t_n. Don't simplify.

$t_n = 10 + (n - 1) \times 4$

3 Start a new document by pressing ctrl + N. Select **3:Add Lists & Spreadsheet**.

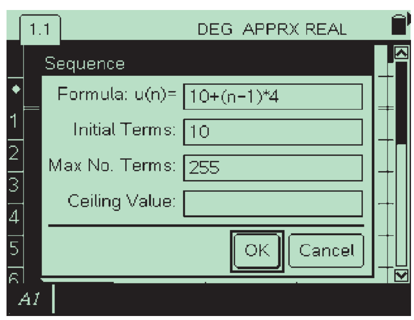
a Place the cursor in any cell in column A and press menu/**3:Data** to generate the screen opposite.



b With the cursor on **1:Generate Sequence**, press enter to display the pop-up screen shown opposite.

Type in the entries as shown. Use tab to move between entry boxes. Leave the **Max No. Terms** at 255.

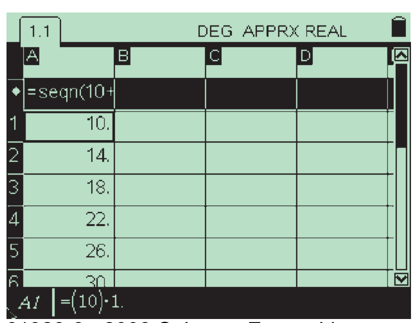
Note: The calculator uses u(n) for the nth term.



c Press enter to close the pop-up screen and display the sequence of terms.

The term number can be read directly from the row number (left-hand side) of the spreadsheet. For example, the 5th term would be 26.

Use the ▼ arrow to move down through the sequence to see further terms.



How to generate the terms of a sequence using the ClassPad

Generate the terms in an arithmetic sequence with $a = 10$ and $d = 4$.

Steps

Strategy: Find an expression for the nth term of the sequence as for Example 5. A graphics calculator can then be used to display the sequence in a table.

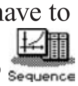
1 For this sequence, $a = 10$ and $d = 4$.

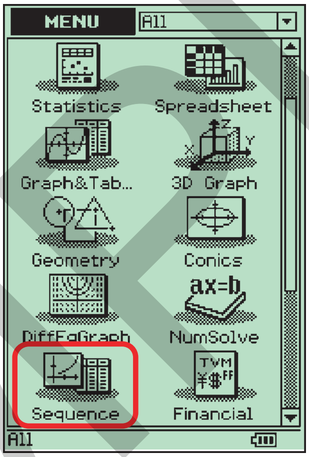
$a = 10,\ d = 4$

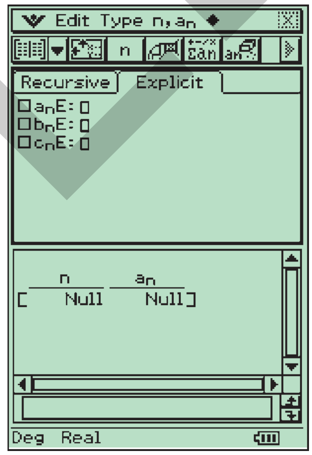
2 Use $t_n = a + (n - 1)d$ to write down an expression for the nth term, t_n. Don't simplify.

$t_n = 10 + (n - 1) \times 4$

3 Creating sequences on the ClassPad calculator is done via the **Sequence** application.

a From the application menu screen, locate the **Sequence** application. (You may have to scroll down.) Tap  to open.

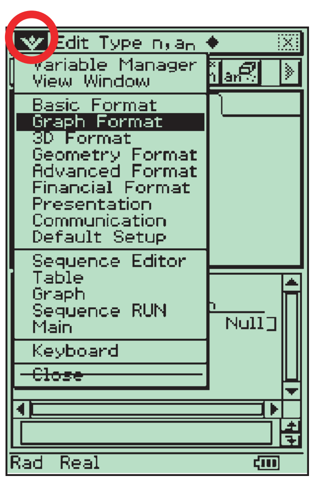


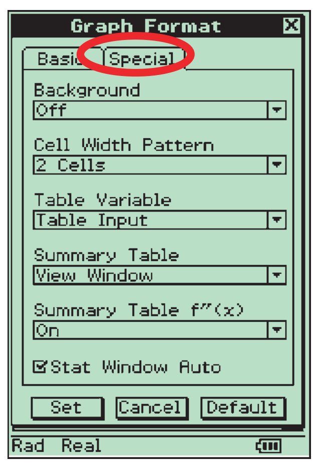


b The **Sequence** application opens with two half-screens. The top half requires information about the rule defining the sequence. The bottom half will generate values for the defined sequence in a table format.

c To ensure the best accuracy with table values, tap the ⩔ icon and choose **Graph Format**. Tap the **Special** tab and choose **2 Cells** for the **Cell Width Pattern**.

Tap **Set** to confirm your selection. This returns you to the **Sequence** application screen.



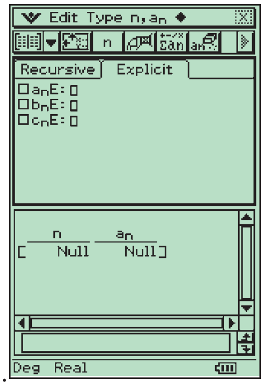


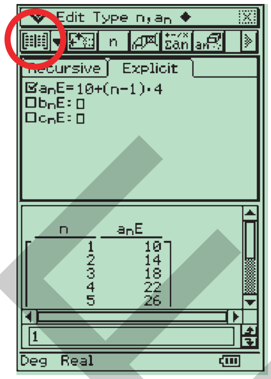
Back to Menu >>>

d Tap the **Explicit** tab because you want to enter the expression for the *n*th term of the sequence.

To enter this expression, move the cursor to the box opposite $a_n E$: and type **10** + (*n* − **1**) × **4**. Press (EXE) to confirm your entry, which is indicated by a tick in the square to the left of $a_n E$:.

Note: ***n*** is found in the toolbar (n).





e To display the terms of the sequence in table format tap the icon.

The first column *n* displays the term numbers 1, 2, 3, . . . The second column $a_n E$ displays the values of the terms in the sequence: 10, 14, 18, 22, 26, . . .

Note: To move up and down through the sequence, tap the arrows in the sidebar.

Having mastered the above process, you are now in a position to work the following example.

Example 7 Finding when a term in a sequence first exceeds a given value

How many terms would we have to write down in the arithmetic sequence 10, 14, 18, 22, . . . before we found a term greater than 51?

Solution

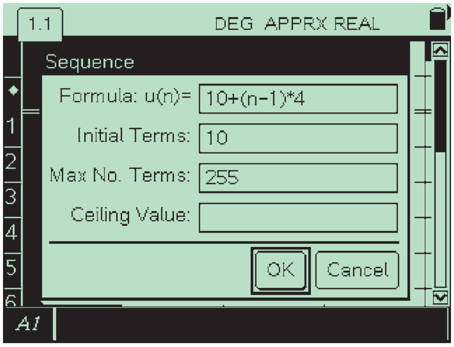
Strategy: Find an expression for the *n*th term of the sequence as for Example 5. A graphics calculator can then be used to display the sequence in a table. The first term that exceeds 51 can then be found.

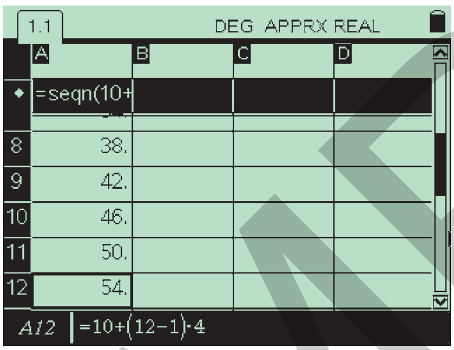
1 For this sequence, $a = 10$ and $d = 4$

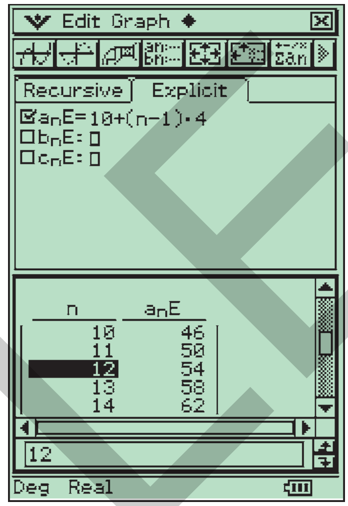
$a = 10, d = 4$

$t_n = 10 + (n - 1) \times 4$

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the first term that exceeds 51; in this case, the 12th term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	10	11	12	...
t_n	10	14	...	46	50	54	...

The first term to exceed 51 is t_{12}.

Application

Any situation where you start with a fixed amount and add or subtract a fixed amount at regular intervals can be modelled by an arithmetic sequence. For example, the increase in weight of a bag of apples as additional apples are added to the bag or the amount of wine left in a bottle as glasses of wine are poured. The following example involves a person on a weight-loss program.

Example 8 **Application of the *n*th term of an arithmetic sequence**

Before starting on a weight-loss program a man weighs 124 kg. He plans to lose weight at a rate of 1.5 kg a week until he reaches his recommended weight of 94 kg.

a Write down a rule for the man's weight, W_n, at the start of week n.

b If he keeps to his plan, how many weeks will it take the man to reach his target weight of 94 kg?

Back to Menu >>>

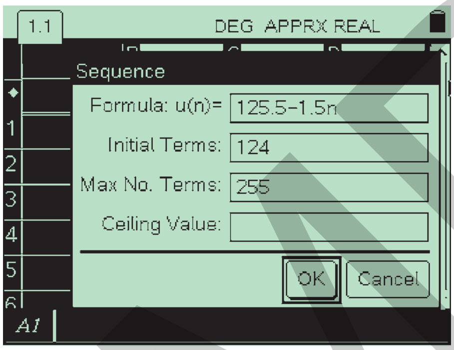
Solution

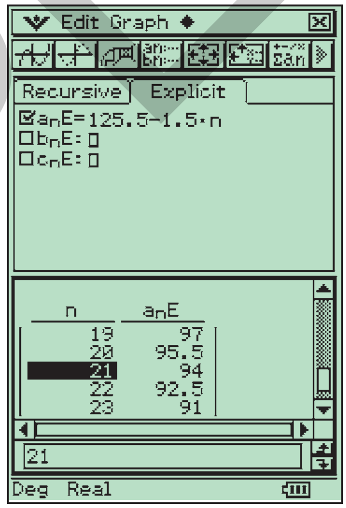
Strategy: You need to recognise that by losing a constant amount of weight each week, the man's weekly weight follows an arithmetic sequence. Using this information, you can write down an expression for his weight in the *n*th week. You can then use this expression to display the sequence of weights in a table and hence determine when the target weight is reached.

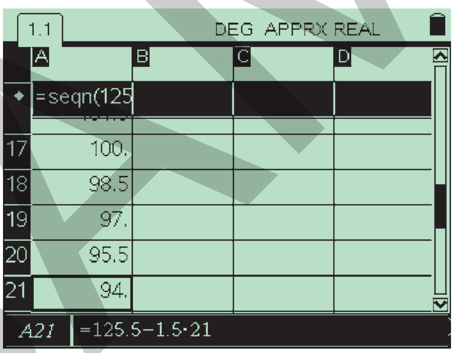
1 Arithmetic sequence with $a = 124$ and $d = -1.5$
Use the rule $W_n = a + (n - 1)d$ to write down an expression for W_n.

Arithmetic sequence
$a = 124,\ d = -1.5$
$W_n = 124 + (n - 1) \times (-1.5)$
$= 124 - 1.5n + 1.5$
$\therefore W_n = 125.5 - 1.5n$

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the first term that is 94 or less; in this case, the 21st term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	19	20	21	...
W_n	124	122.5	...	97	95.5	94	...

If the man keeps to his plan, he will reach his target weight by the start of week 21, or after 20 weeks of being on the program.

Exercise 9C

1 For the arithmetic sequence:

a 3, 6, 9, 12, ... write down the values of a, d, and t_2
b 23, 30, 37, 44, ... write down the values of a, d, and t_3
c 100, 75, 50, 25, ... write down the values of a, d, and t_4
d 13, 11, 9, 7, ... write down the values of a, d, and t_1
e −20, −25, −30, −35, ... write down the values of a, d, and t_2
f −12, −10, −8, −6, ... write down the values of a, d, and t_2

Hint: As you are told these are arithmetic sequences, you can use any two terms to find d.

2 For an arithmetic sequence with:

a $a = 200$ and $d = 10$, determine the value of the 3rd term
b $a = 2$ and $d = 5$, determine the value of the 20th term
c $a = 250$ and $d = 50$, determine the value of the 5th term
d $a = 26$ and $d = -2$, determine the value of the 12th term
e $a = 25$ and $d = -5$, determine the value of the 7th term
f $a = 0$ and $d = 2$, determine the value of the 53rd term

3 Use the rule to determine the value of:

a the 11th term of the arithmetic sequence 5, 10, 15, 20, 25, ...
b the 8th term of the arithmetic sequence 12, 8, 4, 0, ...
c the 27th term of the arithmetic sequence 0.1, 0.11, 0.12, 0.13, ...
d the 13th term of the arithmetic sequence −55, −42, −29, −16, ...
e the 10th term of the arithmetic sequence −1.0, −1.5, −2.0, −2.5, ...
f the 95th term of the arithmetic sequence 130, 123, 116, 109, ...
g the 7th term of the arithmetic sequence $\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{4}, \ldots$

4 Write down the first three terms of the arithmetic sequence in which:

a the 7th is 37 and the 9th term is 47
b the 11th term is 31 and the 15th term is 43
c the 6th term is 0 and the 11th term is −20
d the 8th term is 134 and the 13th term is 159
e the 7th is 60 and the 12th term is 10
f the 10th term is 20 and the 21st term is 75

5 Write down the nth term of an arithmetic sequence with:

a $a = 3$ and $d = 2$ **b** $a = 12$ and $d = 4$ **c** $a = 5$ and $d = 3$
d $a = 10$ and $d = -2$ **e** $a = 25$ and $d = -5$ **f** $a = -4$ and $d = 2$

6 Write down the nth term of the following arithmetic sequences:

a 2, 4, 6, 8, ... **b** 30, 34, 38, 42, ... **c** 12, 15, 18, 21, ...
d 10, 8, 6, 4, ... **e** 100, 75, 50, 25, ... **f** −5, 0, 5, 10, ...

Back to Menu >>>

7 How many terms would we have to write down in the arithmetic sequence:

a $5, 7, 9, 11, \ldots$ before we found a term greater than 25?
b $132, 182, 232, 282, \ldots$ before we found a term greater than 1000?
c $100, 96, 92, 88, \ldots$ before we found a term less than 71?
d $10, 16, 22, 28, \ldots$ before we found a term equal to 52?
e $0.33, 0.66, 0.99, 1.32, \ldots$ before we found a term greater than 2?
f $-17, -15, -13, -11, \ldots$ before we found a positive term?
g $127, 122, 117, 112, \ldots$ before we found a term less than zero?

8 To make up a coloured poster, a printer charges \$25 for the first poster plus \$4 for each additional poster.

a Complete the table.

Number of posters	1	2	3	4
Cost (\$)				

b Write down a rule for the cost, C_n, of making up n posters.
c What would it cost to make up:
i 10 posters? **ii** 35 posters?
d How many posters were produced if the printer charged:
i \$65? **ii** \$105?

9 When a garbage truck starts collecting rubbish it first stops at a corner store where it collects 86 kg of rubbish. It then travels down a long suburban street where it picks up 40 kg of rubbish at each house.

a Complete the table.

Number of pickups	1	2	3	4
Amount (kg)				

b Write down a rule for the amount of garbage, A_n, collected after n pick-ups.
c How much garbage would be carried by the truck after:
i 15 pick-ups? **ii** 27 pick-ups?
d The maximum amount of garbage that can be carried by the truck is 1500 kg. After picking up from the corner store, what is the maximum number of houses it can pick up from before it is fully loaded?

10 You are offered a job with a company at a starting salary of \$20 500 per year with yearly increments of \$450.

a Write down a rule for determining your salary, S_n, at the start of each year.
b What would your salary be:
i at the start of the 5th year? **ii** at the start of the 8th year?
c At this rate, how many years would you have to be with the company to have a salary of at least \$50 000 per year?

11 You have \$860 to spend on yourself while on an overseas holiday. To make the money last as long as possible, you budget for spending \$51.50 per day.

Back to Menu >>>

a Write down a rule for determining the amount of spending money, M_n, you will have left at the start of the nth day of your holiday.

b How much spending money would you have left:

i at the start of the 7th day? **ii** at the start of the 13th day?

c At this spending rate, for how many days can you afford to stay on holidays?

9.4 The sum of an arithmetic sequence and its applications

At the age of 9, the mathematician Gauss was asked to find the sum of the first 100 counting numbers. His method was as follows:

$$1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100 = S_{100}$$

By reversing the order of terms:

$$100 + 99 + 98 + 97 + \cdots + 4 + 3 + 2 + 1 = S_{100}$$

Adding the two expressions we have:

$$101 + 101 + 101 + 101 + \cdots + 101 + 101 + 101 + 101 = 2 \times S_{100}$$

or

$$100(101) = 2 \times S_{100}$$

so

$$S_{100} = \frac{100(101)}{2} = 5050$$

We can use the method of Gauss to help us find an expression for the sum of the first n terms in an arithmetic sequence.

Consider the sum, S_7, of the first 7 terms of an arithmetic series with first term a and common difference d.

We can write:

$$S_7 = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) + (a + 6d)$$

By reversing the order of terms, it is also true that:

$$S_7 = (a + 6d) + (a + 5d) + (a + 4d) + (a + 3d) + (a + 2d) + (a + d) + a$$

Adding the two expressions we have:

$$2S_7 = (2a + 6d) + (2a + 6d) + (2a + 6d) + (2a + 6d) + (2a + 6d) + (2a + 6d) + (2a + 6d) = 7(2a + 6d)$$

$$\therefore 2S_7 = 7(2a + 6d)$$

or

$$S_7 = \tfrac{7}{2}(2a + 6d)$$

Back to Menu >>>

If we did the same thing for eight terms we would find that:

$$S_8 = \tfrac{8}{2}(2a + 7d)$$

for nine terms:

$$S_9 = \tfrac{9}{2}(2a + 8d) \quad \text{and so on.}$$

Generalising this expression to n terms we have:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Rule for finding the sum of the first *n* terms of an arithmetic sequence

The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

where a is the value of the first term and d is the common difference.

Example 9 Using the rule to find the sum of an arithmetic sequence

Use the rule to find the sum of the first six terms of an arithmetic sequence with $a = 5$ and $d = 3$. Check your answer by writing out the first six terms in the sequence and adding.

Solution

1 Use the formula $S_n = \dfrac{n}{2}[2a + (n - 1)d]$ with $n = 6$, $a = 5$ and $d = 3$.

$$S_6 = \frac{6}{2}[2 \times 5 + (6 - 1) \times 3] = 75$$

2 Write down your answer.

The sum of the first six terms is 75.

3 Check answer.

$5 + 8 + 11 + 14 + 17 + 20 = 75$

Example 10 Finding an expression for the sum of the first *n* terms of an arithmetic sequence

For the arithmetic sequence 2, 7, 12, . . . , find an expression for the sum of the first n terms.

Solution

1 Use the rule $S_n = \dfrac{n}{2}[2a + (n - 1)d]$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

2 For this sequence, $a = 2$ and $d = 5$.

$$a = 2,\ d = 5$$

3 Substitute these values to obtain an expression for the sum of n terms. Simplify.

$$\therefore S_n = \frac{n}{2}(2 \times 2 + (n - 1)5)$$
$$= \frac{n}{2}(4 + 5n - 5)$$
$$= \frac{n}{2}(5n - 1).$$

4 Write down your answer.

The sum of the first n terms is $S_n = \dfrac{n}{2}(5n - 1)$.

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Example 11 Finding when the sum of a sequence first exceeds a given value

How many terms are required for the sum of the arithmetic sequence 5, 15, 25, . . . to first exceed 200?

Solution

Strategy: Find an expression for the sum of n terms of the sequence. A graphics calculator can then be used to display the sequence in a table. The first term that exceeds 200 can then be found.

1 For this sequence, $a = 5$ and $d = 10$. Use this information and the rule $S_n = \frac{n}{2}[2a + (n - 1)d]$ to find an expression for the sum.

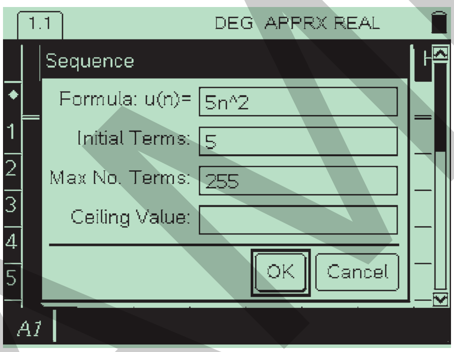
$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

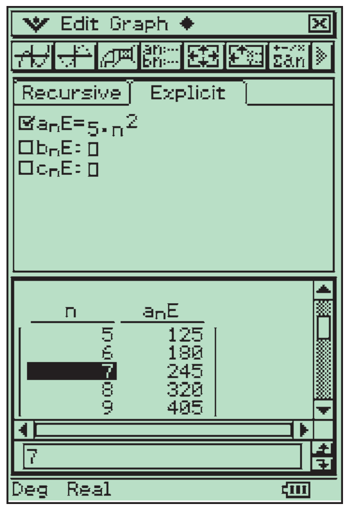
$$a = 5,\ d = 10$$

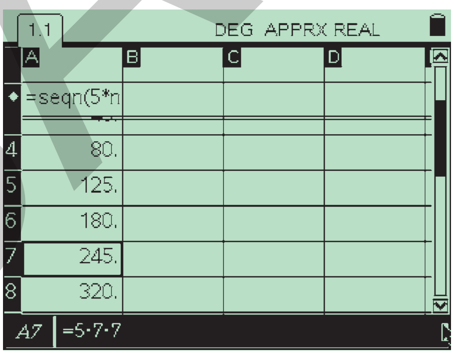
$$\therefore S_n = \frac{n}{2}[2 \times 5 + (n - 1)10]$$

$$= \frac{n}{2}(10 + 10n - 10) = 5n^2$$

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the first term in the sequence that exceeds 200; in this case, the 7th term.







Back to Menu >>>

3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	5	6	7	...
S_n	5	20	...	125	180	245	...

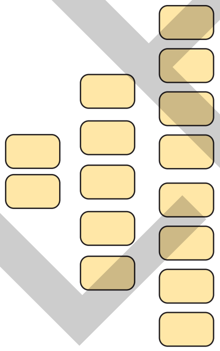
The sum of the sequence first exceeds 200 after adding 7 terms.

Example 12 Application of the sum of n terms of an arithmetic sequence

A child makes a pattern with blocks on the floor. She puts two blocks in the first row, five in the second row, eight in the third row and so on as shown.

a Write down an expression for the number of blocks, b_n, in Row n. How many blocks are there in Row 6?

b Only 100 blocks are available. What is the maximum number of rows of blocks that she can have in her pattern?



Solution

a **1** Arithmetic sequence with $a = 2$ and $d = 3$. Use the rule $b_n = a + (n - 1)d$ to write down an expression for b_n.

Arithmetic sequence: $a = 2$, $d = 3$

$b_n = 2 + (n - 1) \times 3 = 3n - 1$

2 For the sixth row, $n = 6$.

When $n = 6$,

$b_6 = 3 \times 6 - 1 = 17$

3 Write down your answer.

There are 17 blocks in Row 6.

b **1** Use the rule $S_n = \frac{n}{2}[2a + (n - 1)d]$ to write down an expression for S_n when $a = 2$ and $d = 3$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

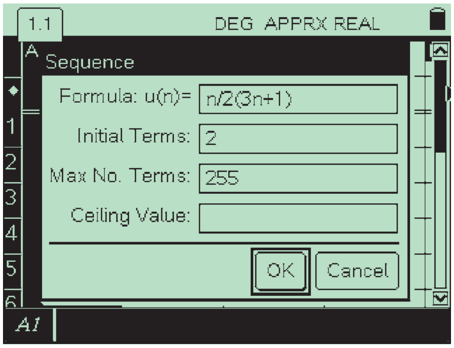
$$\therefore S_n = \frac{n}{2}[2 \times 2 + (n - 1)3]$$

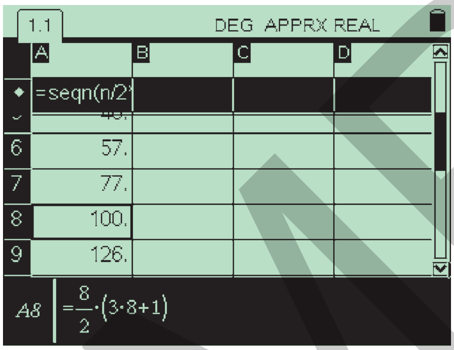
$$= \frac{n}{2}(4 + 3n - 3) = \frac{n}{2}(3n + 1)$$

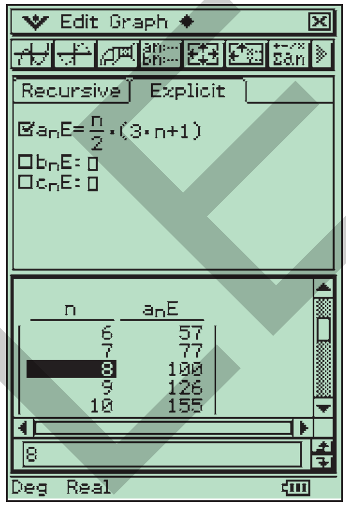
Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Back to Menu >>>

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the last term in the sequence that is 100 or less; in this case, the 8th term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	7	8	9	...
S_n	2	7	...	77	100	126	...

The girl can build a pattern with eight rows of blocks.

Exercise 9D

1 Use the rule to find the sum of the first:

a six terms of an arithmetic sequence with $a = 5$ and $d = 3$

b five terms of an arithmetic sequence with $a = 12$ and $d = -2$

c seven terms of an arithmetic sequence with $a = -5$ and $d = 5$

d four terms of an arithmetic sequence with $a = 0.1$ and $d = 0.2$

e six terms of an arithmetic sequence with $a = -9$ and $d = 3$

In each case, write out the series and add up the terms to check your answer.

Back to Menu >>>

2 Use the rule to find the sum of the arithmetic sequence:

a 4, 8, 12, . . . to 20 terms
b 10, 7, 4, . . . to 8 terms
c 120, 110, 100, . . . to nine terms
d 1, 2, 3, . . . to 1000 terms
e 1.000, 1.005, 1.010, . . . to 100 terms
f −8, −6, −4, . . . to 15 terms

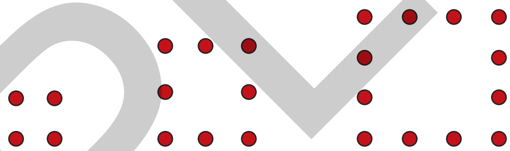
3 Find an expression for the sum of the first n terms of the following arithmetic sequences:

a 8, 16, 24, . . .
b 20, 25, 30, . . .
c 0, 15, 30, . . .
d 2, 6, 10, . . .
e 100, 200, 300, . . .
f 200, 150, 100, . . .

4 How many terms are required for the sum of the arithmetic sequence:

a 4, 7, 10, . . . to equal 50?
b 10, 15, 20, . . . to equal 100?
c 0.2, 0.4, 0.6, . . . to first exceed 5?
d 100, 200, 300, . . . to first exceed 10 000?
e 1.00, 1.05, 1.10, . . . to first exceed 10?

5 You are given a box of counters. Using the counters, you lay out a pattern of squares as shown.



a First focus on the number of counters to make each square, and complete the table.

Square number	1	2	3	4
Number of counters in square				

b If you continue the pattern, write down an expression for N_n, the number of counters needed to make the nth square.

c How many counters will you need to make:
i the 6th square? **ii** 10th square?

d Now focus on the number of counters (the sum) needed to make several squares, and complete the table.

Number of squares	1	2	3	4
Total number of counters				

e If you continue the pattern, write down an expression for S_n, the total number of counters needed to make n squares.

f How many counters will you need to make:
i six squares? **ii** ten squares?

g What is the maximum number of squares you can make following this pattern if you have 100 counters?

6 A child makes a pattern with blocks on the floor. She puts 55 blocks in the first row, 50 in the second, 45 in the third and so on.

a Write down an expression for the number of blocks in the nth row.
b Determine the number of blocks in the 7th row.
c Determine the total number of blocks in the first five rows.
d How many rows can she have in her pattern if she has 300 blocks to play with?

7 In your new 'get-fit' program, you plan to jog 1500 metres around the oval on the first night and then increase this distance by 250 metres each subsequent night. Assume you stick with your plan.

a First focus on the distance jogged each night, and complete the table.

Night number	1	2	3	4
Distance jogged (m)				

b If you continue the pattern, write down an expression for D_n, the distance jogged on the nth night.

c How far do you jog:

i on the 7th night? **ii** 12th night?

d Now focus on the total distance jogged over several nights, and complete the table.

Night number	1	2	3	4
Total distance jogged				

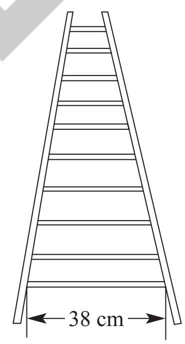
e Write down an expression for S_n, the total distance jogged after n nights.

f Determine the total distance you expect to jog after ten nights.

g Determine the total distance you expect to jog over the first fortnight.

h Determine the number of nights you will need to stick to your program to ensure that you jog a total distance of 54 km.

8 A ladder has 10 rungs. The ladder tapers from top to bottom. The length of the bottom rung is 38 cm. Each successive rung is then 1.2 cm shorter.

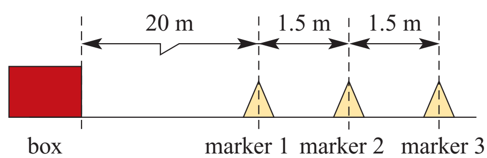


a Write down an expression for the length L_n of the nth rung of the ladder.

b Determine the length of the 6th rung.

c Determine the total length of the rungs in the ladder.

9 You have 25 markers to be placed on a line, 1.5 metres apart. The box containing the markers is 20 metres from the point at which the first marker is to be placed.



a Write down a rule for determining the distance, D_n, of the nth marker from the box.

b How far will the 9th marker be from the box?

c If the last marker is to be 50 metres from the box, how many markers will you need to place on the line?

The markers are very heavy and you can only carry one at a time. As a result you have to collect the first marker from the box, carry it to its position, return to the box, get the second marker, carry it to its position, return to the box, get the third marker, and so on until you have placed all the markers on the line. Assume you start and finish at the box.

d What is the minimum distance you will have to walk if you are required to place 30 markers on the line?

Back to Menu >>>

9.5 Geometric sequences

A sequence in which each successive term can be found by **multiplying the previous term** by a constant factor is called a **geometric** sequence.

For example, the sequence 1, 2, 4, 8, 16, . . . is geometric because each successive term can be found by multiplying the previous term by 2.

term 1	term 2	term 3	term 4	term 5
1	2	4	8	16
×2	×2	×2	×2	×2

The sequence 40, 20, 10, 5, 2.5, . . . is also geometric because each successive term can be found by multiplying the previous term by 0.5.

term 1	term 2	term 3	term 4	term 5
40	20	10	5	2.5
×0.5	×0.5	×0.5	×0.5	×0.5

The common ratio

Because of the way in which a geometric sequence is formed, the **ratio** between successive terms is constant. In the language of geometric sequences, we call it the **common ratio**.
In the sequence

1, 2, 4, 8, 16, . . .

the common ratio is 2, while in the sequence

40, 20, 10, 5, 2.5, . . .

the common ratio is 0.5.
Common ratios can also be negative. For example, for the geometric sequence

1, −2, 4, −8, 16, . . .

the common ratio is −2.
Note: When the common ratio is negative, consecutive terms are alternatively positive and negative.

Determining the common ratio

Often the common ratio of a geometric sequence can be determined by inspection. For example, it is easy to see that for the sequence

1, 3, 9, 27, . . .

the common ratio is 3.

In other cases, the value of the common ratio is not so obvious. In these cases we can determine the value of the common ratio by dividing one of the terms in the sequence by its immediate predecessor.
For example, the common ratio for the geometric series

$$0.8, 0.32, 0.128, \ldots$$

is

$$r = \frac{0.32}{0.8} = 0.4 \qquad \left(\text{or } \frac{0.128}{0.32} = 0.4\right)$$

Once you know the first term in a geometric sequence and its common ratio, the rest of the terms in the sequence can be readily generated. If you want to generate a large number of terms, your graphics calculator will do this with little effort.

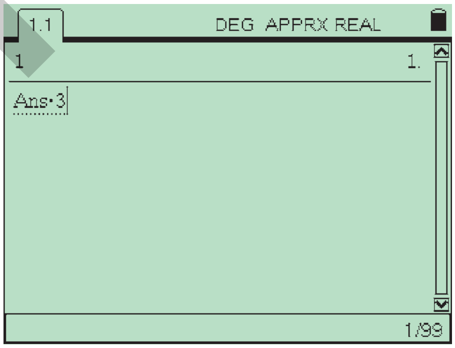
How to generate the terms of a geometric sequence using the TI-Nspire CAS

Generate the first five terms of the geometric sequence 1, 3, 9, 27, . . .

Steps

1 Start a new document by pressing ctrl + Ⓝ.

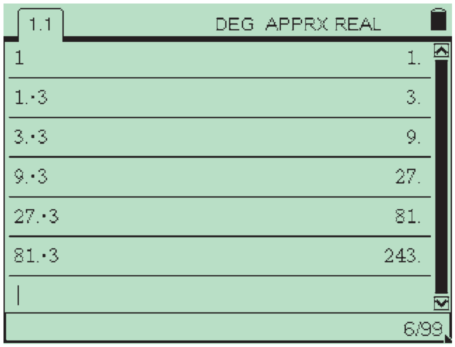
2 Select **1:Add Calculator**.
Enter the value of the first term **1**. Press enter.



3 The common ratio for the sequence is 3. So, type × **3**. Press enter. The second term in the sequence, 3, is generated.

4 Pressing enter again generates the next term, 9.

5 Keep pressing enter until the desired number of terms is generated.

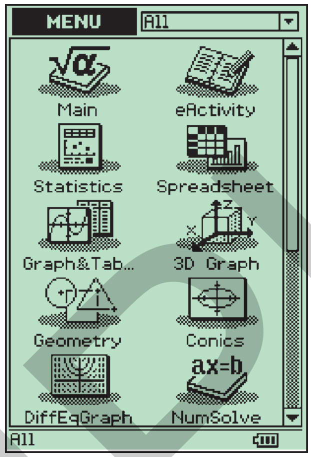


How to generate the terms of a geometric sequence in the main application using the ClassPad

Generate the first five terms of the geometric sequence 1, 3, 9, 27, . . .

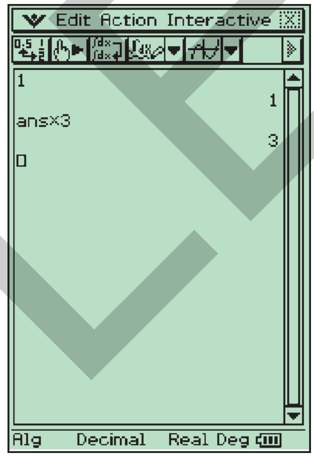
Steps

1 From the application menu screen, locate the built-in **Main** application. Tap Main to open, giving the screen shown opposite.

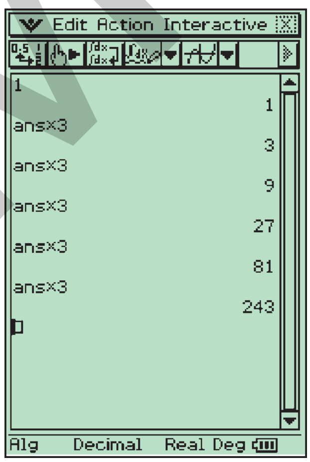


2 **a** Starting with a clean screen, enter the value of the first term, **1**.

b The common ratio for this sequence is 3. So, type × **3**. Then press EXE. The second term in the sequence, 3, is displayed.



3 Pressing EXE again generates the next term, 9. Keep pressing EXE until the required number of terms is generated.



Being able to recognise a geometric sequence is another skill that you need to develop. The key idea here is that the ratio between successive terms in a geometric sequence is constant (the common ratio).

Example 13 Testing for a geometric sequence

a Is the sequence 100, 50, 25, 12.5, . . . geometric?

Solution

Strategy: Divide successive terms in the sequence to see whether they have a constant ratio. If they do, the sequence is geometric.

1 Write down the terms of the sequence.

100, 50, 25, 12.5, . . .

2 Divide successive terms.

$$\frac{50}{100} = 0.5$$

$$\frac{25}{50} = 0.5 \quad \text{and so on}$$

3 Write down your conclusion.

Sequence is geometric as terms differ by a constant ratio.

b Is the sequence 2, 4, 6, 8, . . . geometric?

Solution

1 Write down the terms of the sequence.

2, 4, 6, 8, . . .

2 Divide successive terms.

$$\frac{4}{2} = 2$$

$$\frac{6}{4} = 1.5$$

3 Write down your conclusion.

Sequence is not geometric as successive terms have different ratios.

Exercise 9E

1 Which of the following sequences are geometric?

a 1, 2, 3, 4, . . . **b** 1, 2, 4, 8, . . . **c** 10, 100, 1000, . . .
d 1, 1, 0, 1, . . . **e** 6, 18, 24, . . . **f** 16, 8, 4, . . .
g 20, 30, 40, . . . **h** 2, 4, 8, 16, . . . **i** 10, 20, 40, 80, . . .
j 10, −20, 40, . . . **k** −100, 50, −25, . . . **l** −2, −4, −8, . . .

2 Which of the following sequences are geometric?

a 1.1, 1.21, 1.331, . . . **b** 9.8, 8.8, 7.8, 6.8, . . . **c** 1.01, 1.02, 2.02, . . .
d 7.7, 8.8, 9.9, . . . **e** 36.3, 54.45, 81.675, . . . **f** 0.01, 0.02, 0.04, . . .
g 6000, 4000, 2000, . . . **h** −2000, 4000, −8000, . . . **i** −10.9, −109, −1090, . . .

3 Consider the sequence 2, 20, 200, 2000, . . .

a Why is the sequence geometric? **b** What is the common ratio?
c What is the next term in the sequence?

Back to Menu >>>

d Starting with 2, how many times do you have to multiply 2 by the common ratio to get to the 5th term? What is the value of the 5th term?

e Starting with 2, how many times do you have to multiply 2 by the common ratio to get to the 15th term? What is the value of the 15th term?

4 Consider the sequence $1024, 256, 64, 16, \ldots$

a Why is the sequence geometric?

b What is the common ratio?

c What is the next term in the sequence?

d Starting with 1024, how many times do you have to multiply 1024 by the common ratio to get to the 7th term? What is the value of the 7th term?

e What is the value of the 10th term?

5 Consider the sequence $-1, 5, -25, 125, \ldots$

a Why is the sequence geometric?

b What is the common ratio?

c What is the next term in the sequence?

d Starting with -1, how many times do you have to multiply -1 by the common ratio to get to term 6? What is the value of term 6?

9.6 The *n*th term of a geometric sequence

Some notation

To find a general rule for finding the *n*th term of a geometric sequence, we first need to introduce some notation.

- As for arithmetic sequences, we use t_n to indicate the *n*th term in the sequence and a to represent the **first term** in the sequence.
- We use the symbol r to represent the **common ratio**.

Thus, for the sequence

$$2, 4, 8, 16, \ldots$$

the first term $a = 2$ and the common ratio $r = 2$.

The rule

Consider a geometric sequence with first term a and common ratio r.

Then:

$$t_1 = a$$
$$t_2 = t_1 \times r = ar$$
$$t_3 = t_2 \times r = ar^2$$
$$t_4 = t_3 \times r = ar^3$$
$$t_5 = t_4 \times r = ar^4 \quad \text{and so on.}$$

Thus, following the pattern, we can write:

$$t_n = ar^{n-1}$$

Back to Menu >>>

This gives us the following rule for the nth term of a geometric sequence.

Rule for finding the nth term in a geometric sequence

The nth term of a geometric sequence is given by

$$t_n = ar^{n-1}$$

where a is the value of the first term and r is the common ratio.

Example 14 Identifying a and r in a geometric sequence

For the geometric sequence 12, 18, 27, 36, . . . , write down the values of a, r and t_3.

Solution

1 a is the first term. $a = 12$

2 r is the common ratio. $r = \frac{18}{12} = 1.5$

3 t_3 is the third term. $t_3 = 27$

Example 15 Using the rule to find the nth term of a geometric sequence

The first term of a geometric sequence is $a = 6$ and the common ratio is $r = 2$. Use the rule to determine the 7th term of the sequence.

Solution

1 Use the rule $t_n = ar^{n-1}$. $t_n = ar^{n-1}$

2 In this example, $a = 6$, $r = 2$. For the 7th term, $n = 7$. $a = 6,\ r = 2,\ n = 7$

3 Substitute these values in the rule and evaluate.

$$\therefore t_7 = 6 \times 2^{7-1} = 6 \times 2^6 = 384$$

4 Write down your answer. The 7th term is 384.

Example 16 Using the rule to find the nth term of a geometric sequence

Use the rule to determine the 8th term of the geometric sequence 100, 50, 25, 12.5, . . .

Solution

1 Use the rule $t_n = ar^{n-1}$. $t_n = ar^{n-1}$

2 Here, $a = 100$, $r = \frac{50}{100} = 0.5$ and $n = 8$. $a = 100,\ r = 0.5,\ n = 8$

3 Substitute these values in the rule and evaluate.

$$\therefore t_7 = 100 \times 0.5^{8-1} = 100 \times 0.5^7 = 0.781\,25$$

4 Write down your answer. The 8th term is 0.781 25.

Back to Menu >>>

Example 17 **Using two terms to determine a and r**

In a geometric sequence, the 4th term is 24 and the 9th term is 768. Write down the first three terms of the sequence.

Solution

1 As $t_4 = 24$ we can write:

$24 = ar^{4-1} = ar^3 \quad (1)$

2 As $t_9 = 768$ we can write:

$768 = ar^{9-1} = ar^8 \quad (2)$

3 Solve equations (1) and (2) for a and r. Divide equation (2) by equation (1) to solve for r.

$$\frac{768}{24} = \frac{ar^8}{ar^3} \quad (2) \div (1)$$

$$\therefore 32 = r^5$$

$$\therefore r = \sqrt[5]{32} = 32^{\frac{1}{5}} = 2$$

4 Substitute $r = 2$ in (1) to find a.

$24 = a \times 2^3 = 8a$

$\therefore a = 3$

5 The first three terms are a, ar, ar^2. Use the values of a and r to write down your answer.

The first three terms of the sequence are 3, 6, 12.

Example 18 **Finding when a term in a sequence first exceeds a given value**

How many terms would we have to write down in the geometric sequence 2, 10, 50, 250, . . . before we found a term greater than 30 000?

Solution

Strategy: Find an expression for the nth term of the sequence. A graphics calculator can then be used to display the sequence in a table. The first term that exceeds 30 000 can then be found.

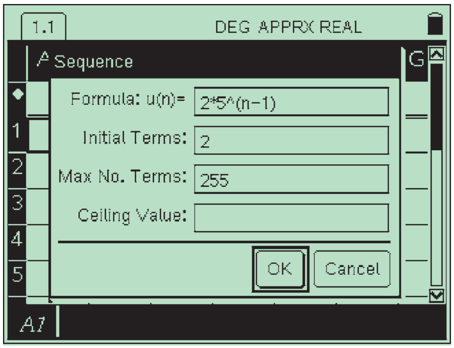
1 For this sequence, $a = 2$ and $r = 5$. Use this information and the rule $t_n = ar^{n-1}$ to write down an expression for the nth term.

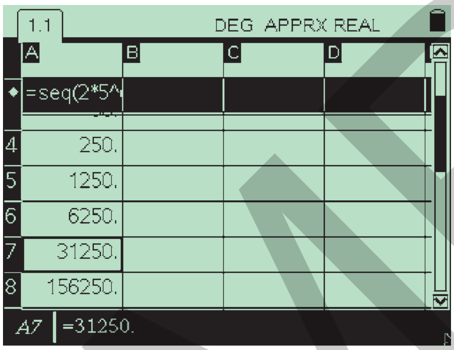
$$t_n = ar^{n-1}$$

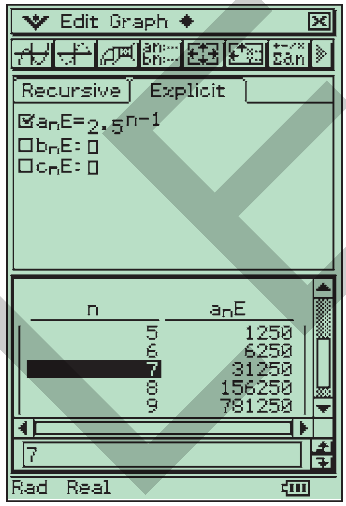
$$a = 2, \ r = 5$$

$$\therefore t_n = 2 \times 5^{n-1}$$

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the last term in the sequence that exceeds 30 000; in this case, the 7th term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	5	6	7	...
S_n	2	10	...	1250	6250	31250	...

The 7th term is the first to exceed 30 000.

Exercise 9F

1 For the geometric sequence:

a $3, 6, 12, \ldots$ write down the values of a, r, and t_3

b $15, 45, 135, \ldots$ write down the values of a, r, and t_2

c $200, 100, 50, 25, \ldots$ write down the values of a, r, and t_4

d $130, 13, 1.3, \ldots$ write down the values of a, r, and t_1

e $20, -24, 28.8, -34.56, \ldots$ write down the values of a, r, and t_3

f $-120, 30, -7.5, 1.875, \ldots$ write down the values of a, r, and t_4

Hint: As you are told these are geometric sequences, you can use any two terms to find *r*.

2 For a geometric sequence with:

a $a = 20$ and $r = 10$, determine the value of the 3rd term

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Back to Menu >>>

b $a = 3$ and $r = 2$, determine the value of the 7th term
c $a = 512$ and $r = 0.5$, determine the value of the 5th term
d $a = 1024$ and $r = -0.5$, determine the value of the 8th term
e $a = 1000$ and $r = -1.01$, determine the value of the 4th term
f $a = -100$ and $r = -2.2$, determine the value of the 3rd term

3 The first term of a geometric sequence is $a = 7.5$ and the common ratio is $r = 4$. Use the rule to determine the value of:

a the 3rd term **b** the 7th term **c** the 15th term

4 Use the rule to determine the value of:

a the 7th term of the geometric sequence 1, 5, 25, 125, 625, . . .
b the 8th term of the geometric sequence 10 000, 2000, 400, . . .
c the 10th term of the geometric sequence −1, 2, −4, 8, . . .
d the 9th term of the geometric sequence −20, −60, −180, . . .
e the 8th term of the geometric sequence $2,\ 1,\ \frac{1}{2}, \ldots$

5 Write down the first three terms of the geometric sequence in which:

a the 5th term is 81 and the 8th term is 2187
b the 2nd term is 10 000 and the 5th term is 1250
c the 3rd term is 40 and the 6th term is −320
d the 2nd term is 160 and the 4th term is 250

6 Write down the *n*th term of a geometric sequence with:

a $a = 3$ and $r = 2$ **b** $a = 12$ and $r = 1.3$ **c** $a = 5$ and $r = 0.2$
d $a = -10$ and $r = 1.75$ **e** $a = 25$ and $r = -0.5$ **f** $a = -4$ and $r = -2$

7 Write down the *n*th term of the following geometric sequences:

a 2, 4, 8, . . . **b** 30, 45, 67.5, . . . **c** 24, 6, 1.5, . . .
d −1.0, −0.8, −0.64, . . . **e** 1000, −500, 250, . . . **f** −1.1, 1.21, −1.331, . . .

8 How many terms would we have to write down in the geometric sequence:

a 2, 4, 8, 16, . . . before we found a term greater than 250?
b 1, 1.1, 1.21, . . . before we found a term greater than 2?
c 100, 80, 64, . . . before we found a term less than 10?
d −8, −16, −32, . . . before we found a term equal to −4096?
e 0.9, 0.81, 0.729, . . . before we found a term less than 0.1?

9.7 Applications modelled by geometric sequences

Populations grow in size and genuine antiques gain value in time, while most cars depreciate in value with time. These are examples of growth and decay of quantities that can be modelled with a geometric sequence.

Back to Menu >>>

Example 19 Growth of a population of bacteria

A population of a bacteria doubles its numbers every minute. If we start off with five bacteria, how many will we have at the start of the 31st minute (that is, after half an hour)?

Solution

1 Identify, name and define the key variable.

Let b_n be the number of bacteria at the start of the nth minute.

2 Use the information given to write down the number of bacteria, b_n, at the start of the 1st, 2nd, 3rd . . . minutes.

$b_1 = 5$
$b_2 = 5 \times 2 = 10$
$b_3 = 10 \times 2 = 20$ and so on.

3 The sequence 5, 10, 20, . . . is geometric, $a = 5, r = 2$.

Geometric sequence: $a = 5$, $r = 2$

4 Use the rule $t_n = ar^{n-1}$, with $a = 5$, $r = 2$ and $n = 31$, to write down an expression for the number of bacteria at the start of the 31st minute. Evaluate.

$$t_n = ar^{n-1}$$
$$\therefore b_n = 5 \times 2^{31-1} = 5 \times 2^{30} = 5\,368\,709\,120$$

5 Write down your answer.

After 30 minutes there will be 5 368 709 120 bacteria.

Example 20 Car depreciation

Cars depreciate in value over time. Depreciation is regarded as negative growth. One method of working out a secondhand car's value is to assume that it has depreciated at a constant rate since new. For example, suppose a car cost \$30 000 when new. What will be the value of the car after 10 years if we assume that it loses 10% of its value each year? Give your answer to the nearest dollar.

Solution

1 Identify, name and define the key variable. Use the information given to write down the value of the car, V_n, at the start of the 1st, 2nd, 3rd . . . years.
For example:

$$V_2 = 30\,000 - 10\% \text{ of } 30\,000$$

or

$$V_2 = 30\,000 - \frac{10}{100} \times 30\,000$$

Let V_n be the value of the car at the start of the nth year.

$V_1 = 30\,000$

$V_2 = 30\,000 - \frac{10}{100} \times 30\,000 = 27\,000$

$V_3 = 27\,000 - \frac{10}{100} \times 27\,000 = 24\,300$

and so on.

2 The sequence is geometric with $a = 30\,000$ and $r = 0.9$ $\left(r = \frac{27000}{30000} = 0.9\right)$

Geometric sequence: $a = 30\,000$ and $r = 0.9$

3 To determine the value of the car after 10 years, use the rule $V_n = ar^{n-1}$ with $a = 30\,000$, $r = 0.90$ and $n = 11$.
Note: In the sequence, the value of the car *after* 10 years is given by the 11th term.

$$V_n = ar^{n-1}$$
$$\therefore V_{11} = 30\,000 \times (0.9)^{11-1}$$
$$= 30\,000 \times (0.9)^{10}$$
$$= 10\,460.353\ldots$$

4 Write down your answer correct to the nearest dollar.

After 10 years, the value of the car will be $10 460.

Percentage change and the common ratio

As we have seen from the previous examples, when we generate a sequence of terms by adding or subtracting a fixed percentage of each term's value, the sequence is geometric. We can then calculate the common ratio by listing some terms and dividing successive terms to find r.

We can shorten the process by making use of a rule that gives the value of the common ratio of a geometric sequence in terms of percentage change and vice versa. We now derive these rules.

Consider the geometric sequence

$$a, ar, ar^2, \ldots$$

For this sequence, the percentage change R% between terms 1 and 2 is given by:

$$\begin{aligned} R &= \frac{\textit{Term } 2 - \textit{Term } 1}{\textit{Term } 1} \times 100\% \\ &= \frac{ar - a}{a} \times 100\% \\ &= \frac{a(r-1)}{a} \times 100 \\ &= (r-1) \times 100\% \end{aligned}$$

Rule 1: For determining percentage change from the common ratio

The percentage change R% is given by

$$R = (r-1) \times 100\%$$

where r is the common ratio.

Making r the subject of the formula in Rule 1 gives Rule 2.

Rule 2: For determining the common ratio from the percentage change

The common ratio r is given by

$$r = 1 + \frac{R}{100}$$

where R% is the percentage change.

Back to Menu >>>

Example 21 **Converting a common ratio (r) to a percentage change (R)**

Common ratio (r) to percentage change (R)

$r = 1.1$ (*increasing* sequence)

$$R = (r - 1) \times 100\%$$
$$= (1.1 - 1) \times 100\% = 10\%$$

$r = 0.9$ (*decreasing* sequence)

$$R = (r - 1) \times 100\%$$
$$= (0.9 - 1) \times 100\% = -10\%$$

Percentage change (R) to common ratio (r):

$R = 5\%$ (*increasing* sequence)

$$r = 1 + \frac{R}{100}$$
$$= 1 + \frac{5}{100} = 1.05$$

$R = -5\%$ (*decreasing* sequence)

$$r = 1 + \frac{R}{100}$$
$$= 1 + \frac{-5}{100} = 0.95$$

These rules, particularly Rule 2, are worth remembering as they can save a lot of time in solving application problems involving the use of geometric sequences. The rules will also be useful in the next chapter, when you meet difference equations.

Example 22 **Spread of an illness**

During an outbreak of gastroenteritis, a medical clinic reported six cases of the illness. Each day after that, the number of cases reported at the clinic increased by 9%. If this pattern continued, how many cases of gastroenteritis would you expect the clinic to report on the 8th day? Give your answer to the nearest whole number.

Solution

1 Identify, name and define the key variable.

Let G_n be the number of cases of gastroenteritis reported on the nth day.

2 As the number of cases reported increases by a fixed percentage each day, we have a geometric sequence. For this sequence, $a = 6$. Use $r = 1 + \frac{R}{100}$ to determine r. ($R = +9\%$)

Geometric sequence with:
$a = 6$
$$r = 1 + \frac{9}{100} = 1.09$$

3 The number of cases reported on the 8th day is given by the 8th term. Use $G_n = ar^{n-1}$ with $a = 6$, $r = 1.09$ and $n = 8$ to evaluate.

$$G_8 = 6 \times (1.09)^{8-1}$$
$$= 6 \times (1.09)^7$$
$$= 10.97$$

4 Write down your answer correct to the nearest whole number.

Eleven cases are expected on the 8th day.

Exercise 9G

1 a Convert the following percentage changes into a common ratio:

i $R = 10\%$ **ii** $R = 5\%$ **iii** $R = 13\%$ **iv** $R = 20\%$ **v** $R = 1\%$

vi $R = 100\%$ **vii** $R = 150\%$ **viii** $R = -10\%$ **ix** $R = -13\%$ **x** $R = -20\%$

xi $R = -5\%$ **xii** $R = -1\%$ **xiii** $R = 15\%$ **xiv** $R = 25\%$ **xv** $R = -25\%$

b Convert the following common ratios into percentage changes:

i $r = 1.10$ **ii** $r = 1.20$ **iii** $r = 1.05$ **iv** $r = 1.50$

v $r = 1.13$ **vi** $r = 0.95$ **vii** $r = 0.90$ **viii** $r = 0.87$

ix $r = 0.50$ **x** $r = 0.99$

2 A bacteria population doubles in size every minute.

a If we start off with 10 bacteria, how many bacteria will there be after one minute (at the start of the 2nd minute)?

b Complete the table.

Time (minutes)	1	2	3
Number of bacteria	10		

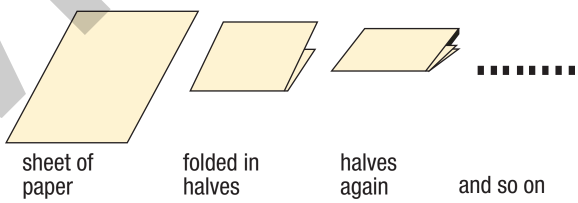
c If this pattern of growth continues, the sequence of terms generated is geometric. Write down the value of the first term and the common ratio.

d Write down an expression for the number of bacteria at the start of the nth minute.

e Determine the number of bacteria:

i at the start of the fifth minute **ii** after 20 minutes (at the start of the 21st minute)

3 A sheet of paper is 0.1 mm thick. It is folded in halves and the halves are folded again and so on.



a Complete the table.

b Assuming the sheet of paper is large enough, how thick will the folded paper be when it has been folded 15 times. Give your answer in metres.

Number of folds	1	2	3	4
Thickness (mm)	0.2	0.4		

4 Provided the conditions are right, a fish population increases its size by 40% every six months.

a Starting with a population of 1000 fish, how many fish would there be after six months?

b Calling the first six months Time period 1, the second six months Time period 2, etc., complete the table.

Start of time period	1	2	3	4
Number of fish				

c If this pattern of growth continues, the sequence of terms generated is geometric. Write down the value of the first term and the common ratio.

d Write down an expression for the number of fish, N_n, in the population at the start of the nth time period.

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Back to Menu >>>

e Determine the number of fish:

i at the start of the 7th time period **ii** after 5 years **iii** after 10 years

Give answers correct to the nearest whole number.

5 When you purchase an antique table for \$12 000, you are told its value will increase by 125% every 25 years.

a Making this assumption, how much do you expect the table to be worth in 25 years' time?

b Calling the first 25 years Time period 1, the second 25 years Time period 2, etc., complete the table.

Start of time period	1	2	3
Value of antique table (\$)			

c Write down an expression for the value of the table, V_n, at the start of the nth time period.

d Determine the value of the table:

i at the start of the 5th time period **ii** after 150 years **iii** after 250 years

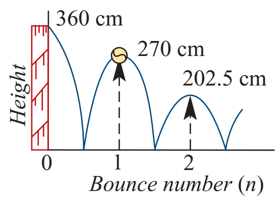
6 Suppose a car cost \$20 000 when new. Assume that it loses 7.5% of its value each year. Assuming that it continues to depreciate in value at the same rate,

a What is its value

i at the start of the 5th year of its life? **ii** after 8 years?

b How long will it take for the value of the car to depreciate to \$1000?

7 A ball is dropped vertically from a tower 3.6 metres high and the height of its rebound is recorded for four successive bounces. The results are shown in a table.



Bounce number (n)	0	1	2	3
Height (centimetres)	360.00	270.00	202.50	151.875

a Do the heights of the bouncing ball given in the table form a geometric sequence? Explain.

b Assuming that the heights of the bouncing ball follow a geometric sequence:

i predict the height of the 4th bounce

ii write down an expression for the height of the nth bounce

iii predict the height of the 15th bounce

9.8 The sum of the terms in a geometric sequence

Consider the geometric sequence

$$a, ar^1, ar^2, ar^3, \ldots,$$

The sum of the first n terms of this sequence, S_n, is given by

$$S_n = a + ar^1 + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} \qquad (1)$$

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Back to Menu >>>

Now, multiplying both sides of equation (1) by r:

$$rS_n = ar^1 + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \qquad (2)$$

Subtracting equation (2) from equation (1) gives:

$$S_n - rS_n = a + \cancel{ar^1} + \cancel{ar^2} + \cancel{ar^3} + \ldots + \cancel{ar^{n-2}} + \cancel{ar^{n-1}}$$
$$-\cancel{ar^1} - \cancel{ar^2} - \cancel{ar^3} - \ldots - \cancel{ar^{n-2}} - \cancel{ar^{n-1}} - ar^n$$

$$\therefore S_n - rS_n = a - ar^n$$

or $$S_n(1-r) = a(1-r^n)$$

so $$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \qquad \text{for } r \neq 1$$

Rule for finding the sum of a geometric sequence

The sum S_n of the first n terms of a geometric sequence is given by

$$S_n = \frac{a(1-r^n)}{1-r} \qquad \text{(the best rule to use when } 0 < r < 1\text{)}$$

or $$S_n = \frac{a(r^n-1)}{r-1} \qquad \text{(the best rule to use when } r > 1\text{)}$$

where a is the first term of the sequence and r is the common ratio ($r \neq 1$).

Example 23 Using the rule to find the sum of a geometric sequence

Use the rule to find the sum of the first five terms of the geometric sequence 2, 6, 18, . . .
Check your answer by writing out the first five terms in the sequence and adding.

Solution

1 In this example, $a = 2$, $r = 3$ $(6 \div 2)$, and $n = 5$.

$a = 2, r = 3, n = 5$

2 As $r > 1$, use the rule $S_n = \dfrac{a(r^n-1)}{r-1}$

$$S_n = \frac{a(r^n-1)}{r-1}$$

3 Substitute these values in the rule and evaluate.

$$\therefore S_5 = \frac{2(3^5-1)}{3-1} = \frac{2(3^5-1)}{2} = 242$$

4 Write down your answer.

The sum of the first five terms is 242.

5 Check answer.

$2 + 6 + 18 + 54 + 162 = 242$

Back to Menu >>>

Example 24 Using the rule to find the sum of a geometric sequence

For the geometric sequence 2, 8, 32, . . . , find an expression for the sum of the first n terms.

Solution

1 For this sequence, $a = 2$ and $r = 4$. $\quad a = 2,\ r = 4$

2 As $r > 1$, use the rule $S_n = \dfrac{a(r^n - 1)}{r - 1}$

3 Substitute these values to obtain an expression for the sum of n terms. Simplify.

$$\therefore S_n = \frac{2(4^n - 1)}{4 - 1} = \frac{2(4^n - 1)}{3}$$

4 Write down your answer. The sum of the first n terms is $\dfrac{2(4^n - 1)}{3}$.

Example 25 Finding when the sum of a sequence first exceeds a given value

When does the sum of the geometric sequence 2, 10, 50, . . . first exceed 7500?

Solution

Strategy: Find an expression for the sum of n terms of the sequence. A graphics calculator can then be used to display the sequence in a table. The first term that exceeds 7500 can then be found.

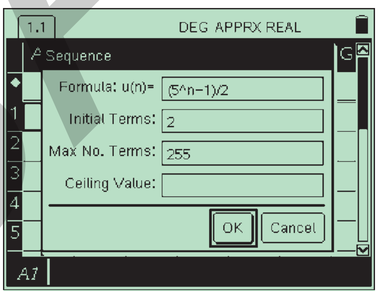
1 Use the rule $S_n = \dfrac{a(r^n - 1)}{r - 1}$ $\qquad S_n = \dfrac{a(r^n - 1)}{r - 1}$

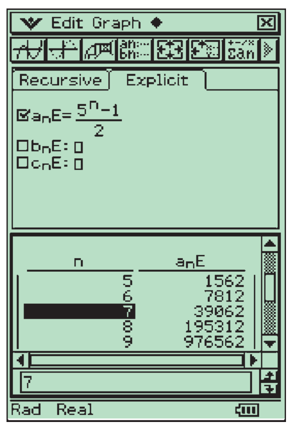
In this example, $a = 2, r = 5$ (10 ÷ 2) $\qquad a = 2, r = 5$

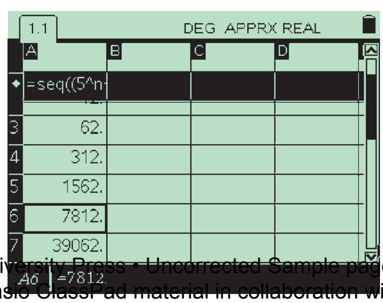
Use this information and the rule to find an expression for the sum in terms of n.

$$\therefore S_n = \frac{2(5^n - 1)}{5 - 1} = \frac{(5^n - 1)}{2}$$

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the last term in the sequence that exceeds 7500; in this case, the 6th term.







Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	4	5	6	...
S_n	2	12	...	312	1562	7812	...

The sum of the sequence first exceeds 7500 after adding 6 terms.

Exercise 9H

1 Use the rule to find the sum of the first:

a five terms in a geometric sequence with $a = 5$ and $r = 3$
b four terms in a geometric sequence with $a = 10$ and $r = 0.1$
c three terms in a geometric sequence with $a = -5$ and $r = 1.2$
d three terms in a geometric sequence with $a = 10\,000$ and $r = 1.06$
e five terms in a geometric sequence with $a = 512$ and $r = 0.5$

In each case, write out the terms and add to check your answer.

2 Use the rule to find the sum of the geometric sequence:

a 2, 4, 8, . . . to 20 terms
b 1000, 100, 10, . . . to 15 terms
c 1, 1.2, 1.44, . . . to 9 terms
d 2, 1, 0.5, . . . to 20 terms
e 1.000, 1.05, 1.1025, . . . to 10 terms
f 1.1, 1.21, 1.331, . . . to 5 terms

3 Use the rule to write down an expression for the sum of the first n terms of the following:

a a geometric sequence with $a = 10$ and $r = 1.5$
b a geometric sequence with $a = 50$ and $r = 0.2$
c 4, 20, 100, . . .
d 8, 4, 2, . . .
e 0.9, 0.3, 0.1, . . .

4 When does the sum of the geometric sequence:

a 2, 4, 8, . . . first exceed 50?
b 10, 100, 1000, . . . first exceed 1 000 000?
c 0.25, 0.5, 1, . . . first exceed 10?
d 2000, 2100, 2205, . . . first exceed 100 000?
e 0.1, 0.5, 2.5, . . . first exceed 12?
f 0.1, 1, 10, . . . first exceed 1000?

9.9 The sum of an infinite geometric sequence

Imagine taking a piece of string of length one metre, and cutting it in half. Put one piece aside and cut the other piece in half again. Keep on repeating the process.

What you will be left with (in theory, but not in practice) will be an infinite number of pieces of string with the following lengths (in metres).

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \ldots$$

This is a geometric sequence with first term $a = \frac{1}{2}$ and common ratio $r = \frac{1}{2}$. We are interested in its sum, that is,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \cdots$$

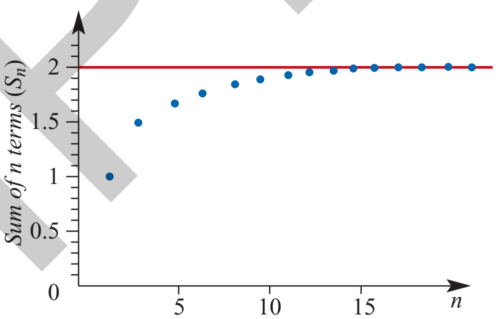
From practical considerations we know it must be one metre, because we started with a piece of string of length one metre. But how does this come about in theory?

As you may have noticed, the terms in a geometric sequence with a common ratio $0 < r < 1$ become smaller and smaller as we progress through the sequence. This is clearly seen in the geometric sequence with $a = 1$ and $r = 0.5$:

$$1, 0.5, 0.25, 0.125, 0.0625, 0.03125, 0.015625, 0.0078125, \ldots$$

As a result, when we add such a sequence, successive terms contribute successively less to the sum.

$S_1 = 1$
$S_2 = 1.5$
$S_3 = 1.75$
$S_4 = 1.875$
$S_5 = 1.9375$
$S_6 = 1.968\,75$
$S_7 = 1.984\,375$
$S_8 = 1.992\,187\,5$ and so on.



The other thing we notice is that, even if we add a lot more terms, the sum is not going to get a whole lot bigger. In fact, we might guess that, no matter how many more terms we added in, the sum will never exceed 2 (see graph). This is, in fact, the case and we can see this by looking at what happens to our rule for finding the sum of a geometric sequence with $0 < r < 1$.

Earlier we saw that the sum to n terms of a geometric sequence with first term a and common ratio r is:

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{for} \quad r \neq 1$$

If $0 < r < 1$ then as n, the number of terms we add up in the sequence, increases then r^n gets smaller and smaller. For example, $(0.1)^5 = 0.000\,01$, $(0.1)^{10} = 0.000\,000\,000\,1$, $(0.1)^{15} = 0.000\,000\,000\,000\,001$, and so on.

So, if n is very large we can write:

$$S_n \approx \frac{a(1 - 0)}{1 - r}$$

Now, we let n become as large as we possibly can. We now have what we call the **sum to infinity**, and the approximation is now exact, so we can write:

$$S_\infty = \frac{a}{1 - r} \quad \text{(provided } 0 < r < 1\text{)}$$

If we use this rule to find the sum to infinity of the geometric series we began with:

$$1 + 0.5 + 0.25 + 0.125 + 0.0625 + 0.03125 + 0.015625 + 0.0078125 + \cdots$$

we find:

$$S_\infty = \frac{1}{1-0.5} = \frac{1}{0.5} = 2$$

which is as we suspected.

Rule for finding the sum of an infinite geometric sequence

The sum of an infinite geometric sequence is given by

$$S_\infty = \frac{a}{1-r} \quad (\text{provided } 0 < r < 1 \text{ or } -1 < r < 0)$$

where a is the value of the first term and r is the common ratio.

Example 26 Using the rule to sum an infinite geometric sequence

Sum the geometric sequence 24, 6, 1.5, . . . to infinity.

Solution

1 Use the rule $S_\infty = \dfrac{a}{1-r}$. Here:

$a = 24$ and $r = 0.25$ $\left(r = \frac{6}{24} = 0.25\right)$

$$S_\infty = \frac{a}{1-r}$$

$$a = 24, r = 0.25$$

2 Substitute these values in the rule and evaluate.

$$\therefore S_\infty = \frac{24}{1-0.25} = 32$$

3 Write down your answer.

The sum to infinity is 32.

Example 27 An application of an infinite geometric sequence

Find the exact value of the repeated decimal $0.\dot{3} = 0.333\ldots$.

Solution

1 The repeated decimal $0.\dot{3} = 0.333\ldots$ can be written as the sum of an infinite geometric sequence.

$$0.333\ldots = 0.3 + 0.03 + 0.003 + \ldots$$

2 Sum the series using $S_\infty = \dfrac{a}{1-r}$. Here:

$a = 0.3$ and $r = 0.1$ $\left(r = \dfrac{0.03}{0.3} = 0.1\right)$

$$S_\infty = \frac{a}{1-r}$$

$$a = 0.3, r = 0.1$$

3 Substitute these values in the rule and evaluate.

$$\therefore S_\infty = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}$$

4 Write down your answer.

The exact value of $0.333\ldots$ is $\frac{1}{3}$.

Example 28 **An application of an infinite geometric sequence**

A stone is thrown so that it skips across the surface of a lake. The first skip is 5 metres in length, the second 3.5 metres, the third 2.45 metres and so on. If the stone keeps on skipping (and doesn't sink in the meantime), how far across the water will the stone eventually travel? Give your answer correct to the nearest metre.

Solution

1 The total distance, d, travelled by the stone is the sum: $5 + 3.5 + 2.45 + \cdots$
This is the sum to infinity of the terms of a geometric sequence with $a = 5$ and $r = \frac{3.5}{5} = 0.7$

$d = 5 + 3.5 + 2.45 + \cdots$

Geometric sequence:

$a = 5,\ r = \frac{3.5}{5} = 0.7 = 0.7$

2 Find the sum using $S_\infty = \frac{a}{1-r}$. Here:
$a = 5$ and $r = 0.2$ ($r = 3.5 \div 5 = 0.7$).
Evaluate.

$S_\infty = \frac{a}{1-r}$

$\therefore S_\infty = \frac{5}{1-0.7} = \frac{5}{0.3} = 16.66\ldots$

3 Write down your answer correct to the nearest metre.

The stone will eventually travel a distance of 17 metres.

Example 29 **An application of an infinite geometric sequence**

A cup of coffee is spilt on a carpet. It makes a stain that is 200 cm^2 in area. The stain continues to grow in size. In the first minute, the area of the stain grows by 40 cm^2. In successive minutes it grows by 20% of the previous minute's growth. What is the final area of the stain?

Solution

Strategy: In problems like this we begin by focusing on the growth of the stain, as this is what we model with an infinite geometric sequence. Once we have worked this out, we add this amount to the original area of the stain (200 cm^2) to get the final total area of the stain.

1 Let G_n be the growth in area of the stain during the nth minute.
Write down the *growth* in the area of the stain, G_n, at the start of the 1st, 2nd, 3rd, . . . minutes.
The total *growth* in the area is the sum of the infinite geometric sequence:
$40 + 8 + 1.6 + \cdots$

Let G_n be the growth in area of the stain during the nth minute.

$G_1 = 40$

$G_2 = 40 \times \frac{20}{100} = 8$

$G_3 = 8 \times \frac{20}{100} = 1.6$ and so on.

2 Find the sum using $S_\infty = \dfrac{a}{1-r}$. Here: $a = 40$ and $r = 0.2$ ($r = 8 \div 40 = 0.2$). Evaluate.

$$S_\infty = \frac{a}{1-r}$$

$$a = 40,\ r = 0.2$$

$$\therefore S_\infty = \frac{40}{1-0.2} = \frac{40}{0.8} = 50$$

3 Write down your answer for the total growth.

The stain will eventually grow by a further 50 cm^2.

4 The final area of the stain, A, is the original area (200 cm^2) plus the growth.

$$A_{final} = 200 + 50 = 250$$

5 Write down your answer for the final area of the stain.

The final area of the stain is 250 cm^2.

Exercise 9I

1 Sum each of the following geometric sequences to infinity:

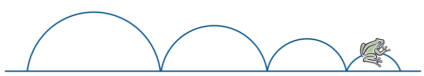
- **a** $4, 1, 0.25, \ldots$ correct to four decimal places
- **b** $1000, 100, 10, \ldots$ correct to two decimal places
- **c** $1.0, 0.9, 0.81, \ldots$
- **d** $6, 2, \frac{2}{3}, \ldots$
- **e** $0.01, 0.001, 0.0001, \ldots$ correct to four decimal places

2 Find the fractional form of the following repeated decimals:

a $0.\dot{1}$ **b** $0.1\dot{6}$ **c** $0.\dot{0}\dot{9}$ **d** $0.1\dot{3}$ **e** $0.\dot{5}$ **f** $0.8\dot{3}$

3 A frog hops 20 cm, then 10 cm, then 5 cm, and so on.

- **a** How far will the frog travel in six hops?
- **b** How far will the frog eventually hop (assuming it has the energy to keep hopping forever)?



4 A large post is driven into the ground by a pile driver. On the first hit, the post is driven 30 cm into the ground. On the second strike it is driven 24 cm into the ground. On the third strike it is driven 19.2 cm into the ground and so on.

- **a** How far will the post be driven into the ground after six hits?
- **b** What is the maximum distance the post can be driven into the ground?

5 A pine sapling was 3 metres tall when it was first planted. In its first year it grew 1 metre. In each successive year it grew 75% of the previous year's **growth**. If it continues to grow in this manner, what is the maximum height the tree can grow to?

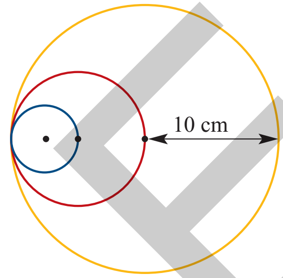
Hint: Remember to add in the original height of the seedling.

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Back to Menu >>>

6 Rob weighs 82 kg. He goes on a diet and loses 1 kg in one week. On the same diet he finds that the next week he loses 0.9 kg and the week after 0.81 kg. If he keeps on losing weight in the same manner, what is the lowest weight he can expect to get down to?

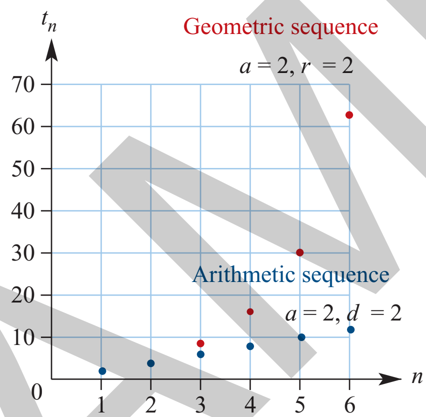
7 Han drew a circle of radius 10 cm. Inside it she drew a second circle whose diameter was the radius of the first. Inside the second circle she drew a circle whose diameter was the radius of the second, and so on as shown in the diagram. If she were able to continue the process indefinitely, show that the total circumference of the circles is 40π cm.



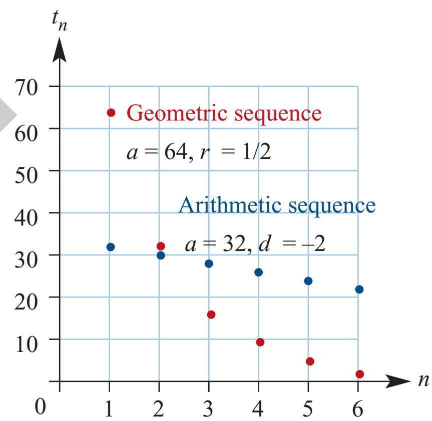
9.10 Rates of growth of arithmetic and geometric sequences

Graph 1 below plots the first few terms of:

- an *arithmetic* sequence with first term $a = 2$ and common difference $d = 2$, and
- a *geometric* sequence with first term $a = 2$ and common ratio $r = 2$



Graph 1



Graph 2

Note: The value of the terms in the *arithmetic* sequence grow in a *linear* manner, while the terms in the *geometric* sequence grow in an *exponential* manner. As a result, the terms in the geometric sequence rapidly exceed the value of the corresponding terms in the arithmetic sequence. Furthermore, no matter what the starting value of the arithmetic sequence, provided $r > 1$, the terms in a geometric sequence will always eventually catch up and pass the value of the terms in an arithmetic sequence. The reverse is true for geometric sequences with a positive common ratio less than 1 (see Graph 2).

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

How to graph the terms of a sequence using the TI-Nspire CAS

Plot the terms of the following sequences on the same graph:

- sequence 1: arithmetic with $a = 2$ and $d = 2$
- sequence 2: geometric with $a = 2$ and $r = 2$

for $n = 1, 2, \ldots 6$. These are the sequences plotted previously.

Steps

1 Write down an expression for the *n*th term of the two sequences using the rules.
Set the calculator to **Seq**uence mode.

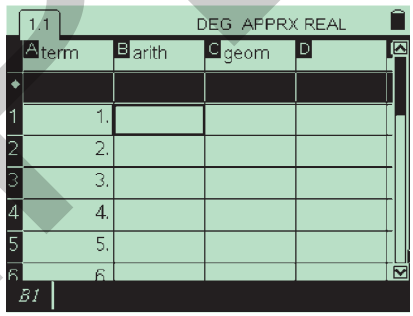
arithmetic: $t_n = 2 + (n - 1) \times 2$
geometric: $t_n = 2 \times 2^{(n-1)}$

2 Start a new document by pressing ctrl + N.

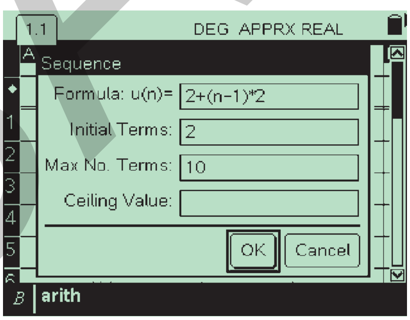
3 Select **3:Add Lists & Spreadsheet**.

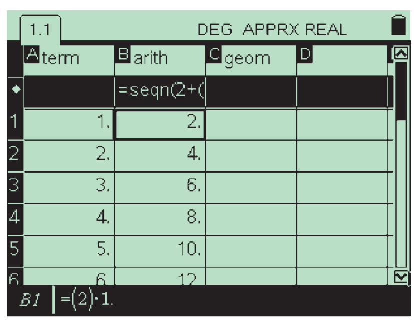
a. Enter the numbers **1** to **6** into a list named *term*, as shown.
Note: You can also use the sequence command to do this.

b. Name column B *arith* and column C *geom*.



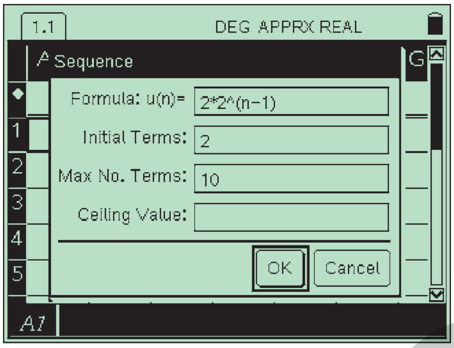
4. a. Place the cursor in any cell in column B and press menu **/3:Data/1:Generate Sequence** and type in the entries as shown (below left). Use tab to move between entry boxes. Press enter to close the pop-up screen and display the values of the first six terms in the arithmetic sequence in column B (below right).

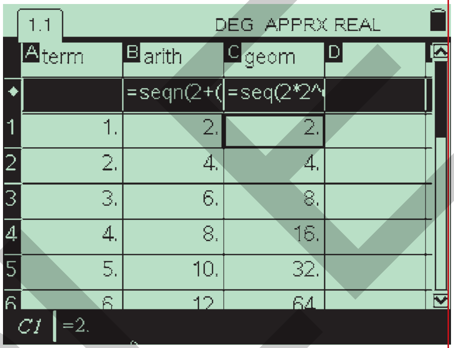




Back to Menu >>>

b Place the cursor in any cell in column C and press (menu)/**3:Data/1:Generate Sequence** and type in the entries shown (below left). Press (enter) to close the pop-up screen and display the values of the first six terms in the geometric sequence in column C (below right).



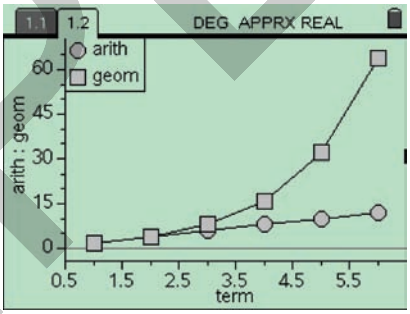


5 Press (home) and select **5:Data & Statistics**.

a. Construct a scatterplot using *term* as the independent variable and *arith* as the dependent variable.

b. Press (menu)/**2:Plot Properties/6:Add Y Variable**. Select the variable *geom*. Press (enter). This will show the terms of the geometric sequence on the same plot.

Note: The arithmetic sequence increases in a linear manner, whereas the geometric sequence increases in an exponential manner.



Back to Menu >>>

How to graph the terms of a sequence using the ClassPad

Plot the terms of the following sequences on the same graph:

- sequence 1: arithmetic with $a = 2$ and $d = 2$
- sequence 2: geometric with $a = 2$ and $r = 2$

for $n = 1, 2, \ldots 6$. These are the sequences plotted previously.

Steps

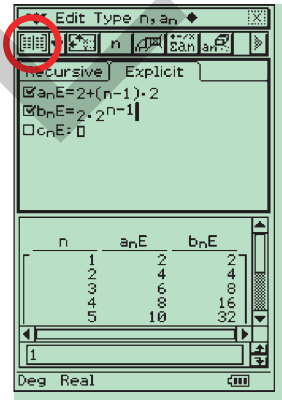
1 Write down an expression for the nth term of the two sequences using the rules.
Set the calculator to **Seq**uence mode.

arithmetic: $t_n = 2 + (n - 1) \times 2$
geometric: $t_n = 2 \times 2^{(n-1)}$

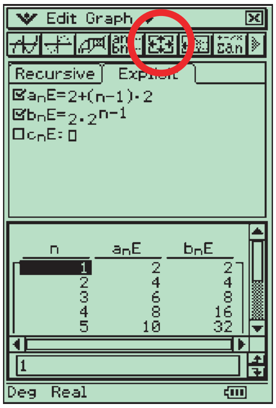
2 Open the built-in **Sequence** application.

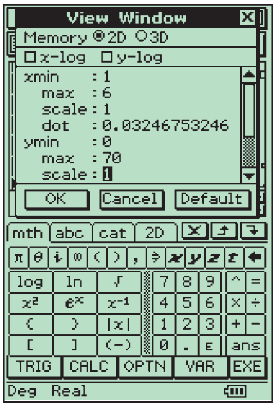
To enter the expression for the nth terms, tap the **Explicit** tab to enter the explicit sequence equation screen. For **a$_n$E** type $\mathbf{2 + (n - 1) \times 2}$, the expression for t_n for the arithmetic sequence, and press EXE to confirm your entry.
b$_n$E type $\mathbf{2 \times 2^{(n-1)}}$, the expression for t_n for the geometric sequence, and press EXE to confirm your entry.

Tap the [table] icon to display the terms of the sequence in table format.



3 Tap [View Window] to open the **View Window**. Enter the values as shown opposite. Tap OK to confirm your selection.
Note: The **dot** value indicates the **trace** increment and may be different on other calculator screens.



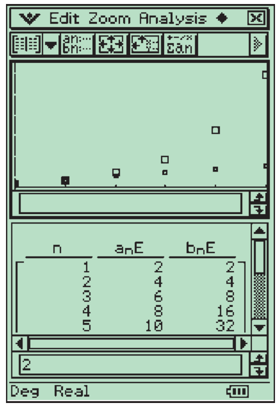


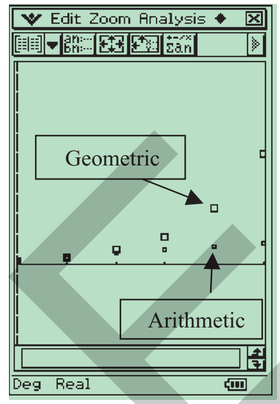
Back to Menu >>>

4 Tap to plot the sequence values.

Tapping Resize from the icon panel will allow the graph to fill the entire screen.

Note: The arithmetic sequence increases in a linear manner, whereas the geometric sequence increases in an exponential manner.





Exercise 9J

Using a graphics calculator, on the same axes plot the first five terms of the following pairs of sequences. In each case, comment on the differences that you notice in the two plots.

Note: You will need to readjust the window settings as you move through the exercises.

1 a arithmetic sequence: $a = 32$ and $d = 0.5$ geometric sequence: $a = 32$ and $r = 0.5$

b arithmetic sequence: $a = 1$ and $d = 2$ geometric sequence: $a = 1$ and $r = 2$

c arithmetic sequence: $a = 100$ and $d = -5$ geometric sequence: $a = 100$ and $r = 0.5$

d arithmetic sequence: $a = 100$ and $d = 5$ geometric sequence: $a = 1$ and $r = 5$

Key ideas and chapter summary

Sequence A **sequence** is a list of numbers or symbols written down in succession, as for example: 5, 15, 25, ...

Arithmetic sequence Each successive term in an **arithmetic sequence** can be found by *adding* or *subtracting* the same amount from the preceding term. This amount is called the **common difference**.

Example: 5, 15, 25, ... is an arithmetic sequence with a common difference of 10

The **rule** for determining the **value of t_n**, the nth term of an arithmetic sequence is:

$$t_n = a + (n - 1)d$$

where a is the first term in the sequence and d is the common difference.

Example: The 12th term of the arithmetic sequence 5, 15, 25, ... is $t_{12} = 5 + (12 - 1)10 = 115$.

The **rule** for determining the **sum**, S_n, of the first n terms of an arithmetic sequence is:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

where a is the first term in the sequence and d is the common difference.

Example: The sum of the first six terms of the arithmetic sequence 5, 15, 25, ... is $S_6 = \frac{6}{2}(2 \times 5 + (6 - 1) \times 10) = 180$.

Geometric sequence Successive terms in a **geometric sequence** can be found by *multiplying* the previous term by a constant amount. This amount is called the **common ratio**.

Example: 5, 10, 20, ... is a geometric sequence with a common ratio of $2 \left(\frac{10}{5} = 2\right)$.

The **rule** for determining the **value of t_n**, the nth term of a geometric sequence is:

$$t_n = ar^{n-1}$$

where a is the first term in the sequence and r is the common ratio.

Example: The 7th term of the geometric sequence 5, 10, 20, ... is:

$$t_7 = 5 \times 2^{7-1} = 5 \times 2^6 = 320.$$

Back to Menu >>>

Review

The **rule** for determining the **sum**, S_n, of the first n terms of a geometric sequence is:

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{for } r \neq 1.$$

The first rule is most convenient to use when $r > 1$. The second rule is most convenient to use when $r < 1$.

Example: The sum of the first 7 terms of the geometric sequence 5, 10, 20, . . . is $S_7 = \frac{5(2^7 - 1)}{2 - 1} = 635$.

Infinite geometric sequence

When the common ratio is less than one, a geometric sequence can be summed to infinity.

The **rule** for determining S_∞, the **sum to infinity** of a geometric sequence is:

$$S_\infty = \frac{a}{1 - r} \quad \text{for } 0 < r < 1 \text{ or } -1 < r < 0.$$

Example: The sum to infinity of the sequence: 10, 5, 2.5, . . . is:

$$S_\infty = \frac{10}{1 - 0.5} = 20$$

Skills check

Having completed this chapter you should be able to:

- recognise an arithmetic or geometric sequence
- determine the common difference of an arithmetic sequence from successive terms
- determine the common ratio of a geometric sequence from successive terms
- find the nth term of an arithmetic sequence given a and d
- find the nth term of a geometric sequence given a and r
- find the values of a and d of an arithmetic sequence given two terms
- find the values of a and r of a geometric sequence given two terms
- use a table of values to determine when the terms in a sequence first exceed a given value
- find the sum of the first n terms of an arithmetic sequence given a and d
- find the sum of the first n terms of a geometric sequence given a and r
- find the sum to infinity of a geometric sequence with a common ratio less than one
- convert a percentage change into a common ratio
- solve application problems involving arithmetic and geometric sequences

Cambridge University Press • Uncorrected Sample pages • 978-0-521-61328-6 • 2008 © Jones, Evans, Lipson
TI-Nspire & Casio ClassPad material in collaboration with Brown and McMenamin

Multiple-choice questions

1 Assuming the pattern continues, which one of the following is **not** an arithmetic sequence?

A 2, 32, 62, . . . **B** 3, 9, 15, . . . **C** 99, 97, 95, . . .
D $\frac{1}{3}$, 1, 3, . . . **E** −5, −10, −15, . . .

2 Assuming the pattern continues, which one of the following is **not** a geometric sequence?

A 16, 32, 64, . . . **B** 3, 9, 27, . . . **C** 99, 33, 11, . . .
D $\frac{1}{3}$, 1, 3, . . . **E** −5, −10, −15, . . .

3 The 8th term in the sequence, 2, 7, 12, . . . is:

A 16 **B** 19 **C** 37 **D** 40 **E** 42

4 The nth term in the sequence, 2, 7, 12, . . . is:

A $5n$ **B** $n + 5$ **C** $5n - 3$ **D** $5n + 7$ **E** $7n - 5$

5 The first term in a sequence is 100. Each subsequent term is 0.4 times the previous term. What is the 5th term?

A 1.024 **B** 2.56 **C** 40 **D** 101.6 **E** 102

6 The sum of the first eight terms of the arithmetic sequence: 25, 21, 17, . . . is:

A 85 **B** 88 **C** 90 **D** 91 **E** 168

7 The common ratio of the geometric sequence 0.3, 1.5, 7.5, . . . is:

A 0.3 **B** 0.5 **C** 1.2 **D** 5 **E** 6

8 The successive terms in a geometric sequence decrease in value by 7.5%. The common ratio for this geometric sequence is:

A 0.75 **B** 0.85 **C** 0.925 **D** 1.075 **E** 1.15

9 The 9th term of a geometric sequence with $a = 100$ and $r = 0.8$ is closest to:

A 8.0 **B** 13.4 **C** 16.8 **D** 21.0 **E** 80.0

10 The nth term of a geometric sequence with $a = 4$ and $r = 1.5$ is:

A 4×1.5^n **B** $4 \times 1.5^{n-1}$ **C** 1.5×4^n **D** $1.5 \times 4^{n-1}$ **E** $(4 \times 1.5)^{n-1}$

11 The sum of the first five terms of the geometric sequence 2, 2.5, 3.125, . . . is closest to:

A 2 **B** 4 **C** 13.2 **D** 16.4 **E** 26.4

12 The sum to infinity of the sequence 25, 5, 1, . . . is:

A 25 **B** 30 **C** 31.25 **D** 48 **E** 50

13 A geometric sequence has 1st term 8 and 4th term 64. The 6th term of this sequence is:

A 8 **B** 64 **C** 128 **D** 256 **E** 512

Review

14 In an arithmetic sequence, $a = 250$ and $d = 26$. The first term in this sequence to exceed 500 is the:

A 8th term **B** 9th term **C** 10th term **D** 11th term **E** 12th term

15 If successive terms in a geometric sequence increase by 12%, then the common ratio, $r =$:

A 0.12 **B** 0.88 **C** 1.0 **D** 1.12 **E** 1.2

Extended-response questions

1 A service-station storage tank needs refilling as there are only 1500 litres left in the tank. Petrol is pumped into the tank at the rate of 750 litres per minute.

a How much petrol is in the tank at the start of the third minute?

b Write down an expression for the amount of petrol in the tank, A_n, at the start of the nth minute.

c The tank holds 15 000 litres. How long does it take to fill the tank?

2 A chain email is set up on the internet. The first person sends the email to five people they know (Round 1). Each of these five people is asked to send the email to five people they know (Round 2) and so on.

a Complete the table on the assumption that all people who receive the email send it on to five new people.

Round	1	2	3	4
Number of emails	5	25		

b Is the sequence of terms arithmetic or geometric?

c How many people will be sent emails in:

i Round 6? **ii** Round 13?

d What is the total number of emails sent in the first 11 rounds?

3 In its first year of operation, a company sells \$1 200 000 worth of its product. The company predicts that the value of its sales will increase geometrically by 7% each year.

a Complete the table on this assumption.

Year	1	2	3
Sales (\$)	1 200 000		

b Assuming the growth in sales continues this same geometric pattern,

i write down an expression for the company's sales, S_n, in the nth year of operation

ii determine the value of the company's sales in their 5th year of operation

iii determine when the company's sales will first exceed \$5 000 000

iv find the total value of sales of the company during their first ten years of operation

4 After a long period of heavy rain, water seeps into an underground cave. After a day there are 5000 litres of water in the cave. The next day a further 4000 litres of water seeps into the cave. The day after, 3200 litres, and so on.

a How much water seeps into the cave on the 5th day?

b How much water in total has seeped into the cave after five days?

c How long before 20 000 litres of water in total have seeped into the cave?

d If the water keeps seeping into the cave indefinitely, and none is lost, how much water will be in the cave eventually?