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CHAPTER

9

Ratios and similarity

Objectives

- To divide a quantity in a given ratio
- To determine the ratio in which a quantity has been divided
- To apply the **transformations** which are expansions from the origin
- To define **similarity** of two figures
- To determine when two triangles are similar by using the conditions
  - equal angles (**AAA**)
  - equal ratios (**PPP**)
  - corresponding sides having the same ratio and the included angle equal (**PAP**)
- To apply **similarity** to solve problems
- To determine and apply **expansion factors** for areas and volumes

9.1 Ratios

This section is revision of work of previous years. Several examples are presented.

Example 1

Divide 300 in the ratio 3 : 2.

Solution

$$\begin{aligned} \text{one part} &= 300 \div 5 = 60 \\ \therefore \quad \text{two parts} &= 60 \times 2 = 120 \\ \therefore \quad \text{three parts} &= 60 \times 3 = 180 \end{aligned}$$

Example 2

Divide 3000 in the ratio 3 : 2 : 1.

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Solution

$$\begin{aligned} \text{one part} &= 3000 \div 6 = 500 \\ \therefore \quad \text{two parts} &= 500 \times 2 = 1000 \\ \therefore \quad \text{three parts} &= 500 \times 3 = 1500 \end{aligned}$$

Example 3

A day is divided into 10 new-hours, each new-hour is divided into 100 new-minutes and each new-minute is divided into 100 new-seconds. What is the ratio of a new-second to an ordinary second?

Solution

There are $10 \times 10^2 \times 10^2$ new-seconds in a day

and $24 \times 60 \times 60$ ordinary seconds in a day

$\therefore$ the ratio of new-seconds : ordinary seconds

$$\begin{aligned} &= \frac{1}{10^5} : \frac{1}{8.64 \times 10^4} \\ &= 864 : 1000 \\ &= 108 : 125 \end{aligned}$$

Example 4

Two positive integers are in the ratio 2 : 5. If the product of the integers is 40 find the larger integer.

Solution

Let a and b denote the integers

$$\frac{a}{b} = \frac{2}{5} \quad \ldots \boxed{1}$$

and $$ab = 40 \quad \ldots \boxed{2}$$

From $\boxed{1}$ $$a = \frac{2}{5}b \quad \text{Substitute in } \boxed{2}$$

$$\begin{aligned} \therefore \quad \frac{2}{5}b^2 &= 40 \\ \therefore \quad b^2 &= 100 \\ \therefore \quad b &= \pm 10 \end{aligned}$$

and as b is a positive integer, $b = 10$ and $a = 4$

The larger integer is 10.

Skillsheet TEACHER

Exercise 9A

Example 1 **1** Divide 9000 in the ratio 2 : 7.

Example 2 **2** Divide 15 000 in the ratio 2 : 2 : 1.

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3 $x : 6 = 9 : 15$. Find x.

4 The ratio of the numbers of orange flowers to pink flowers in a garden is 6 : 11. There are 144 orange flowers. How many pink flowers are there?

5 $15 : 2 = x : 3$. Find x.

6 The angles of a triangle are in the ratio 6 : 5 : 7. Find the sizes of the three angles.

7 Three men X, Y and Z share an amount of money in the ratio 2 : 3 : 7. If Y receives \$2 more than X, how much does Z obtain?

8 An alloy consists of copper, zinc and tin in the ratios 1 : 3 : 4 (by weight). If there is 10 g of copper in the alloy, find the weights of zinc and tin.

9 In a bag the ratio of red beads to white beads to green beads is 7 : 2 : 1. If there are 56 red beads, how many white beads and how many green beads are there?

10 On a map the length of a road is represented by 45 mm. If the scale is 1 : 125 000, find the actual length of the road.

11 Five thousand two hundred dollars was divided between a mother and daughter in the ratio 8 : 5. Find the difference between the sums they received.

12 Points A, B, C and D are placed in that order on a line so that $AB = 2BC = CD$. Express BD as a fraction of AD.

13 If the radius of a circle is increased by two units, find the ratio of the new circumference to the new diameter.

14 In a class of 30 students the ratio of boys to girls is 2 : 3. If six boys join the class, find the new ratio of boys to girls in the class.

15 If $a : b = 3 : 4$ and $a : (b + c) = 2 : 5$, find the ratio $a : c$.

16 The scale of a map reads 1 : 250 000. Find the distance, in kilometres, between two towns which are 3.5 cm apart on the map.

9.2 An introduction to similarity

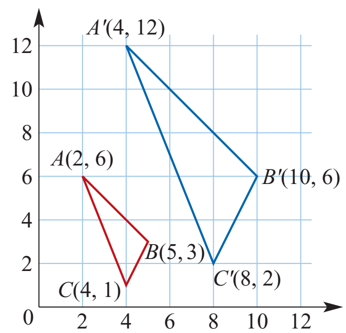
The two triangles ABC and $A'B'C'$ are similar.

Note: $OA' = 2OA$, $OB' = 2OB$, $OC' = 2OC$.

Triangle $A'B'C'$ can be considered as the image of triangle ABC under a mapping of the plane in which the coordinates are multiplied by 2.

This mapping is called an **expansion** from the origin of factor 2.

This can be written in transformation notation: $(x, y) \rightarrow (2x, 2y)$.



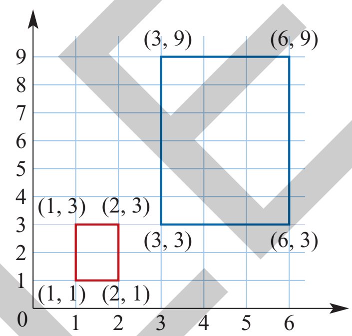
There is also a mapping from $\Delta A'B'C'$ to ΔABC which is an expansion from the origin of factor $\frac{1}{2}$.

The rule for this is $(x, y) \to \left(\frac{1}{2}x, \frac{1}{2}y\right)$.

Two figures are **similar** if one is congruent to an image of the other under an expansion from the origin of factor k.

For example, the rectangle of side lengths 1 and 2 is similar to the rectangle with side lengths 3 and 6.

Note here the expansion factor is 3 and the rule is $(x, y) \to (3x, 3y)$.



Note:

- any two circles are similar
- any two squares are similar
- any two equilateral triangles are similar

For a triangle ABC with side lengths a, b, c and a similar triangle $A'B'C'$ with corresponding side lengths a', b', c' it can be seen that

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = k$$

where k is the appropriate expansion factor.

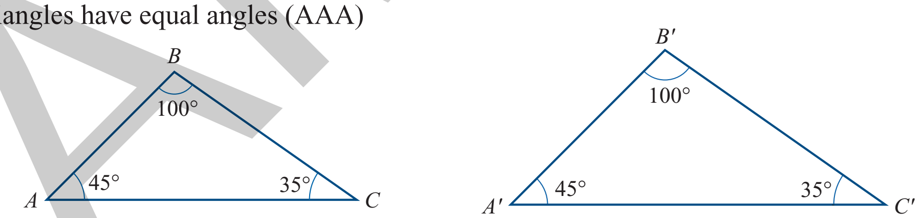
Similar statements can be made about other pairs of similar polygons.

Note also that the measure of an angle does not change under an expansion: i.e., for two similar figures, corresponding angles are equal.

Similar triangles

Two triangles are similar if one of the following conditions holds:

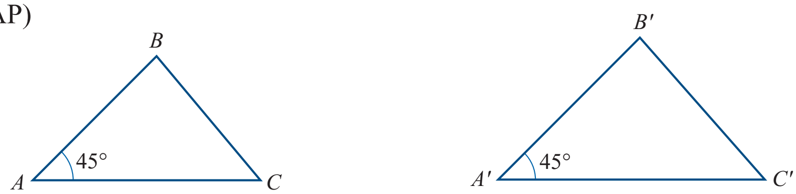
- triangles have equal angles (AAA)



- corresponding sides are in the same ratio (PPP)

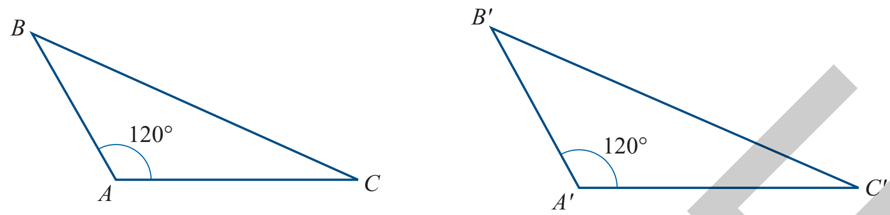
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = k, \text{ where } k \text{ is the expansion (enlargement) factor}$$

- two pairs of corresponding sides have the same ratio and the included angles are equal, (PAP)



$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$

- two pairs of corresponding sides have the same ratio and two corresponding non-included angles are equal, provided these angles are right angles or obtuse.



Triangle *ABC* is similar to triangle *A′B′C′* can be written symbolically as

$$\Delta ABC \sim \Delta A'B'C'$$

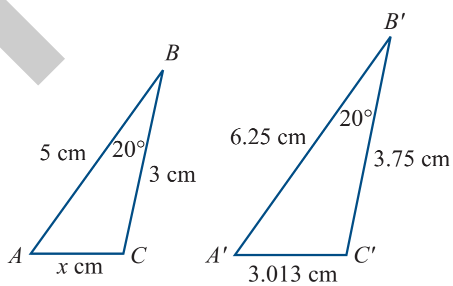
The triangles are named so that angles of equal magnitude hold the same position i.e., *A* corresponds to *A′*, *B* corresponds to *B′*, *C* corresponds to *C′*.

i.e. $\frac{BC}{B'C'} = \frac{AC}{A'C'}$ or $\frac{BC}{B'C'} = \frac{BA}{B'A'}$

Example 5

a Give the reason for triangle *ABC* being similar to triangle *A′B′C′*.

b Find the value of *x*.



Solution

a Triangle *ABC* is similar to triangle *A′B′C′* as

$$\frac{5}{6.25} = \frac{3}{3.75} = 0.8$$

and the magnitude of $\angle ABC$ = magnitude of $\angle A'B'C' = 20^\circ$

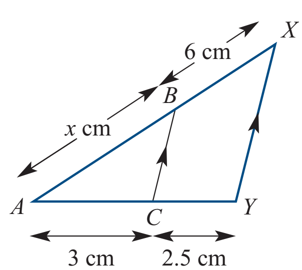
PAP is the condition for similarity.

b
$$\frac{x}{3.013} = \frac{5}{6.25}$$
$$\therefore \quad x = \frac{5}{6.25} \times 3.013$$
$$= 2.4104$$

Example 6

a Give the reason for triangle *ABC* being similar to triangle *AXY*.

b Find the value of *x*.



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Solution

a Corresponding angles are of equal magnitude (AAA).

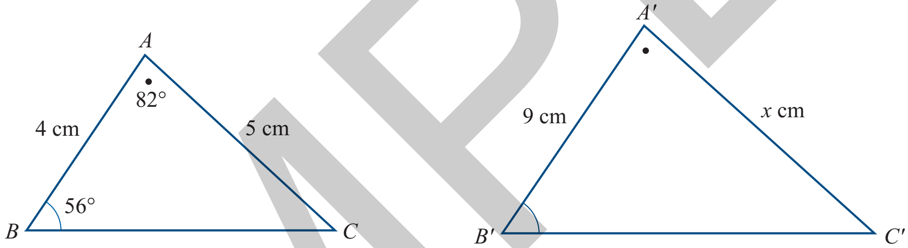
b
$$\frac{AB}{AX} = \frac{AC}{AY}$$
$$\text{i.e.,} \quad \frac{x}{x+6} = \frac{3}{5.5}$$
$$5.5x = 3(x+6)$$
$$2.5x = 18$$
$$\therefore \quad x = 7.2$$

Exercise 9B

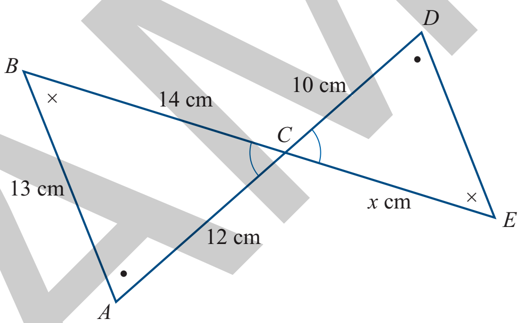
Example 5

1 Give reasons why the following pairs of triangles are similar and find the value of x in each case.

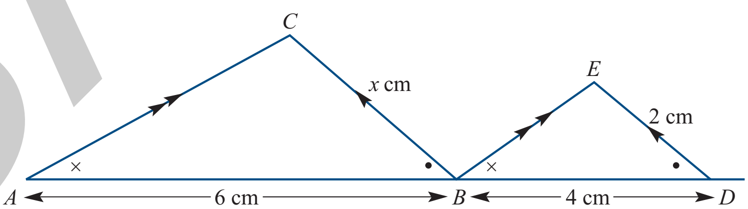
a



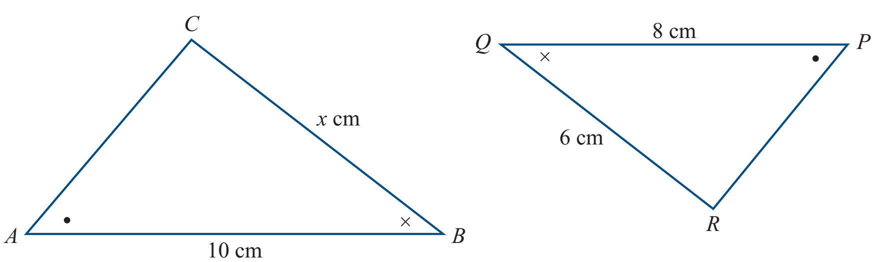
b



c



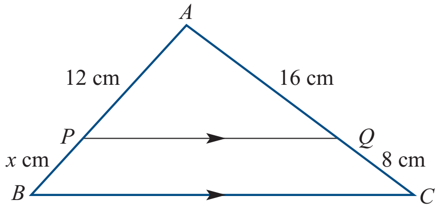
d



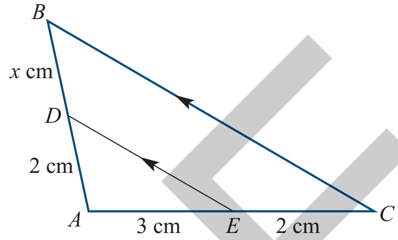
Example 6

2 Give reasons why the following pairs of triangles are similar and find the value of x in each case.

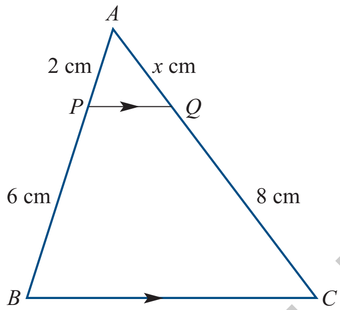
a



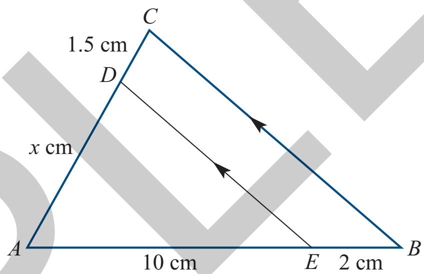
b



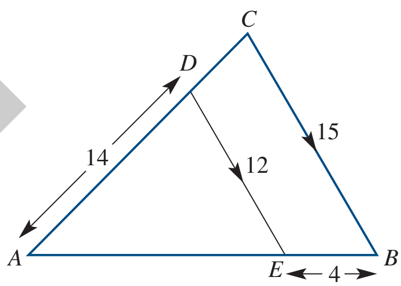
c



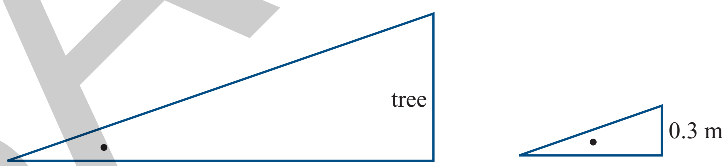
d



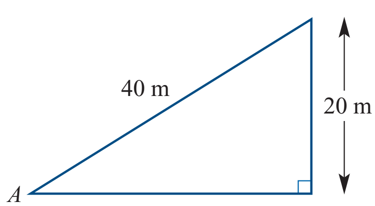
3 Given that $AD = 14$, $ED = 12$, $BC = 15$ and $EB = 4$, find AC, AE and AB.



4 A tree casts a shadow of 33 m and at the same time a stick 30 cm long casts a shadow 224 cm long. How high is the tree?

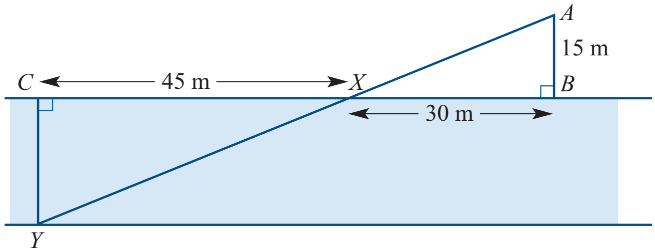


5 A 20 m high neon sign is supported by a 40 m steel cable as shown. An ant crawls along the cable starting at A. How high is the ant when it is 15 m from A?



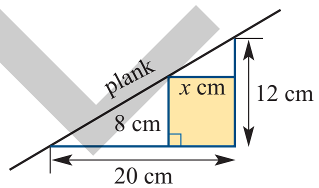
6 A hill has gradient of 1 in 20, i.e. for every 20 m horizontally there is a 1 m increase in height. If you go 300 m horizontally, how high up will you be?

7 A man stands at A and looks at point Y across the river. He gets a friend to place a stone at X so that A, X and Y are collinear. He then measures AB, BX and XC to be 15 m, 30 m and 45 m respectively. Find CY, the distance across the river.

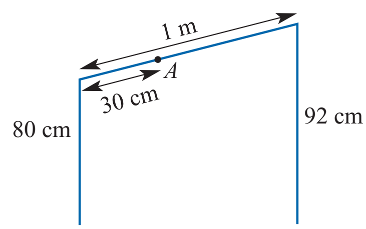


8 Find the height, h m, of a tree that casts a shadow 32 m long at the same time that a vertical straight stick 2 m long casts a shadow 6.2 m long.

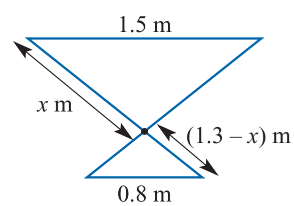
9 A plank is placed straight up stairs that are 20 cm wide and 12 cm deep. Find x, where x cm is the width of the widest rectangular box of height 8 cm that can be placed on a stair under the plank.



10 The sloping edge of a technical drawing table is 1 m from front to back. Calculate the height above the ground of a point A, which is 30 cm from the front edge.



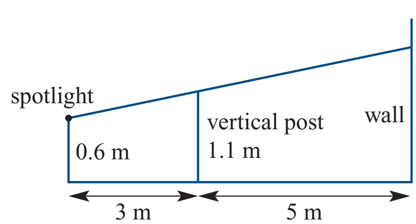
11 Two similar rods 1.3 m long have to be hinged together to support a table 1.5 m wide. The rods have been fixed to the floor 0.8 m apart. Find the position of the hinge by finding the value of x.



12 A man whose eye is 1.7 m from the ground, when standing 3.5 m in front of a wall 3 m high, can just see the top of a tower that is 100 m away from the wall. Find the height of the tower.

13 A man is 8 m up a 10 m ladder, the top of which leans against a vertical wall and touches it at a height of 9 m above the ground. Find the height of the man above the ground.

14 A spotlight is at a height of 0.6 m above ground level. A vertical post 1.1 m high stands 3 m away and 5 m further away there is a vertical wall. How high up the wall does the shadow reach?



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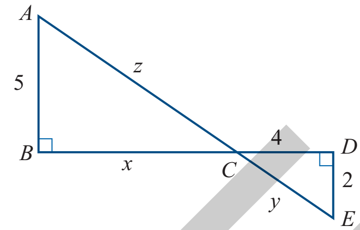
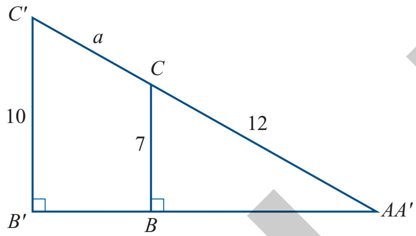
15 Measurements in the diagram shown are in cm.



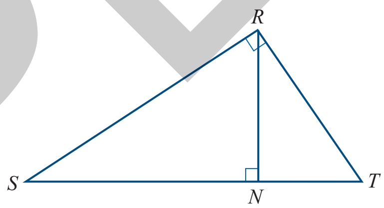
a Prove that $\Delta ABC \sim \Delta EDC$.
b Find x.
c Use Pythagoras' theorem to find y and z.
d Verify $y : z = ED : AB$.

16 Find a.



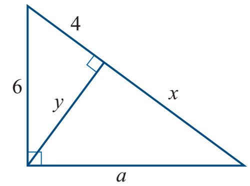
17 A man who is 1.8 m tall casts a shadow of 0.76 m in length. If at the same time a telephone pole casts a 3 m shadow, find the height of the pole.

18 In the diagram shown, $RT = 4$ cm, $ST = 10$ cm. Find the length NT.



19 ABC is a triangular frame with $AB = 14$ m, $BC = 10$ m, $CA = 7$ m. A point P on AB, 1.5 m from A, is linked by a rod to a point Q on AC, 3 m from A. Calculate the length of PQ.

20 Using this diagram, find a, x and y.



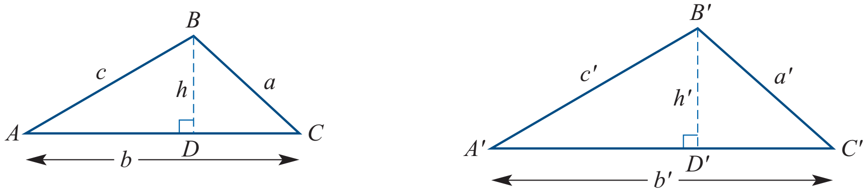
9.3 Areas, volumes and similarity

If two shapes are similar and the expansion (enlargement) factor is k, i.e., for any length AB of one shape, the length of the corresponding length $A'B'$ of the similar shape has length kAB, then the

$$\text{area of similar shape} = k^2 \times \text{area of the original shape}$$

For two triangles ABC and $A'B'C'$ which are similar, i.e., $\Delta ABC \sim \Delta A'B'C'$ with $A'B' = kAB$,

$$\text{area of triangle } A'B'C' = k^2 \times \text{area of triangle } ABC$$



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This can be shown by observing that $\Delta ABC \sim \Delta A'B'C'$ and

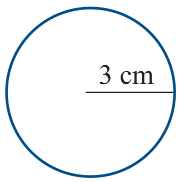
$$\text{area of triangle } A'B'C' = \frac{1}{2} \times b' \times h' = \frac{1}{2} \times kb \times kh,$$

(where $AC = b$ and $A'C' = b'$)

$$= k^2 \left(\frac{1}{2} bh\right)$$

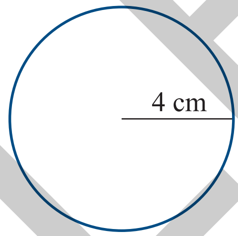
$$= k^2 \times \text{area of triangle } ABC$$

Some examples of similar shapes and the ratio of their areas are considered in the following.



Similar circles

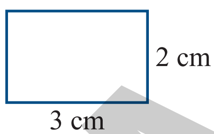
Scale factor $= \frac{4}{3}$



Area $= \pi.3^2$

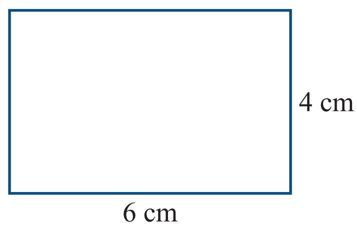
Area $= \pi.4^2$

$$\text{Ratio of areas} = \frac{\pi.4^2}{\pi.3^2} = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2$$



Similar rectangles

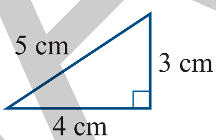
Scale factor = 2



Area $= 6 \text{ cm}^2$

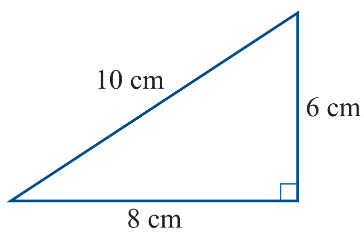
Area $= 24 \text{ cm}^2$

$$\text{Ratio of areas} = \frac{24}{6} = 4 = 2^2$$



Similar triangles

Scale factor = 2



Area $= 6 \text{ cm}^2$

Area $= 24 \text{ cm}^2$

$$\text{Ratio of areas} = \frac{24}{6} = 4 = 2^2$$

Example 7

The two rectangles shown below are similar. The area of rectangle $ABCD$ is 20 cm^2. Find the area of rectangle $A'B'C'D'$.

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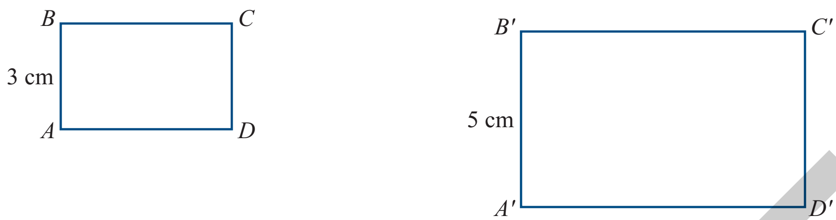


Solution

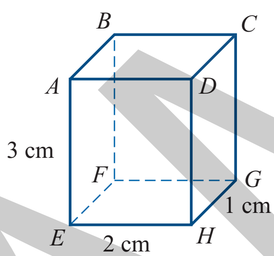
$$\text{The ratio of the length of their bases} = \frac{A'B'}{AB} = \frac{5}{3}$$

$$\text{The ratio of their areas} = \frac{\text{Area of } A'B'C'D'}{\text{Area of } ABCD} = \left(\frac{5}{3}\right)^2 = \frac{25}{9}$$

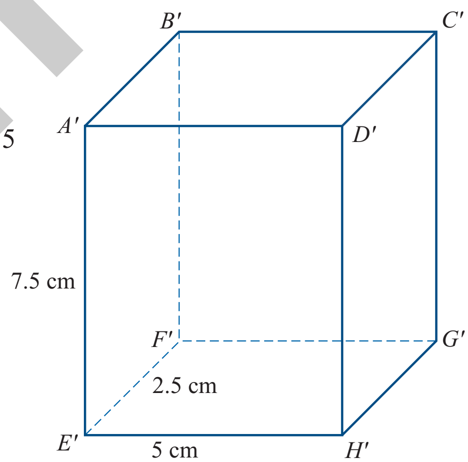
$$\therefore \quad \text{Area of } A'B'C'D' = \frac{25}{9} \times 20\ \text{cm}^2$$

$$= 55\frac{5}{9}\ \text{cm}^2$$

Two solids are considered to be similar if they have the same shape and the ratio of their corresponding linear dimensions are equal.



Scale factor = 2.5



The cuboids $ABCDEFGH$ and $A'B'C'D'E'F'G'H'$ are similar.
For similar solids, if the scale factor is k then the

$$\text{volume of the similar solid} = k^3 \times \text{volume of the original solid}$$

For example,

$$\text{Volume of } ABCDEFGH = (3 \times 2 \times 1)\ \text{cm}^3 = 6\ \text{cm}^3$$

$$\text{Volume of } A'B'C'D'E'F'G'H' = (5 \times 2.5 \times 7.5)\ \text{cm}^3 = 93.75\ \text{cm}^3$$

$$\text{The ratio of volumes} = \frac{93.75}{6} = 15.625 = 2.5^3$$

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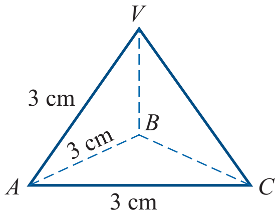
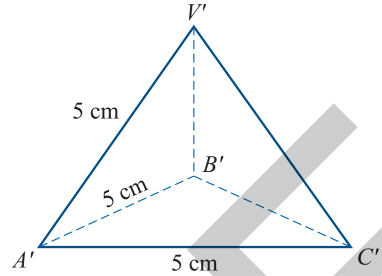
Here is another example.



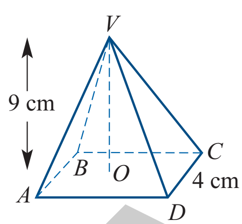
Scale factor $= \frac{5}{3}$

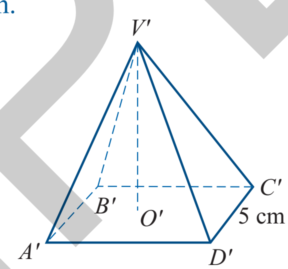


$$\text{Ratio of volumes} = \frac{5^3}{3^3} = \left(\frac{5}{3}\right)^3$$

Example 8

The two square pyramids are similar. $VO = 9$ cm.





a Find the ratio of the length of their bases, and hence the height, $V'O'$, of the pyramid $V'A'B'C'D'$.

b The volume of $VABCD$ is 48 cm^3. Find the ratio of their volumes, and hence find the volume of $V'A'B'C'D'$.

Solution

a The ratio of the length of their bases $= \frac{C'D'}{CD} = \frac{5}{4}$

$$\therefore \quad V'O' = \frac{5}{4} \times 9$$
$$= \frac{45}{4}$$

The length of $V'O'$ is 11.25 cm.

b The volume of $VABCD$ is 48 cm^3

$$\text{The ratio of their volumes} = \frac{\text{Volume of } V'A'B'C'D'}{\text{Volume of } VABCD} = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$$

$$\therefore \text{Volume of } V'A'B'C'D' = \frac{125}{64} \times 48 \text{ cm}^3$$
$$= 93.75 \text{ cm}^3$$

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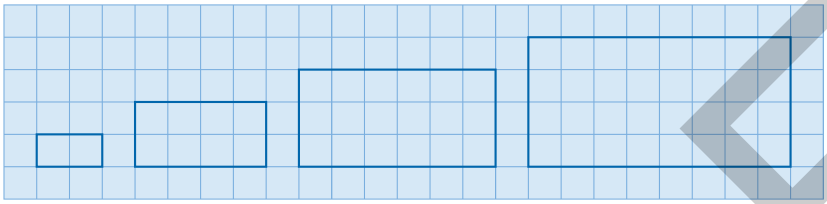
Skillsheet TEACHER

Exercise 9C

1 These four rectangles are similar.

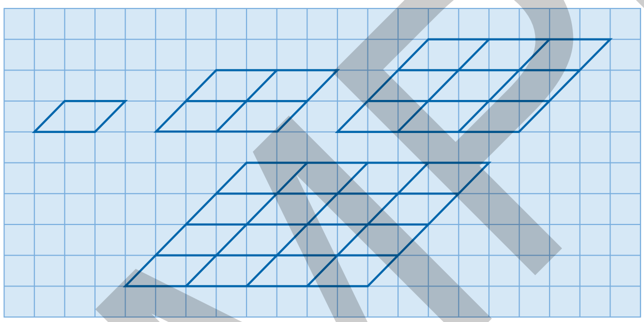


a Write down the ratio of the lengths of their bases.

b By counting rectangles, write down the ratio of their areas.

c Is there a relationship between these two ratios?

2 These four parallelograms are similar.



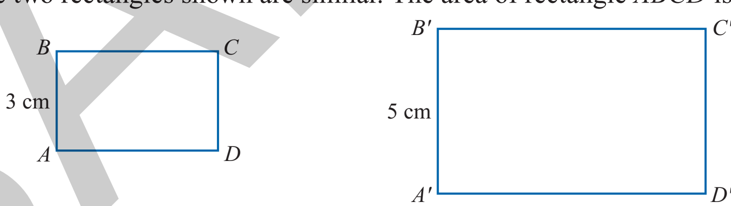
a Write down the ratio of the lengths of their bases.

b By counting parallelograms, write down the ratio of their areas.

c Is there a relationship between these two ratios?

Example 7

3 The two rectangles shown are similar. The area of rectangle $ABCD$ is 7 cm^2.



Find the area of rectangle $A'B'C'D'$.

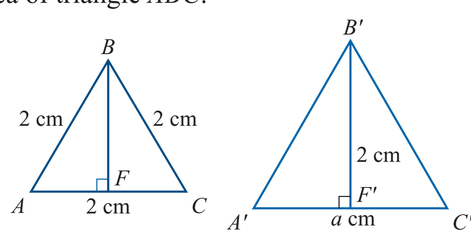
4 Triangle ABC is similar to triangle XYZ. $\dfrac{XY}{AB} = \dfrac{YZ}{BC} = \dfrac{ZX}{CA} = 2.1$

The area of triangle XYZ is 20 cm^2. Find the area of triangle ABC.

5 Triangles ABC and $A'B'C'$ are equilateral triangles.

a Find the length of BF. **b** Find a.

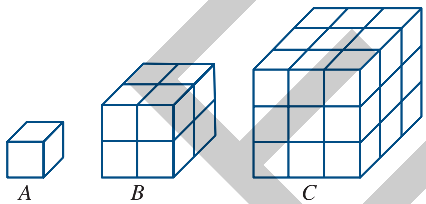
c Find the ratio $\dfrac{\text{Area of triangle } A'B'C'}{\text{Area of triangle } ABC}$



6 The areas of two similar triangles are 16 and 25. What is the ratio of a pair of corresponding sides?

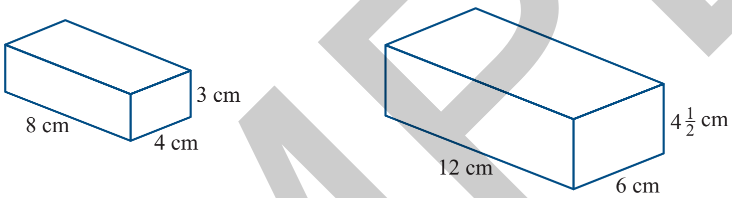
7 The areas of two similar triangles are 144 and 81. If the base of the large triangle is 30, what is the corresponding base of the smaller triangle?

8 These three solids are similar.



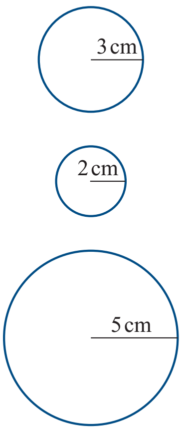
a Write down the ratio of the lengths of the bases.
b Write down the ratio of the lengths of the heights.
c By counting cuboids equal in shape and size to the cuboid given in *A*, write down the ratio of the volumes.
d Is there a relationship between the answers to **a**, **b** and **c**?

9 These are two similar rectangular blocks.



a Write down the ratio of their
i longest edges **ii** depths **iii** heights.
b By counting cubes of side 1 cm, write down the ratio of their volumes.
c Is there any relationship between the ratios in **a** and **b**?

Example 8 **10** These three solids are spheres.



a Write down the ratio of the radii of the three spheres.
b The volume of a sphere of radius r is given by the formula $V = \frac{4}{3}\pi r^3$. Express the volume of each sphere as a multiple of π.
Hence write down the ratio of their volumes.
c Is there any relationship between the ratios found in **a** and **b**?

*In **11** to **20**, objects referred to in the same question are mathematically similar.*

11 The sides of two cubes are in the ratio 2 : 1. What is the ratio of their volumes?

12 The radii of two spheres are in the ratio 3 : 4. What is the ratio of their volumes?

13 Two regular tetrahedrons have volumes in the ratio 8 : 27. What is the ratio of their sides?

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14 Two right cones have volumes in the ratio 64 : 27. What is the ratio of

a their heights **b** their base radii?

15 Two similar bottles are such that one is twice as high as the other. What is the ratio of

a their surface areas **b** their capacities?

16 Each linear dimension of a model car is $\frac{1}{10}$ of the corresponding car dimension. Find the ratio of

a the areas of their windscreens **b** the capacities of their boots
c the widths of the cars **d** the number of wheels they have.

17 Three similar jugs have heights 8 cm, 12 cm and 16 cm. If the smallest jug holds $\frac{1}{2}$ litre, find the capacities of the other two.

18 Three similar drinking glasses have heights 7.5 cm, 9 cm and 10.5 cm. If the tallest glass holds 343 millilitres, find the capacities of the other two.

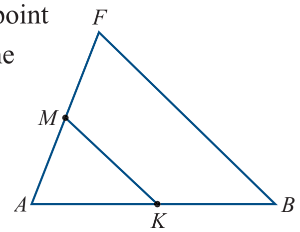
19 A toy manufacturer produces model cars which are similar in every way to the actual cars. If the ratio of the door area of the model to the door area of the car is 1 : 2500, find

a the ratio of their lengths **b** the ratio of the capacities of their petrol tanks
c the width of the model, if the actual car is 150 cm wide
d the area of the rear window of the actual car if the area of the rear window of the model is 3 cm^2.

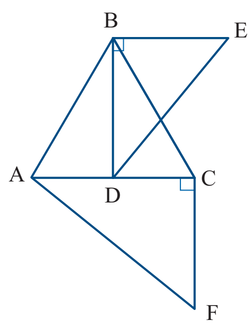
20 The ratio of the areas of two similar labels on two similar jars of coffee is 144 : 169. Find the ratio of

a the heights of the two jars **b** their capacities.

21 **a** In the figure, if M is the midpoint of AF and K is the midpoint of AB, the area of ΔABF is how many times as great as the area of ΔAKM?
b If the area of ΔABF is 15, find the area of ΔAKM.



22 In the diagram, ΔABC is equilateral. $\angle BDE = \angle CAF$ and D is the midpoint of AC. Find the ratio of area of ΔBDE : area ofΔACF.



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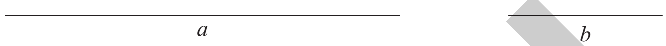
23 The areas of two similar triangles are 144 cm^2 and 81 cm^2. If the length of one side of the first triangle is 6 cm, what is the length of the corresponding side of the second?

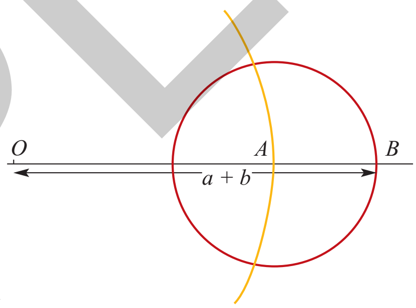
9.4 Geometric representation of arithmetic operations

Simple arithmetic operations correspond to elementary geometrical constructions. In many cases the validity of these constructions can be established through similar triangles. If two segments are given with lengths a and b (as measured by a given unit segment) then $a + b$, $a - b$, ra (where r is any rational number), ab, $\frac{a}{b}$ and $\sqrt{a}$ can be constructed.



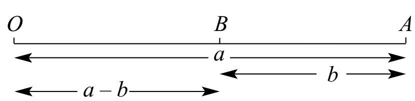
Construction of $a + b$

Draw a straight line and mark off with a compass, as shown in the diagram, the distance OA and AB where $OA = a$ and $AB = b$. Then $OB = a + b$.



Construction of $a - b$

Draw a straight line and mark off with a compass the distance OA and AB where $OA = a$ and $AB = b$, but this time AB is constructed in the other direction. Then $OB = a - b$.

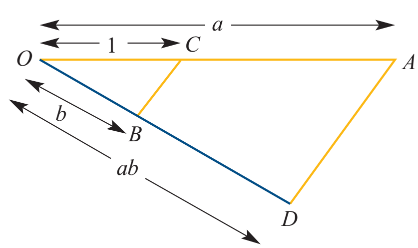


Construction of ra

To construct $3a = a + a + a$, three copies of the line segment of length a are constructed. For $na = a + a + \cdots + a$, where n is a natural number, n copies of the line segment of length a are constructed.

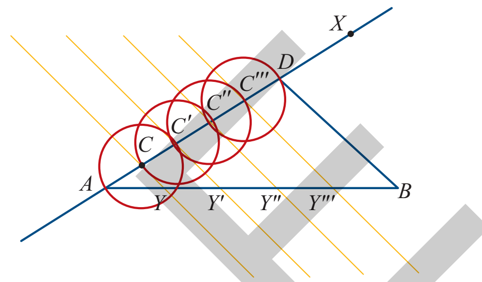
Construction of ab

Mark off line segments OA and OB of length a units and b units respectively. Construct OC of length 1 unit. Join points C and B and draw a line parallel to the line CB through A. The line segment OD has length ab.
Note that triangle OAD is similar to triangle OCB and $OA = aOC$. Therefore $OD = aOB = ab$.



Construction of $\frac{1}{b}$

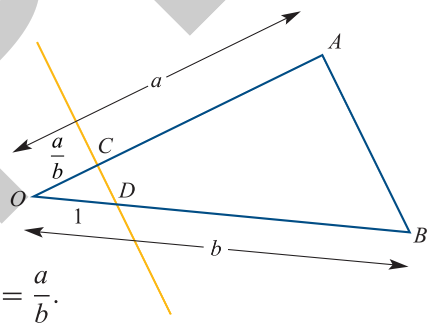
This will be done for $b = 5$.



Line segment AB is of unit length. Draw any line AX. Choose a line segment AC and then replicate this line segment four times to form line segments CC', $C'C''$, $C''C'''$ and $C'''D$. Draw line segment DB and then parallel line segments CY, $C'Y'$, $C''Y''$ and $C'''Y'''$ to divide line segment AB into five equal segments. Each of these segments has length $\frac{1}{5}$ of a unit.

Note that triangle ACY is similar to triangle ADB. And $5AC = AD$. Hence $AB = 5AY$

Construction of $\frac{a}{b}$



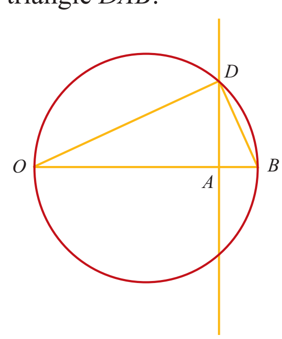
One way of constructing $\frac{a}{b}$ is to mark off line segments OA and OB of length a units and b units respectively. Construct OD of length 1 unit. Join points A and B and draw a line parallel to the line AB through D. The line segment OC has length $\frac{a}{b}$

Note that triangle OAB is similar to triangle OCD and $OB = bOD$. Therefore $OA = bOC$ and this implies $OC = \frac{a}{b}$.

Construction of $\sqrt{a}$

Construct line segments of length a and 1, and a circle of diameter $a + 1$. In the diagram $OA = a$ and $AB = 1$. Angle ODB is a right angle (right angle subtended at the circle by a diameter), and OAD is a right angle by construction.

Therefore triangle ODB is similar to triangle OAD and to triangle DAB.



Hence $\frac{OA}{AD} = \frac{AD}{AB}$

$\frac{a}{AD} = \frac{AD}{1}$

Therefore $AD^2 = a$

and hence $AD = \sqrt{a}$

Exercise 9D

1 Construct a line segment of length $\sqrt{3}$ units.

2 Construct a line segment of length $\sqrt{5}$ units.

3 Draw a line segment of length 10 cm and use a construction described above to divide it into three equal intervals.

4 Draw a line segment of length 20 cm and use a construction described above to divide it into nine equal intervals.

5 Draw two line segments OA and OB of lengths 4 cm and 14 cm respectively. Use a construction described above to construct a line segment of length $\frac{2}{7}$ units.

6 Draw two line segments OA and OB of lengths 9 cm and 13 cm respectively. Use a construction described above to construct a line segment of length $\frac{9}{13}$ units.

7 Describe the method for constructing a line of length $\frac{10}{3}$ units.

8 Illustrate the construction of a line segment of length 3×4 units, given line segments of length 3 units, 4 units and 1 unit.

9.5 Golden ratio

If $\frac{a}{b} = \frac{c}{a}$ then a is said to be the **geometric mean** of c and b (or sometimes the mean proportional of b and c).

Let AB be a line segment length a units and C a point on AB such that $\frac{AB}{AC} = \frac{AC}{CB}$.
Let $AC = x$. Therefore $CB = a - x$

The relation $\frac{a}{x} = \frac{x}{a-x}$ holds. AC is the geometric mean of AB and CB.

If $\frac{a}{x} = \frac{x}{a-x}$ then $a(a - x) = x^2$

Which implies that $x^2 + ax - a^2 = 0$

Therefore using the general quadratic formula
$$x = \frac{-a \pm \sqrt{a^2 - 4 \times 1 \times -a^2}}{2 \times 1}$$
$$= \frac{-a \pm \sqrt{5a^2}}{2 \times 1}$$
$$x = a \times \frac{-1 \pm \sqrt{5}}{2}$$

Only one of these is possible as AC is a length. Thus $AC = a \times \frac{-1 + \sqrt{5}}{2}$ (which is positive)

Therefore
$$\frac{AB}{AC} = a \div \frac{(-1+\sqrt{5})a}{2}$$
$$= \frac{2}{-1+\sqrt{5}}$$
$$= \frac{2}{-1+\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}}$$
$$= \frac{1+\sqrt{5}}{2}$$

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Hence the ratio $\frac{AB}{AC}$ is independent of the length of AB and is always the same number. This number is known as the **golden ratio** or section and is denoted by ϕ, i.e., $\phi = \frac{1+\sqrt{5}}{2}$

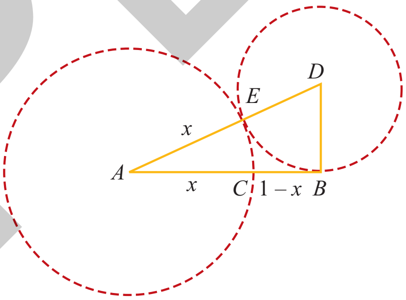
ϕ is the only number which when diminished by one becomes its own reciprocal,

i.e., $\phi - 1 = \frac{1}{\phi}$

This is shown as
$$\begin{aligned}\phi - 1 &= \frac{1+\sqrt{5}}{2} - 1\\ &= \frac{-1+\sqrt{5}}{2}\\ &= \frac{-1+\sqrt{5}}{2} \times \frac{1+\sqrt{5}}{1+\sqrt{5}}\\ &= \frac{2}{1+\sqrt{5}}\end{aligned}$$

A construction of the golden ratio is as follows.

Let AB be a segment of unit length. Draw BD of length $\frac{AB}{2}$, perpendicular to AB. Draw line segment AD. With centre D draw an arc of radius DB cutting AD at E. Draw an arc of radius AE with centre A cutting AB at C.

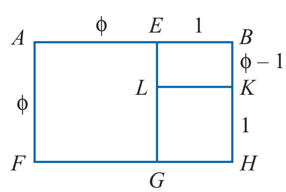


$$\frac{AB}{AC} = \frac{1}{x} = \phi$$

The golden rectangle

The rectangle $ABHF$ shown is known as the **golden rectangle**.

The ratio of the side lengths $AB : AF = 1 + \phi : \phi$



and
$$\begin{aligned}\frac{1+\phi}{\phi} &= \left(1 + \frac{1+\sqrt{5}}{2}\right) \div \left(\frac{1+\sqrt{5}}{2}\right)\\ &= \frac{3+\sqrt{5}}{2} \times \frac{2}{1+\sqrt{5}}\\ &= \frac{3+\sqrt{5}}{\sqrt{5}+1}\\ &= \frac{2+2\sqrt{5}}{4}\\ &= \frac{1+\sqrt{5}}{2} = \phi\end{aligned}$$

That is, the ratio of the side lengths is ϕ.

This rectangle has some very pleasant properties, as observed in the following explorations.

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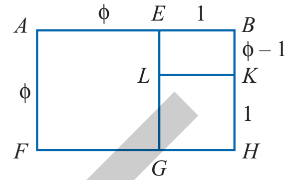
Forming a sequence of similar golden rectangles

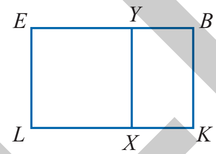


In the golden rectangle *ABHF*, construct square *AEGF* with side length ϕ.

The remaining rectangle *EBHG* has side lengths 1 and ϕ. Construct the square *LKHG* with side length 1. The sides of the remaining rectangle *EBKL* are 1 and $\phi - 1$.

It was established earlier in the section that $\phi - 1 = \frac{1}{\phi}$.

Thus the rectangles *ABHF*, *EBHG*, *EBKL* are all similar as they all have sides in the ratio $\phi : 1$. This pattern continues. Consider the golden rectangle *EBKL*.



Now rectangle *YBKX* has sides $\frac{1}{\phi}$ and $1 - \frac{1}{\phi}$.

The ratio $\quad \frac{1}{\phi} : 1 - \frac{1}{\phi} = 1 : \phi - 1$

and as shown previously

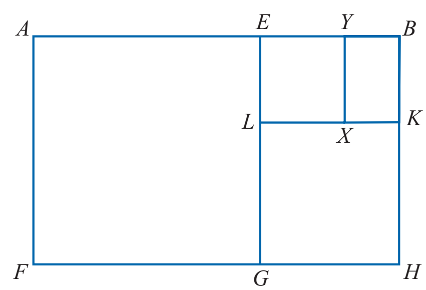
$$\begin{aligned} \phi - 1 &= \frac{1 + \sqrt{5}}{2} - 1 \\ &= \frac{-1 + \sqrt{5}}{2} \\ &= \frac{1}{\phi} \end{aligned}$$

Therefore $1 : \phi - 1 = \phi : 1$

It can be shown that all the rectangles formed in this way are similar to each other.

Forming a sequence of squares and rectangles, the areas of which are in geometric sequence with common ratio $\frac{1}{\phi}$

The ratio of the areas of the squares and rectangles is also worth considering. The areas in sequence are



Area rectangle *ABHF* $= (\phi + 1)\phi = \phi^3$

Area of square *AEGF* $= \phi^2$

Area of rectangle *EBHG* $= \phi$

Area of square *LKHG* $= 1$

Area of rectangle *EBKL* $= \frac{1}{\phi}$

Area of square *EYXL* $= \frac{1}{\phi^2}$

Area of rectangle *YBKX* $= \frac{1}{\phi^3}$

Exercise 9E

1 For the golden ratio ϕ show that

a $\phi - 1 = \dfrac{1}{\phi}$ **b** $\phi^3 = 2\phi + 1$

c $2 - \phi = (\phi - 1)^2 = \dfrac{1}{\phi^2}$

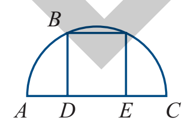
2 ABC is a right-angled triangle with the right angle at C. CX is the altitude of the triangle from C.

a Prove that $\dfrac{AX}{CX} = \dfrac{CX}{XB}$; i.e., the length CX is the geometric mean of lengths AX and XB.

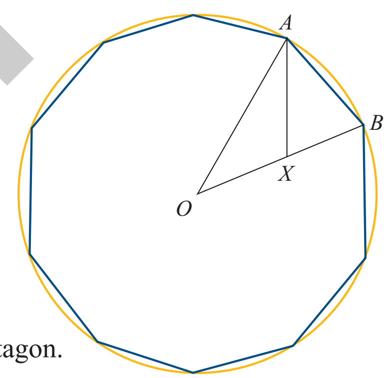
b Find CX if

i $AX = 2$ and $XB = 8$ **ii** $AX = 1$ and $XB = 10$.

3 A square is inscribed in a semicircle as shown. Prove that $\dfrac{AD}{BD} = \dfrac{BD}{CD} = \phi - 1$.



4 A regular decagon is inscribed in a circle with unit radius as shown.



a Find the magnitude of angle

i AOB **ii** OAB

b The line AX bisects angle OAB. Prove that

i triangle AXB is isosceles

ii triangle AXO is isosceles

iii triangle AOB is similar to triangle BXA

c Find the length of AB, to two decimal places.

d Describe a construction for

i a regular decagon **ii** a regular pentagon.

5 Calculate $\phi^0, \phi^1, \phi^2, \phi^3, \phi^4$ and $\phi^{-1}, \phi^{-2}, \phi^{-3}, \phi^{-4}$. Show that each power of ϕ is equal to the sum of the two powers before it, i.e., $\phi^{n+1} = \phi^n + \phi^{n-1}$

6 The Fibonacci sequence is defined by $t_1 = t_2 = 1$ and $t_{n+1} = t_{n-1} + t_n$. The sequence is 1, 1, 2, 3, 5, ... Consider the sequence $\dfrac{t_2}{t_1}, \dfrac{t_3}{t_2}, \dfrac{t_4}{t_3}, \dfrac{t_5}{t_4}$ ... and show that as n gets very large (n approaches infinity), $\dfrac{t_{n+1}}{t_n}$ approaches ϕ.

MC TEST D&D TEST

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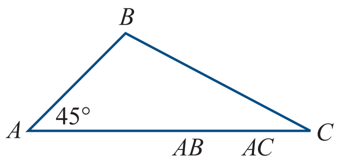
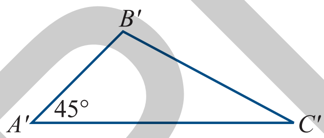
Review

Chapter summary

- Two figures are similar to each other if one is congruent to the other under an expansion from the origin of factor k. An expansion of factor k from the origin has rule $(x, y) \to (kx, ky)$
- **Similar triangles**
  Two triangles are similar if one of the following conditions holds.
  - Triangles have equal angles (AAA)
  - Corresponding sides are in the same ratio (PPP)
  - Two pairs of corresponding sides have the same ratio and the included angles are equal (PAP)



$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$



  If triangle ABC is similar to triangle XYZ, this can be written symbolically as $\Delta ABC \sim \Delta XYZ$. The triangles are named so that angles of equal magnitude hold the same position, i.e., A corresponds to X, B corresponds to Y, C corresponds to Z.
- If two shapes are similar and the scale factor is k, i.e. for any length AB of one shape, the corresponding length $A'B'$ of the similar shape has length kAB, then the area of the similar shape $= k^2 \times$ area of the original shape.
  For similar solids, if the scale factor is k, then the volume of the similar solid is $k^3 \times$ volume of the original solid.

Multiple-choice questions

1 If $5 : 3 = 7 : x$ then x is equal to
A 12 **B** $\frac{35}{3}$ **C** 5 **D** $\frac{21}{5}$ **E** $\frac{5}{21}$

2 Brass is composed of a mixture of copper and zinc. If the ratio copper : zinc is 85 : 15, then the amount of copper in 400 kg of brass is
A 60 kg **B** 340 kg **C** 360 kg **D** 380 kg **E** 150 kg

3 If the total cost of P articles is Q dollars, then the cost of R articles of the same type is
A PQR **B** $\frac{P}{QR}$ **C** $\frac{PQ}{R}$ **D** $\frac{QR}{P}$ **E** $\frac{R}{PQ}$

4 A car is 3.2 m long. The length in cm of a model of the car if the scale is 1 : 100 is
A 0.032 **B** 0.32 **C** 3.2 **D** 320 **E** 32

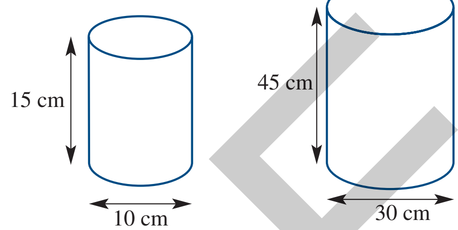
5 An athlete runs 75 m in 9 seconds. If she were to maintain the same average speed for 100 m her time for 100 m in seconds would be
A 11.6 **B** 12.0 **C** 11.8 **D** 12.2 **E** 12.4

6 If 50 is divided into three parts in the ratio 1 : 3 : 6 then the largest part is

A 5 **B** 15 **C** $\frac{50}{3}$ **D** 30 **E** 3

7 Two similar cylinders are shown.

The ratio of the volume of the smaller cylinder to the larger cylinder is



A 1 : 3 **B** 1 : 9 **C** 1 : 27

D 1 : 5 **E** 2 : 9

8 The radius of sphere A is $\frac{4}{5}$ times the radius of sphere B. Hence, the ratio of the volume of sphere A to the volume of sphere B is

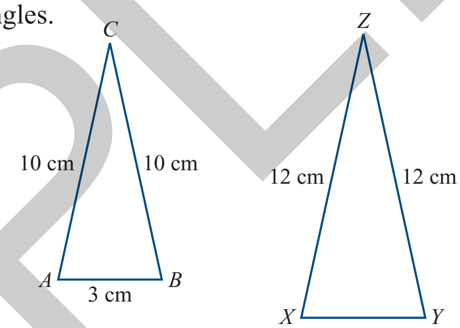
A 16 : 25 **B** 4 : 5 **C** 5 : 4 **D** 25 : 16 **E** 64 : 125

9 Triangles *ABC* and *XYZ* are similar isosceles triangles.

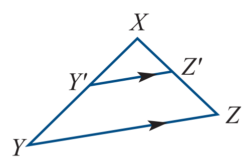
The length of *XY* is

A 4 cm **B** 5 cm **C** 4.2 cm

D 2.5 cm **E** 3.6 cm



10 *YZ* is parallel to *Y′Z′* and $Y'Y = \frac{1}{3}YX$. The area of triangle *XYZ* is 60 cm^2. The area of triangle *XY′Z′* is



A 20 cm^2 **B** 30 cm^2 **C** $\frac{20}{9}$ cm^2

D $\frac{20}{3}$ cm^2 **E** $\frac{80}{3}$ cm^2

Short-answer questions (technology-free)

1 In triangle *XYZ*, *P* is a point on *XY* and *Q* is a point on *XZ* such that *PQ* is parallel to *YZ*.

a Show that the two triangles *XYZ* and *XPQ* are similar.

b If *XY* = 36 cm, *XZ* = 30 cm and *XP* = 24 cm, find

i *XQ* **ii** *QZ*

c Write down the values of *XP* : *PY* and *PQ* : *YZ*.

2 Triangles *ABC* and *DEF* are similar. If the area of triangle *ABC* is 12.5 cm^2, the area of triangle *DEF* is 4.5 cm^2 and *AB* = 5 cm, find

a the length of *DE* **b** the value of *AC* : *DF* **c** the value of *EF* : *BC*.

3 If a 1 m stake casts a shadow 2.3 m long, find the height of a tree (in metres) which casts a shadow 21 m long.

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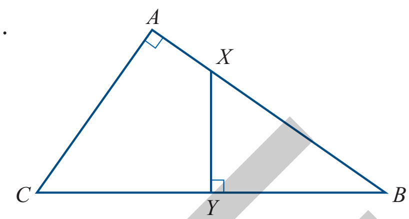
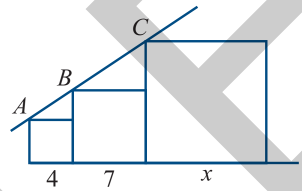
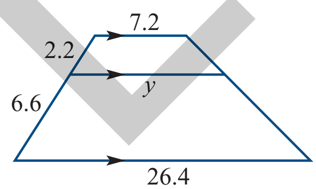
4 *ABC* is a right-angled triangle with $AB = 4$ and $AC = 3$. If the triangle is folded along the line *XY*, vertex *C* coincides with vertex *B*. Find the length of *XY*.



5 Points *A*, *B* and *C* lie on a straight line. The squares are adjacent and have side lengths 4, 7 and *x*. Find the value of *x*.



6 Find the value of *y* in the diagram on the right.

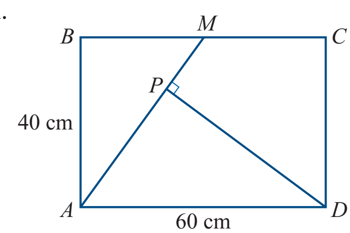


7 An alloy is produced by mixing metal *X* with metal *Y* in the ratio of 5 : 3 by volume. The mass of 1 cm^3 of metal *X* is $\frac{8}{5}$ g and of 1 cm^3 of metal *Y* is $\frac{4}{3}$ g. Calculate

- **a** the mass of a solid cube of alloy of edge 4 cm
- **b** the ratio, in the form $n : 1$, by mass, of metal *X* to metal *Y* in the alloy
- **c** the volume, to the nearest cm^3, of a cubic block of alloy whose mass is 1.5 kg
- **d** the length, in mm, of the edge of this cubic block.

8 *ABCD* is a rectangle in which $AB = 40$ cm and $AD = 60$ cm. *M* is the midpoint of *BC*, and *DP* is perpendicular to *AM*.

- **a** Prove that the triangles *BMA* and *PAD* are similar.
- **b** Calculate the ratio of the areas of the triangles *BMA* and *PAD*.
- **c** Calculate the length of *PD*.



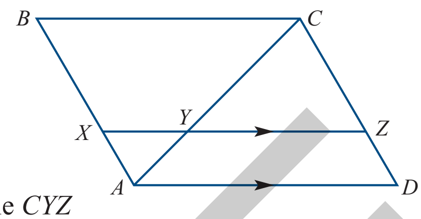
9 A sculptor is commissioned to create a bronze statue 2 m high. He begins by making a clay model 30 cm high.

- **a** Express, in simplest form, the ratio of the height of the completed bronze statue to the height of the clay model.
- **b** If the total surface area of the model is 360 cm^2, find the total surface area of the statue.
- **c** If the total volume of the model is 1000 cm^3, find the volume of the statue.

10 The radius of a spherical soap bubble increases by 1%. Find, correct to the nearest whole number, the percentage increase in

- **a** its surface area
- **b** its volume.

11 AC is the diagonal of a rhombus $ABCD$. The line XYZ is parallel to AD, $AX = 3$ cm and $AB = 9$ cm.



Find

a $\dfrac{XY}{BC}$ **b** $\dfrac{AY}{AC}$ **c** $\dfrac{CY}{AC}$

d $\dfrac{YZ}{AD}$ **e** $\dfrac{\text{area triangle } AXY}{\text{area triangle } ABC}$ **f** $\dfrac{\text{area triangle } CYZ}{\text{area triangle } ACD}$

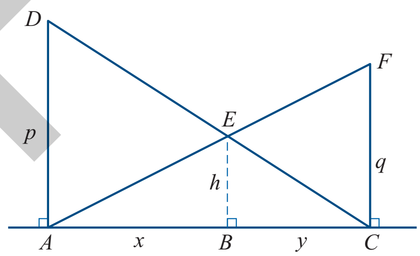
12 AB and DC are parallel sides of a trapezium and $DC = 3AB$. The diagonals AC and DB intersect at O. Prove that $AO = \frac{1}{4} AC$.

13 Triangles ABC and PQR are similar. The medians AX and PY are drawn. (X is the midpoint of BC and Y is the midpoint of QR.) Prove

a that triangles ABX and PQY are similar **b** $\dfrac{AX}{PY} = \dfrac{BC}{QR}$

Extended-response questions

1 a In this diagram which other triangle is similar to ΔDAC?

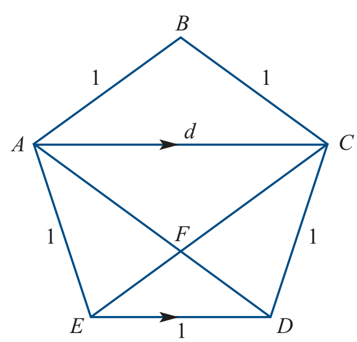


b Explain why $\dfrac{h}{p} = \dfrac{y}{x+y}$.

c Use another pair of similar triangles to write down an expression for $\dfrac{h}{q}$ in terms of x and y.

d Explain why $h\left(\dfrac{1}{p} + \dfrac{1}{q}\right) = 1$. **e** Calculate h when $p = 4$ and $q = 5$.

2 $ABCDE$ is a regular pentagon whose sides are each 1 unit long. Each diagonal is of length d units. In a regular pentagon, each diagonal is parallel to one of the sides of the pentagon.



a What kind of shape is $ABCF$ and what is the length of CF?

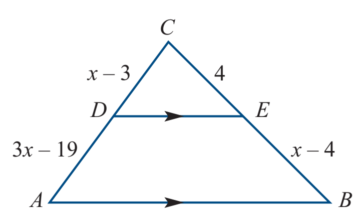
b Explain why the length of EF is $d - 1$.

c Which triangle is similar to ΔEFD?

d Use the pair of similar triangles to write an equation for d and show that the equation can be rewritten as $d^2 - d - 1 = 0$.

e Find d.

3 Place conditions upon x such that DE is parallel to AB given that $CD = x - 3$, $DA = 3x - 19$, $CE = 4$ and $EB = x - 4$.



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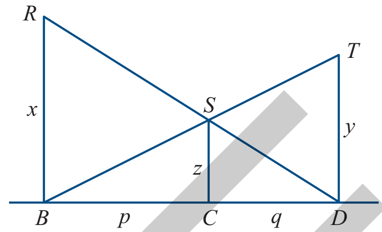
Review

4 **a** If BR, CS and DT are perpendicular to BD, name the pairs of similar triangles.



b Which is correct:

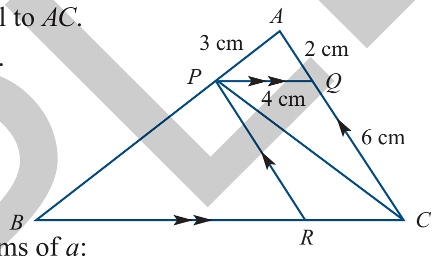
$$\frac{z}{y} = \frac{p}{q} \text{ or } \frac{z}{y} = \frac{p}{p+q}?$$

c Which is correct:

$$\frac{z}{x} = \frac{q}{p} \text{ or } \frac{z}{x} = \frac{q}{p+q}?$$

d Show that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$

5 In the diagram, PQ is parallel to BC and PR is parallel to AC. $AQ = 2$ cm, $QC = 6$ cm, $AP = 3$ cm and $PQ = 4$ cm.



a Calculate

i PB **ii** BR

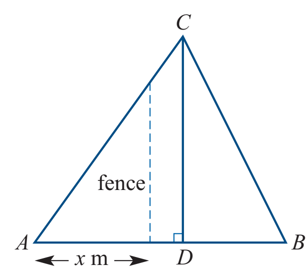
iii $\dfrac{\text{area } \Delta APQ}{\text{area } \Delta ABC}$ **iv** $\dfrac{\text{area } \Delta BPR}{\text{area } \Delta ABC}$

b If the area of triangle APQ is a cm^2, express in terms of a:

i area ΔABC **ii** area ΔCPQ

6 Construct a triangle ABC such that $BC = 10$ cm, $AC = 9$ cm and $AB = 6$ cm. Find a point D on AB and a point E on AC, such that DE is parallel to BC and the area of ΔADE is one-ninth the area of ΔABC.

7 A triangular lot has boundaries of lengths $AB = 130$ m, $BC = 40\sqrt{10}$ m and $CA = 150$ m. The length of CD is 120 m. A fence is to be erected which runs at right angles from AB. If the lot is to be divided into two equal areas, find x.



8 The Greek historian Herodotus wrote that the proportions of the great pyramid at Giza in Egypt were chosen so that the area of a square, for which the side lengths are equal to the height of the great pyramid, is equal to the area of one of the triangular faces.

Let h m be the height of the pyramid, k m the altitude of one of the face triangles, and b m be the length of a side of the square base.

Show that Herodotus' definition gives $k : \frac{b}{2} = \phi$.