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CHAPTER 14

Circle theorems

Objectives

- To establish the following results and use them to prove further properties and solve problems:
  - The angle subtended at the circumference is half the angle at the centre subtended by the same arc
  - Angles in the same segment of a circle are equal
  - A tangent to a circle is perpendicular to the radius drawn from the point of contact
  - The two tangents drawn from an external point to a circle are the same length
  - The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment
  - A quadrilateral is **cyclic** (that is, the four vertices lie on a circle) if and only if the sum of each pair of opposite angles is two right angles
  - If AB and CD are two chords of a circle which cut at a point P (which may be inside or outside a circle) then $PA \cdot PB = PC \cdot PD$
  - If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant then $PT^2 = PA \cdot PB$

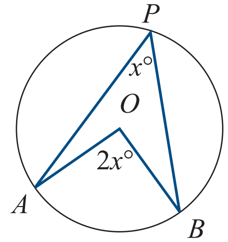
These theorems and related results can be investigated through a geometry package such as Cabri Geometry.

It is assumed in this chapter that the student is familiar with basic properties of parallel lines and triangles.

14.1 Angle properties of the circle

Theorem 1

The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



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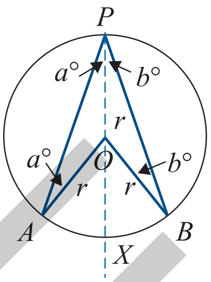
Proof



Join points P and O and extend the line through O as shown in the diagram.

Note that $AO = BO = PO = r$ the radius of the circle. Therefore triangles PAO and PBO are isosceles.

Let $\angle APO = \angle PAO = a^\circ$ and $\angle BPO = \angle PBO = b^\circ$

Then angle AOX is $2a^\circ$ (exterior angle of a triangle) and angle BOX is $2b^\circ$ (exterior angle of a triangle)

$\therefore \quad \angle AOB = 2a^\circ + 2b^\circ = 2(a+b)^\circ = 2\angle APB$

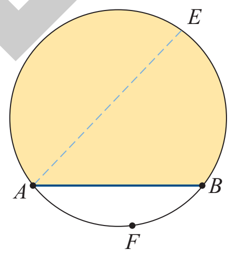
Note: In the proof presented above, the centre and point P are considered to be on the same side of chord AB.

The proof is not dependent on this and the result always holds.

The converse of this result also holds:

i.e., if A and B are points on a circle with centre O and angle APB is equal to half angle AOB, then P lies on the circle.

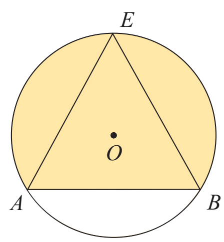
A **segment** of a circle is the part of the plane bounded by an arc and its chord.



Arc AEB and chord AB define a major segment which is shaded.

Arc AFB and chord AB define a minor segment which is not shaded.

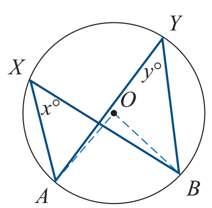
$\angle AEB$ is said to be an angle in segment AEB.



Theorem 2

Angles in the same segment of a circle are equal.

Proof



Let $\angle AXB = x^\circ$ and $\angle AYB = y^\circ$

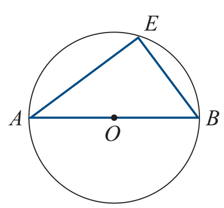
Then by Theorem 1 $\angle AOB = 2x^\circ = 2y^\circ$

Therefore $x = y$

Theorem 3

The angle subtended by a diameter at the circumference is equal to a right angle (90°).

Proof



The angle subtended at the centre is 180°.

Theorem 1 gives the result.

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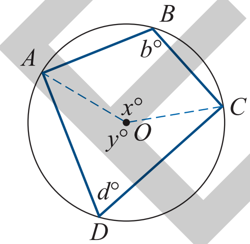
A quadrilateral which can be inscribed in a circle is called a **cyclic quadrilateral.**

Theorem 4

The opposite angles of a quadrilateral inscribed in a circle sum to two right angles (180°). (The opposite angles of a cyclic quadrilateral are supplementary). The converse of this result also holds.

Proof

O is the centre of the circle



By Theorem 1 $\quad y = 2b$ and $x = 2d$

Also $\quad x + y = 360$

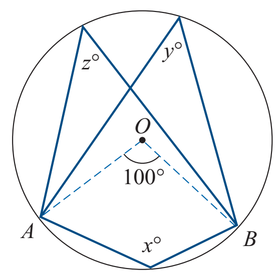
Therefore $\quad 2b + 2d = 360$

i.e. $\quad b + d = 180$

The converse states: if a quadrilateral has opposite angles supplementary then the quadrilateral is inscribable in a circle.

Example 1

Find the value of each of the pronumerals in the diagram. O is the centre of the circle and $\angle AOB = 100°$.



Solution

Theorem 1 gives that $z = y = 50$

The value of x can be found by observing either of the following.

Reflex angle AOB is 260°. Therefore $x = 130$ (Theorem 1)

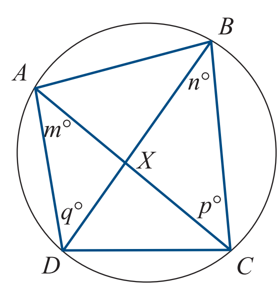
or $y + x = 180$ (Theorem 4)

Therefore $x = 180 - 50 = 130$

Example 2

A, B, C, D are points on a circle. The diagonals of quadrilateral $ABCD$ meet at X. Prove that triangles ADX and BCX are similar.

Solution



$\angle DAC$ and $\angle DBC$ are in the same segment.

Therefore $m = n$

$\angle BDA$ and $\angle BCA$ are in the same segment.

Therefore $p = q$

Also $\angle AXD = \angle BXC$ (vertically opposite).

Therefore triangles ADX and BCX are equiangular and thus similar.

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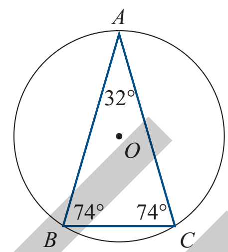
Example 3

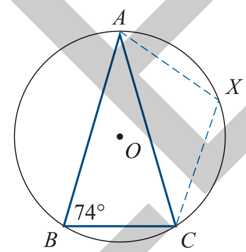
An isosceles triangle is inscribed in a circle. Find the angles in the three minor segments of the circle cut off by the sides of this triangle.



Solution

First, to determine the magnitude of $\angle AXC$ cyclic quadrilateral $AXCB$ is formed. Thus $\angle AXC$ and $\angle ABC$ are supplementary.

Therefore $\angle AXC = 106°$. All angles in the minor segment formed by AC will have this magnitude.



In a similar fashion it can be shown that the angles in the minor segment formed by AB all have magnitude $106°$ and for the minor segment formed by BC the angles all have magnitude $148°$.

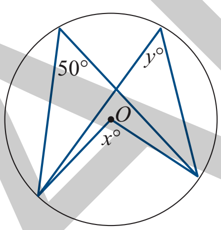
Exercise 14A

Skillsheet TEACHER

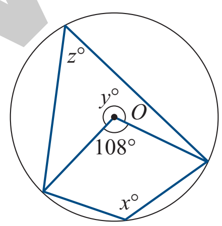
Example 1

1 Find the values of the pronumerals for each of the following, where O denotes the centre of the given circle.

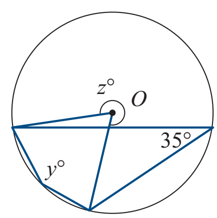
a



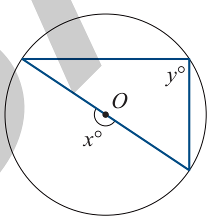
b



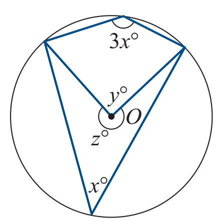
c



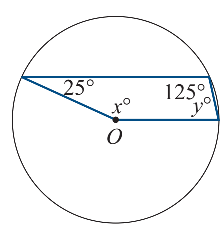
d



e

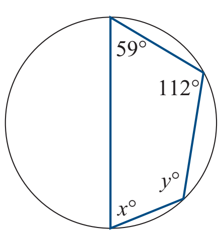


f



2 Find the value of the pronumerals for each of the following.

a



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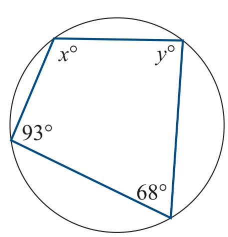
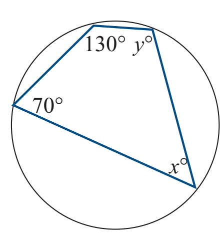
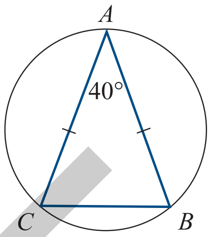


c



3 An isosceles triangle ABC is inscribed in a circle. What are the angles in the three minor segments cut off by the sides of the triangle?



4 $ABCDE$ is a pentagon inscribed in a circle. If $AE = DE$ and $\angle BDC = 20°$, $\angle CAD = 28°$ and $\angle ABD = 70°$, find all of the interior angles of the pentagon.

Example 2 **5** If two opposite sides of a cyclic quadrilateral are equal, prove that the other two sides are parallel.

6 $ABCD$ is a parallelogram. The circle through A, B and C cuts CD (produced if necessary) at E. Prove that $AE = AD$.

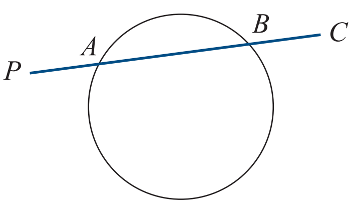
Example 3 **7** $ABCD$ is a cyclic quadrilateral and O is the centre of the circle through A, B, C and D. If $\angle AOC = 120°$, find the magnitude of $\angle ADC$.

8 Prove that if a parallelogram is inscribed in a circle it must be a rectangle.

9 Prove that the bisectors of the four interior angles of a quadrilateral form a cyclic quadrilateral.

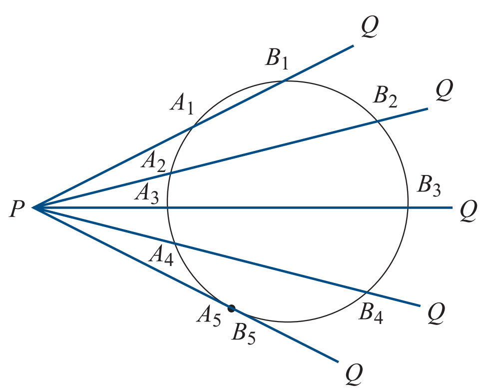
14.2 Tangents

Line PC is called a secant and line segment AB a chord.



If the secant is rotated with P as the pivot point a sequence of pairs of points on the circle is defined. As PQ moves towards the edge of the circle the points of the pairs become closer until they eventually coincide.

When PQ is in this final position (i.e., where the intersection points A and B collide) it is called a **tangent** to the circle. PQ touches the circle. The point at which the tangent touches the circle is called the **point** of contact. The **length of a tangent** from a point P outside the tangent is the distance between P and the point of contact.



Theorem 5

A tangent to a circle is perpendicular to the radius drawn to the point of contact.

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Proof

Let T be the point of contact of tangent PQ.

Let S be the point on PQ, not T, such that OSP is a right angle.

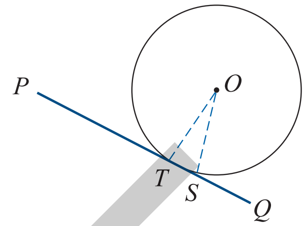
Triangle OST has a right angle at S.

Therefore $OT > OS$ as OT is the hypotenuse of triangle OTS.

$\therefore$ S is inside the circle as OT is a radius.

$\therefore$ The line through T and S must cut the circle again. But PQ is a tangent. A contradiction.

Therefore $T = S$ and angle OTP is a right angle.



Theorem 6

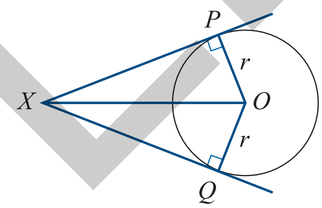
The two tangents drawn from an external point to a circle are of the same length.

Proof

Triangle XPO is congruent to triangle XQO as XO is a common side.

$$\angle XPO = \angle XQO = 90^\circ$$
$$OP = OQ \text{ (radii)}$$

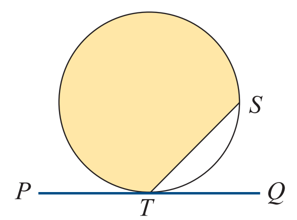
Therefore $XP = XQ$



Alternate segment theorem

The shaded segment is called the alternate segment in relation to $\angle STQ$.

The unshaded segment is alternate to $\angle PTS$



Theorem 7

The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.

Proof

Let $\angle STQ = x^\circ$, $\angle RTS = y^\circ$ and $\angle TRS = z^\circ$ where RT is a diameter.

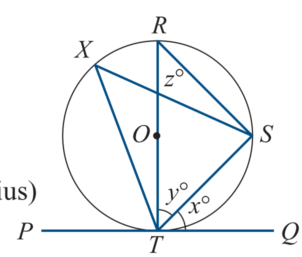
Then $\angle RST = 90^\circ$ (Theorem 3, angle subtended by a diameter)

Also $\angle RTQ = 90^\circ$ (Theorem 5, tangent is perpendicular to radius)

Hence $x + y = 90$ and $y + z = 90$

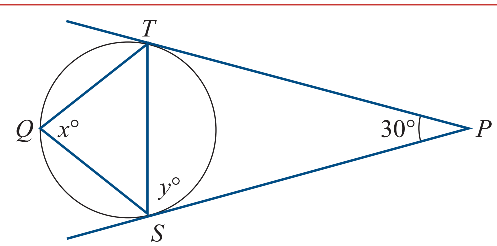
Therefore $x = z$

But $\angle TXS$ is in the same segment as $\angle TRS$ and therefore $\angle TXS = x^\circ$



Example 4

Find the magnitude of the angles x and y in the diagram.



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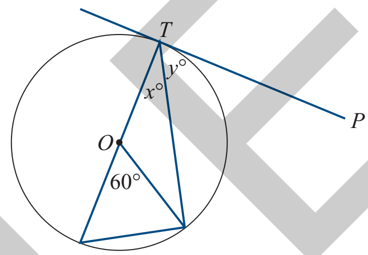
Solution

Triangle PTS is isosceles (Theorem 6, two tangents from the same point) and therefore $\angle PTS = \angle PST$

Hence $y = 75$. The alternate segment theorem gives that $x = y = 75$

Example 5

Find the values of x and y.
PT is tangent to the circle centre O

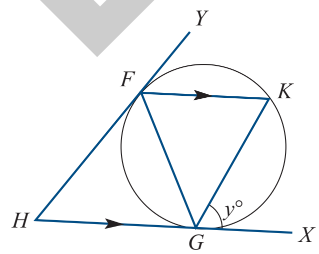


Solution

$x = 30$ as the angle at the circumference is half the angle subtended at the centre and $y = 60$ as $\angle OTP$ is a right angle.

Example 6

The tangents to a circle at F and G meet at H. If a chord FK is drawn parallel to HG, prove that triangle FGK is isosceles.



Solution

Let $\angle XGK = y°$
Then $\angle GFK = y°$ (alternate segment theorem)
and $\angle GKF = y°$ (alternate angles)

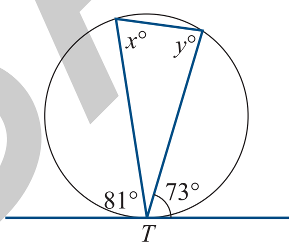
Therefore triangle FGK is isosceles with $FG = KG$

Exercise 14B

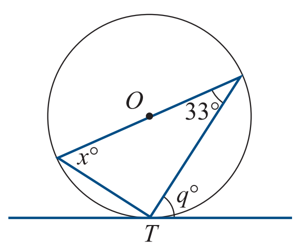
Example 4

1 Find the value of the pronumerals for each of the following. T is the point of contact of the tangent and O the centre of the circle.

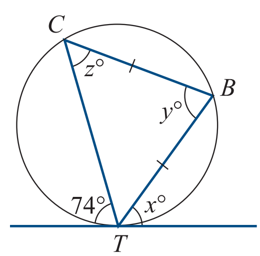
a



b

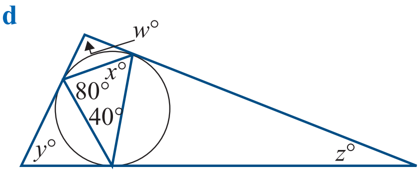


c

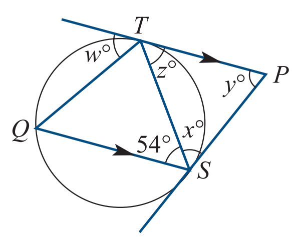


$BC = BT$

d



e



S and T are points of contact of tangents from P.
TP is parallel to QS

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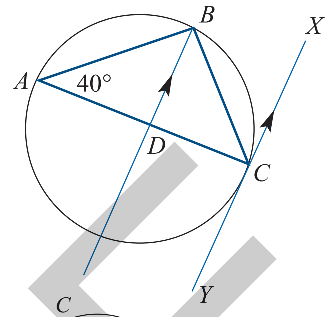
Example 5

2 A triangle ABC is inscribed in a circle, and the tangent at C to the circle is parallel to the bisector of angle ABC.

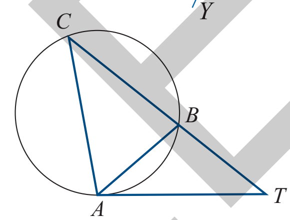
a Find the magnitude of $\angle BCX$.

b Find the magnitude of $\angle CBD$, where D is the point of intersection of the bisector of angle ABC with AC.

c Find the magnitude of $\angle ABC$.



3 AT is a tangent at A and TBC is a secant to the circle. Given $\angle CTA = 30°$, $\angle CAT = 110°$, find the magnitude of angles ACB, ABC and BAT.



4 If AB and AC are two tangents to a circle and $\angle BAC = 116°$, find the magnitudes of the angles in the two segments into which BC divides the circle.

Example 6

5 From a point A outside a circle, a secant ABC is drawn cutting the circle at B and C, and a tangent AD touching it at D. A chord DE is drawn equal in length to chord DB. Prove that triangles ABD and CDE are similar.

6 AB is a chord of a circle and CT, the tangent at C, is parallel to AB. Prove that $CA = CB$.

7 Through a point T, a tangent TA and a secant TPQ are drawn to a circle AQP. If the chord AB is drawn parallel to PQ, prove that the triangles PAT and BAQ are similar.

8 PQ is a diameter of a circle and AB is a perpendicular chord cutting it at N. Prove that PN is equal in length to the perpendicular from P on to the tangent at A.

14.3 Chords in circles

Theorem 8

If AB and CD are two chords which cut at a point P (which may be inside or outside the circle) then $PA \cdot PB = PC \cdot PD$.

Proof

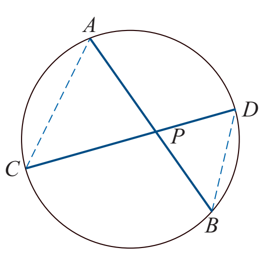
CASE 1 (The intersection point is inside the circle.)

Consider triangles APC and BPD.

$\angle APC = \angle BPD$ (vertically opposite)

$\angle CDB = \angle CAB$ (angles in the same segment)

$\angle ACD = \angle DBA$ (angles in the same segment)



Therefore triangle CAP is similar to triangle BDP.

Therefore

$$\frac{AP}{PD} = \frac{CP}{PB}$$

and $AP \cdot PB = CP \cdot PD$, which can be written $PA \cdot PB = PC \cdot PD$

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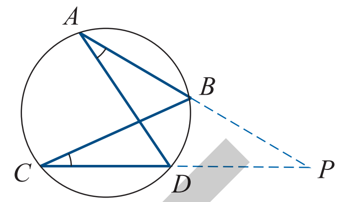
CASE 2 (The intersection point is outside the circle.)
Show triangle APD is similar to triangle CPB
Hence

$$\frac{AP}{CP} = \frac{PD}{PB}$$

i.e. $\quad AP \cdot PB = PD.CP$

which can be written $PA \cdot PB = PC \cdot PD$



Theorem 9

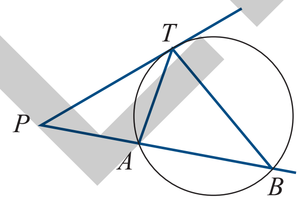
If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant then $PT^2 = PA \cdot PB$

Proof

$$\angle PTA = \angle TBA \text{ (alternate segment theorem)}$$
$$\angle PTB = \angle TAP \text{ (angle sum of a triangle)}$$

Therefore triangle PTB is similar to triangle PAT

$$\therefore \quad \frac{PT}{PA} = \frac{PB}{PT} \text{ which implies } PT^2 = PA \cdot PB$$



Example 7

The arch of a bridge is to be in the form of an arc of a circle. The span of the bridge is to be 25 m and the height in the middle 2 m. Find the radius of the circle.

Solution

By Theorem 8

$$RP \cdot PQ = MP \cdot PN$$

Therefore

$$2PQ = 12.5^2$$

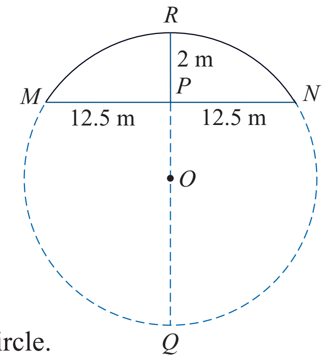
$$\therefore \quad PQ = \frac{12.5^2}{2}$$

Also $\quad PQ = 2r - 2$ where r is the radius of the circle.

Hence $\quad 2r - 2 = \frac{12.5^2}{2}$

and $\quad r = \frac{1}{2}\left(\frac{12.5^2}{2} + 2\right)$

$$= \frac{641}{16} \text{ m}$$



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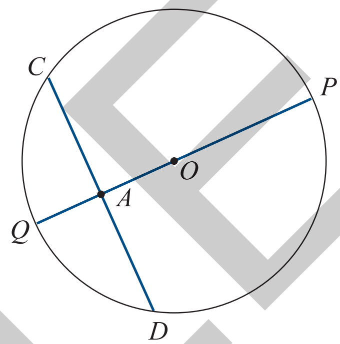
Example 8

If r is the radius of a circle, with center O, and if A is any point inside the circle, show that the product $CA \cdot AD = r^2 - OA^2$, where CD is a chord through A.

Solution



Let PQ be a diameter through A

Theorem 8 gives that

$$CA \cdot AD = QA \cdot AP$$

Also $\quad QA = r - OA$ and $PA = r + OA$

$\therefore \quad CA \cdot AD = r^2 - OA^2$

Exercise 14C

Example 7

1 Two chords AB and CD intersect at a point P within a circle. Given that

a $AP = 5$ cm, $PB = 4$ cm, $CP = 2$ cm, find PD

b $AP = 4$ cm, $CP = 3$ cm, $PD = 8$ cm, find PB.

2 If AB is a chord and P is a point on AB such that $AP = 8$ cm, $PB = 5$ cm and P is 3 cm from the centre of the circle, find the radius.

3 If AB is a chord of a circle with centre O and P is a point on AB such that $BP = 4PA$, $OP = 5$ cm and the radius of the circle is 7 cm, find AB.

Example 8

4 Two circles intersect at A and B and, from any point P on AB produced tangents PQ and PR are drawn to the circles. Prove that $PQ = PR$.

5 PQ is a variable chord of the smaller of two fixed concentric circles. PQ produced meets the circumference of the larger circle at R. Prove that the product $RP.RQ$ is constant for all positions and lengths of PQ.

6 ABC is an isosceles triangle with $AB = AC$. A line through A meets BC at D and the circumcircle of the triangle at E. Prove that $AB^2 = AD \cdot AE$.

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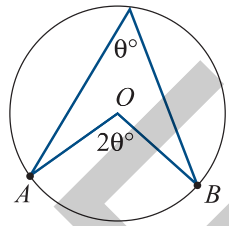
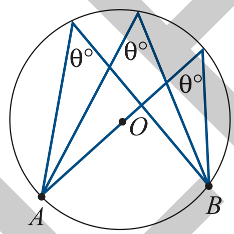
Chapter summary

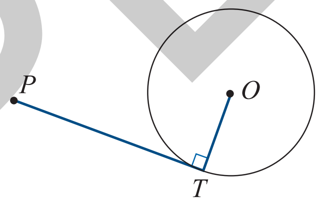
- The angle subtended at the circumference is half the angle at the centre subtended by the same arc.



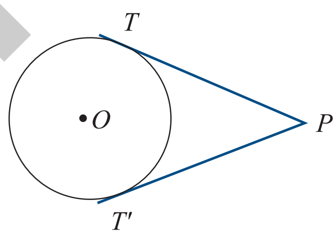
- Angles in the same segment of a circle are equal.



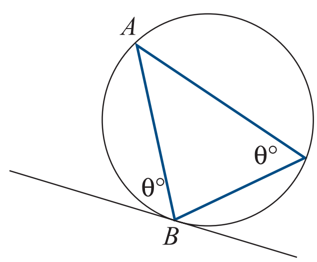
- A tangent to a circle is perpendicular to the radius drawn from the point of contact.



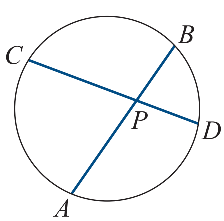
- The two tangents drawn from an external point are the same length i.e. $PT = PT'$.

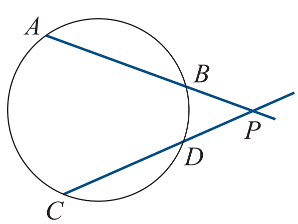


- The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



- A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is two right angles.
- If AB and CD are two chords of a circle which cut at a point P then $PA \cdot PB = PC \cdot PD$.

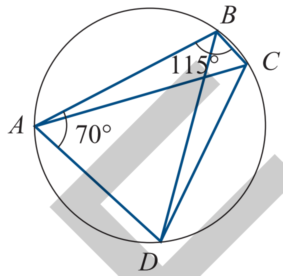




Multiple-choice questions

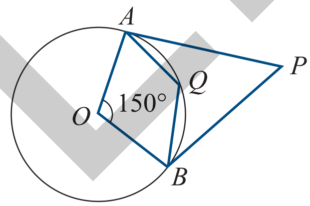
1 In the diagram A, B, C and D are points on the circumference of a circle. $\angle ABC = 115°$, $\angle BAD = 70°$ and $AB = AD$. The magnitude of $\angle ACD$ is

A 45° **B** 55° **C** 40° **D** 70° **E** 50°



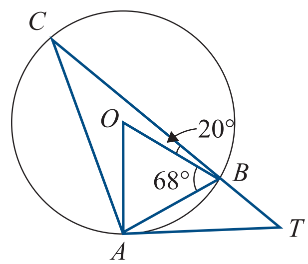
2 In the diagram, PA and PB are tangents to the circle centre O. Given that Q is a point on the minor arc AB and that $\angle AOB = 150°$ the magnitudes of $\angle APB$ and $\angle AQB$ are

A $\angle APB = 30°$ and $\angle AQB = 105°$
B $\angle APB = 40°$ and $\angle AQB = 110°$
C $\angle APB = 25°$ and $\angle AQB = 105°$
D $\angle APB = 30°$ and $\angle AQB = 110°$
E $\angle APB = 25°$ and $\angle AQB = 100°$



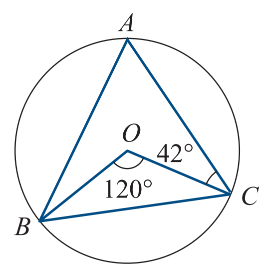
3 A circle centre O, passes through A, B and C. AT is the tangent to the circle at A. CBT is a straight line. Given that $\angle ABO = 68°$ and $\angle OBC = 20°$ the magnitude of $\angle ATB$ is

A 60° **B** 64° **C** 65° **D** 70° **E** 66°



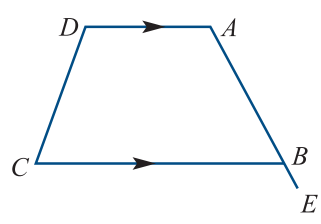
4 In the diagram the points A, B and C lie on a circle centre O. $\angle BOC = 120°$ and $\angle ACO = 42°$. The magnitude of $\angle ABO$ is

A 18° **B** 20° **C** 22° **D** 24° **E** 26°



5 $ABCD$ is a cyclic quadrilateral with AD parallel to BC. $\angle DCB = 65°$. The magnitude of $\angle CBE$ is

A 100° **B** 110° **C** 115° **D** 120° **E** 122°



6 A chord AB of a circle subtends an angle of 50° at a point on the circumference of the circle. The acute angle between the tangents at A and B has magnitude

A 80° **B** 65° **C** 75° **D** 85° **E** 82°

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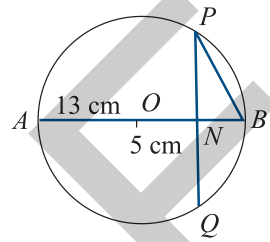
7 Chords AB and CD of circle intersect at P. If $AP = 12$ cm, $PB = 6$ cm and $CP = 2$ cm, the length of PD in centimetres is

A 12 **B** 24 **C** 36 **D** 48 **E** 56

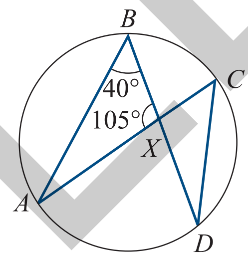
8 In the diagram AB is the diameter of a circle with centre O. PQ is a chord perpendicular to AB. N is the point of intersection of AB and PQ and $ON = 5$ cm. If the radius of the circle is 13 cm the length of chord PB is, in centimetres,

A 12 **B** $4\sqrt{13}$ **C** $2\sqrt{13}$ **D** 14 **E** 8



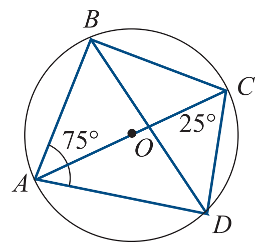
9 In the diagram A, B, C and D are points on the circumference of a circle. $\angle ABD = 40°$ and angle $\angle AXB = 105°$.
The magnitude of $\angle XDC$ is

A 35° **B** 40° **C** 45° **D** 50° **E** 55°



10 A, B, C and D are points on a circle, centre O such that AC is a diameter of the circle. If $\angle BAD = 75°$ and $\angle ACD = 25°$
The magnitude of $\angle BDC$ is

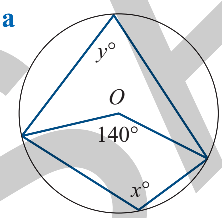
A 10° **B** 15° **C** 20° **D** 25° **E** 30°



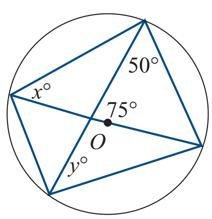
Short-answer questions (technology-free)

1 Find the value of the pronumerals in each of the following.

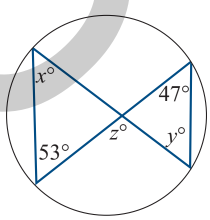
a



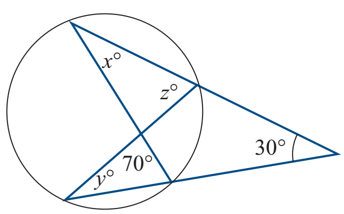
b



c



d



2 O is the centre of a circle and OP is any radius. A chord BA is drawn parallel to OP. OA and BP intersect at C. Prove that

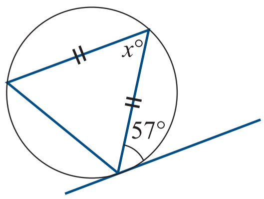
a $\angle CAB = 2\angle CBA$ **b** $\angle PCA = 3\angle PBA$

3 A chord AB of a circle, centre O, is produced to C. The straight line bisecting $\angle OAB$ meets the circle at E. Prove that EB bisects $\angle OBC$

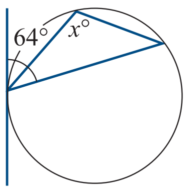
4 Two circles intersect at A and B. The tangent at B to one circle meets the second again at D, and a straight line through A meets the first circle at P and the second at Q. Prove that BP is parallel to DQ.

5 Find the values of the pronumerals for each of the following:

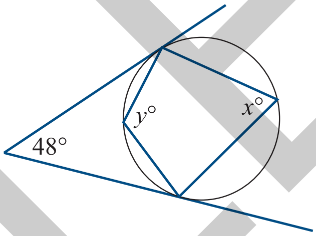
a



b

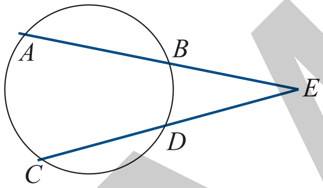


c



6 Two circles intersect at M and N. The tangent to the first at M meets the second circle at P, while the tangent to the second at N meets the first at Q. Prove that $MN^2 = NP \cdot QM$.

7 If $AB = 10$ cm, $BE = 5$ cm and $CE = 25$ cm, find DE.

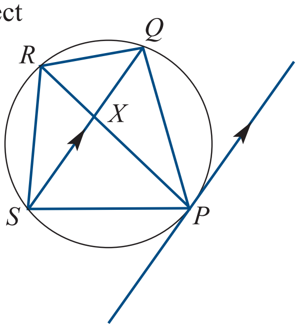


Extended-response questions

1 The diagonals PR and QS of a cyclic quadrilateral $PQRS$ intersect at X. The tangent at P is parallel to QS. Prove that

a $PQ = PS$

b PR bisects $\angle QRS$.

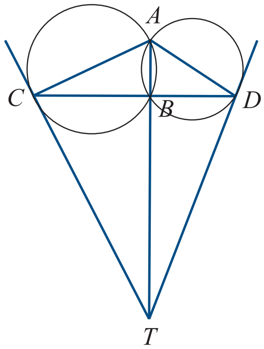


2 Two circles intersect at A and B. The tangents at C and D intersect at T on AB produced. If CBD is a straight line prove that

a $TCAD$ is a cyclic quadrilateral

b $\angle TAC = \angle TAD$

c $TC = TD$.



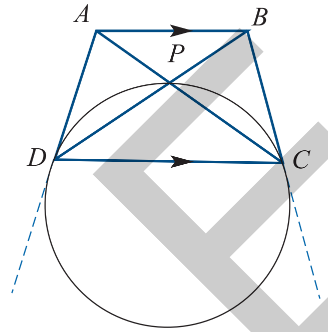
3 $ABCD$ is a trapezium in which AB is parallel to DC and the diagonals meet at P. The circle through D, P and C touches AD, BC at D and C respectively.

Prove that

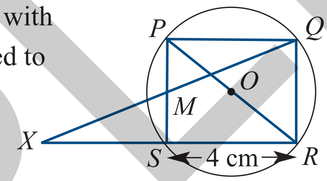
a $\angle BAC = \angle ADB$

b the circle through A, P and D touches BA at A

c $ABCD$ is a cyclic quadrilateral.



4 $PQRS$ is a square of side length 4 cm inscribed in a circle with centre O. M is the midpoint of the side PS. QM is produced to meet RS produced at X.



a Prove that:

i XPR is isosceles

ii PX is a tangent to the circle at P.

b Calculate the area of trapezium $PQRX$.

5 **a** An isosceles triangle ABC is inscribed in a circle. $AB = AC$ and chord AD intersects BC at E. Prove that

$$AB^2 - AE^2 = BE \cdot CE$$

b Diameter AB of circle with centre O is extended to C and from C a line is drawn tangent to the circle at P. PT is drawn perpendicular to AB at T. Prove that

$$CA \cdot CB - TA \cdot TB = CT^2$$